

# THE GEOMETRY OF 1-BASED MINIMAL TYPES

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ABSTRACT. In this paper, we study the geometry of a (non-trivial) 1-based  $SU$  rank-1 complete type. We show that if the (localized, resp.) geometry of the type is modular, then the (localized, resp.) geometry is projective over a division ring. However, unlike the stable case, we construct a locally modular type which is not affine. For the general 1-based case, we prove that even if the geometry of the type itself is not projective over a division ring, it is when we consider a 2-fold or 3-fold of the geometry altogether. In particular, it follows that in any  $\omega$ -categorical, non-trivial, 1-based theory, a vector space over a finite field is interpretable.

Geometric stability theory is one of the most important themes in model theory. Originally developing as a pure subject, it has turned out to be the major technical bridge connecting pure model theory and its applications to algebraic geometry and number theory. It mainly focuses on rank-1 types/structures or regular types where a canonical combinatorial geometry can be assigned. In other words, it is primarily concerned with geometric aspects of Shelah's stability theory [19], the study of stable structures. Arguably, the first major achievements of geometric stability theory are Zilber's results from the early 1980s (the translated version is [23]) on a strongly minimal  $\omega$ -categorical structure. He showed that the geometry assigned to the structure is locally modular, hence, if non-trivial, must either be affine or projective over a finite field. A different proof was discovered independently by Cherlin, Harrington and Lachlan [6]. The results imply both the finiteness in rank of  $\omega$ -categorical,  $\omega$ -stable structures [6] and the non-finite axiomatizability of totally categorical theories. [23, 6] Since then, a number of prime researchers have obtained new enlightening results such as Buechler's dichotomy theorem [3], Hrushovski's group configuration theorems in several contexts [10, 11], his classification of a locally modular type [11], and other generalizations in the context of superstable theories [10, 11, 14]. Refined notions such as 1-basedness, regular types and  $p$ -weight have also been introduced. Pillay, in his book [17], makes a complete exposition of the subject. Hrushovski has now shifted the direction of research towards applications in algebraic geometry and number theory, which has deepened and broadened the subject. It is well known that, using geometric stability theory and in particular Zilber's principle on 'Zariski structures', he solved the Mordell-Lang conjecture [12] and other problems in number theory.

From the mid 1990s, after the initial papers [15], [16] of Kim and Pillay, simplicity theory, introduced by Shelah [18], has developed rapidly and extensively. Simplicity theory is the study of a class of structures, called simple, properly containing that of stable structures. The central organising principle has been the Independence Theorem (type amalgamation), leading to the introduction of Lascar strong types, canonical bases and hyperimaginaries. Definability and the interplay with local stability have also played an important developing

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role. An account on the subject is written by Wagner [22], one of the major contributors. However, there are a number of obstacles to initiating an analogous geometric simplicity theory. This is mainly due to the lack of stationarity. Hrushovski's counterexample shows that Zilber's result on  $\omega$ -categorical strongly minimal structures *cannot* be generalized in the context of  $\omega$ -categorical rank-1 structures. Namely, there is a *non* 1-based, rank-1  $\omega$ -categorical structure [13].

In this paper we attempt to develop initial geometric simplicity theory using classical results in geometry as essential tools. We succeed in obtaining fruitful positive results and examples. We mainly study the solution set  $D$  of a non-trivial 1-based (simple)  $SU$  rank-1 Lascar strong type. Any finite tuple (not in the algebraic closure of the empty set) in a simple structure realizes a rank-1 Lascar strong type over some parameters. In particular, in any 1-based or supersimple theory, we can find a rank-1 type over a *finite* parameter. Moreover, if the 1-based theory is  $\omega$ -categorical, then the theory must have finite  $SU$ -rank (and is therefore supersimple), so every type is coordinatized by rank-1 types [9]. Hence studying such a 1-based  $D$  is important in understanding  $\omega$ -categorical or even general 1-based theories. (As is well known, any  $\omega$ -categorical,  $\omega$ -stable theory is 1-based. The class of  $\omega$ -categorical 1-based structures *properly* contains that of smoothly approximable structures, too.) Furthermore, in the case of a supersimple rank-1 theory, since any model is the disjoint union of  $acl(\emptyset)$  and the solution sets of rank-1 Lascar strong types (over  $\emptyset$ ), we can restrict our attention to the solution set of a Lascar strong type. In [21], using his notion of 'generic pair', Vassiliev investigated the rank-1 theory. Here we study rank-1 types using a straightforward argument so that we can provide, for example, a direct proof of the fact that linearity implies 1-basedness. We also obtain the following more general results.

Local modularity of  $D$  implies 1-basedness (however, the converse is not true in the general simple theory context unlike the stable case). We first prove that the (local, resp.) modularity of the pregeometry of  $D$  implies that the (localized, resp.) geometry is isomorphic to a projective geometry over a division ring (\*). These facts give the impression that, as in the stable case, the geometry of locally modular  $D$  should be affine over a division ring. However, we construct an example *refuting* this surmise.

On the other hand, it follows from (\*) that if  $D$  is (locally) modular and  $\omega$ -categorical then, after possibly naming a point, a sort of  $D^{eq}$  representing the geometry of  $D$  has a (definitional) stable reduct preserving independence. The reduct must be strongly minimal having a projective geometry over a finite field. Then, by the classical result on strongly minimal structures, a definable vector space over a finite field can be recovered inside the reduct. In particular, an infinite group is interpretable. For the more general case of a 1-based  $\omega$ -categorical  $D$ , we also show that a sort representing the geometry on some finite union of sorts in  $D^{eq}$  has a generic (i.e. independence-preserving) strongly minimal stable reduct whose geometry is projective over a finite field. Again, a projective space over a finite field is interpretable. In fact we obtain a more general result. Namely, for *any* 1-based  $D$ , we prove that the canonical geometry, extending (or completing) the geometry of  $D$ , on the set of canonical bases of surfaces in  $D^3$ , is projective over a division ring (Theorem 3.9) (assuming non-triviality). Here, by a surface in  $D^3$ , we mean a rank 2 Lascar strong type of a tuple in  $D^3$ . It follows that in any  $\omega$ -categorical, non-trivial, 1-based theory, a vector space over a finite field is interpretable over a finite parameter.

In this paper we assume that the underlying theory  $T$  is simple, having, for convenience, elimination of hyperimaginaries. This assumption can be removed if  $T$  is supersimple [4], or by working in  $\bar{M}^{heq}$  and replacing notions in  $\bar{M}^{eq}$  with corresponding notions in  $\bar{M}^{heq}$  (e.g.  $acl$  by  $bdd$ , and so on).

## 1. LOCALLY MODULAR TYPE

We start this section by recalling the definitions of (pre)geometry, modularity and local modularity ([17, 2.1]).

**Definition 1.1.** *Let  $S$  be a set. If an operation  $cl : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$  satisfies the following properties, then we say that  $(S, cl)$  is a pregeometry.*

- (1) *For  $A \subseteq S$ ,  $A \subseteq cl(A) = cl(cl(A))$ .*
- (2) *For  $A \subseteq B \subseteq S$ ,  $cl(A) \subseteq cl(B)$ .*
- (3) *For  $A \subseteq S$  and  $a, b \in S$ , if  $a \in cl(Ab) \setminus cl(A)$ , then  $b \in cl(Aa)$ .*
- (4) *If  $a \in cl(A)$ , then  $a \in cl(A_0)$ , for some finite  $A_0 \subseteq A$ .*

The pregeometry  $(S, cl)$  is said to be *homogeneous* if for any closure  $X$  of a finite set and  $a, b \in S \setminus X$ , there is an automorphism of  $(S, cl)$  which fixes  $X$  pointwise and moves  $a$  to  $b$ . We say the pregeometry  $(S, cl)$  is a *geometry* if  $cl(\emptyset) = \emptyset$  and  $cl(a) = \{a\}$  for all  $a \in S$ . If  $(S, cl)$  is a pregeometry, then we can associate a canonical geometry  $(\hat{S}, \hat{cl})$ , where  $\hat{S} = \{cl(a) \mid a \in S \setminus cl(\emptyset)\}$  and for  $A \subseteq S$  we put  $\hat{A} = \{cl(a) \mid a \in A\}$ , and  $\hat{cl}(\hat{A}) = \{cl(b) \mid b \in cl(A)\}$ .

Let  $(S, cl)$  be a pregeometry. We say that  $A(\subseteq S)$  is *independent* if  $a \notin cl((A \setminus \{a\}))$  for all  $a \in A$ . Given  $B$ , a subset  $B_0 \subseteq B$  is said to be a *basis* for  $B$  if  $B \subseteq cl(B_0)$  and  $B_0$  is independent. It follows that any two bases for  $B$  have the same cardinality, denoted by  $dim(B)$ . Any  $A \subseteq S$  gives a *localized* pregeometry on  $S$  defined by  $cl_A(B) = cl(A \cup B)$ , and a notion of dimension *over  $A$*  ( $dim(-/A)$ ). For  $A, B, C \subseteq S$ , if  $dim(A'/C) = dim(A'/B \cup C)$  for any finite  $A' \subseteq A$ , then we say that  $A$  is *independent from  $B$  over  $C$* .

**Definition 1.2.** *Let  $(S, cl)$  be a pregeometry.*

- (1)  *$(S, cl)$  is trivial if  $cl(A) = \bigcup \{cl(\{a\}) : a \in A\}$  for all  $A \subseteq S$ .*
- (2)  *$(S, cl)$  is modular if  $X$  is independent from  $Y$  over  $X \cap Y$  for all closed  $X, Y$ , or equivalently, if  $dim(X) + dim(Y) = dim(X \cup Y) + dim(X \cap Y)$  for finite dimensional closed  $X$  and  $Y$ .*
- (3)  *$(S, cl)$  is locally modular if it is modular over some point in  $S$ .*
- (4)  *$(S, cl)$  is locally finite if the closure of a finite set is finite.*
- (5) *A 3-tuple  $(ca_1a_2) \in S$  is a non-modular triangle if  $dim(ca_1a_2) = 3$ ,  $cl(a_1a_2) = cl(a_1) \cup cl(a_2)$  and there are  $u_i \in cl(ca_i) \setminus (cl(c) \cup cl(a_i))$  for  $i = 1, 2$ .*

Let  $S'$  be the geometry of a pregeometry  $S$ .  $S$  is trivial (modular, or locally modular, resp.) if and only if  $S'$  is (resp.). Moreover, each of the properties is preserved under localization. Thus modularity implies local modularity.

**Fact 1.3.**  *$(S, cl)$  is modular iff for any subset  $A$  and  $b \in S$ , whenever  $c \in cl(Ab)$ , then  $c \in cl(ab)$  for some  $a \in cl(A)$ .*

**Remark 1.4.** (1) *Suppose that  $(S, cl)$  has a non-modular triangle  $(ca_1a_2)$ . Then  $(S, cl)$  is not modular: Let  $u_i$  be as in 1.2.5. First, suppose  $cl(u_1) = cl(u_2)$ . Since  $a_1, a_2 \in$*

$cl(cu_1) \cup cl(cu_2) = cl(cu_1)$ , we get  $2 = \dim(cu_1) = \dim(ca_1a_2u_1) = 3$ , a contradiction. Hence  $\dim(u_1u_2) = 2$ . Now  $a_1 \notin cl(u_1u_2)$ , as otherwise, since  $c \in cl(a_1u_1u_2) = cl(u_1u_2)$  and  $a_2 \in cl(cu_2)$ ,  $\dim(u_1u_2) = \dim(ca_1a_2u_1u_2) = 3$ , a contradiction. Similarly,  $a_2 \notin cl(u_1u_2)$ . Recall that  $cl(a_1a_2) = cl(a_1) \cup cl(a_2)$ . Hence  $cl(a_1a_2) \cap cl(u_1u_2) = cl(\emptyset)$ . Thus

$4 = \dim(a_1a_2) + \dim(u_1u_2) \neq \dim(a_1a_2u_1u_2) + \dim(cl(a_1a_2) \cap cl(u_1u_2)) = 3$ . This shows that  $S$  is not modular.

- (2) If  $(S, cl)$  is a modular geometry and  $cl(cd) = \{c, d\}$  for any two distinct points  $c, d \in S$  then  $S$  is trivial: By (4) of 1.1, it is sufficient to show that the closure of any finite set is itself. Suppose not, say there exists  $d \notin A = \{a_1, \dots, a_n\}$  such that  $d \in cl(A)$ . Then  $cl(a_nd) = \{a_n, d\}$  and  $cl(a_1 \dots a_{n-1}) = \{a_1, \dots, a_{n-1}\}$  by the induction hypothesis. By modularity  $n = \dim(a_1 \dots a_n) = \dim(a_1 \dots a_{n-1}) + \dim(a_nd) = n + 1$ , a contradiction.

Let us first state two important classical results on the geometry which will be used later.

**Fact 1.5.** [1] *A projective (i.e. non-trivial and modular) geometry of dimension  $\geq 4$  in which each closed set of dimension 2 contains at least 3 elements, is isomorphic to a projective geometry over a division ring.*

**Fact 1.6.** [7] *A locally projective (i.e. non-trivial and locally modular), locally-finite geometry of dimension  $> 4$ , in which all closed sets of dimension 2 have the same size, is affine or projective geometry over a finite field.*

(For some reason, in the above Fact 1.6, the dimension condition is erroneously known to be  $\geq 4$  to model theorists. This may be due to the fact that, in Doyen and Hubaut's original paper, they counted *their dimension* 1 less than what is commonly defined by model theorists. They even constructed a counterexample in dimension 4.)

From now on, we consider a pregeometry given on a type. As usual, the saturated model  $\bar{M}$  of the simple theory  $T$  is fixed. We work in  $\bar{M}^{eq}$ . (Tuples and sets are small from  $\bar{M}^{eq}$ .) We also fix a set  $D$ , which is a solution set (in the saturated model) of a Lascar strong type. For notational simplicity, we suppose  $D$  is over  $\emptyset$ . Let us further suppose that  $(D, cl)$  forms a pregeometry where  $cl$  is an  $Aut(\bar{M})$ -invariant closure relation on  $D$  (i.e.  $a \in cl(B)$  if and only if  $f(a) \in cl(f(B))$  for each  $f \in Aut(\bar{M})$ ). For safety, we assume that  $\dim(D) = Card(\bar{M})$ . The typical examples of  $D$  are, of course, rank-1 types with  $cl(-) = acl(-) \cap D$ , or regular types.  $D'$  denotes the geometry of  $D$ . We shall use the same closure notation for the pregeometry and its geometry.

**Remark 1.7.** *The closure of  $\emptyset$  is  $\emptyset$ . The size of  $cl(p)$  does not depend on  $p \in D$ , hence  $(D', cl)$  is homogeneous iff  $(D, cl)$  is.*

Now, we observe that a modular  $(D, cl)$  is well behaved.<sup>1</sup>

**Lemma 1.8.** *( $D$  modular.) Suppose there exist  $a, b \in D$  such that  $\dim\{a, b\} = 2$  and  $cl(ab) = cl(a) \cup cl(b)$ . Then  $D$  is trivial and homogeneous.*

<sup>1</sup>There may exist a reasonable demand for distinguishing between the terminology 'modular type' and 'type having a modular geometry'. Here, we consider the two terms to be equal.

*Proof.* It suffices to show triviality. Moreover, we can assume that  $D$  forms a geometry. Then the condition in the lemma simply says that  $cl(ab) = \{a, b\}$ .

Now, by Remark 1.4.2, it is enough to show that  $cl(cd) = \{c, d\}$  for any two distinct points  $c, d$ : Suppose not, say there is a third point  $u' \in cl(cd)$ . Let  $q(x, d) = tp(c/d)$ . As  $D$  is the solution set of a Lascar strong type, by the Independence Theorem for Lascar strong types, we can amalgamate  $q(x, a)$  and  $q(x, b)$ . Let  $c'$  be a realization of a common nonforking extension of  $q(x, a)$  and  $q(x, b)$ . Then  $dim(a, b, c') = 3$ , and there are  $u, v$  such that  $tp(cdu') = tp(c'bu) = tp(c'av)$ . Hence the tuple  $(c'ab)$  forms a non-modular triangle. By Remark 1.4.1, this contradicts the modularity of  $D$  and finishes the proof.  $\square$

In the above lemma, the assumption that  $D$  is modular is necessary. We shall see in section 2 that even local modularity of  $D$  is insufficient to imply the conclusion. There are other 1-based (3.7) and even a non 1-based example [13], where the closure of some pair of points can be 2 or 3 elements.

By now, we have the following theorems due to Fact 1.5 and the previous lemma.

**Theorem 1.9.** *If  $D$  is modular, then either  $D$  is trivial, or the geometry  $D'$  of  $D$  is projective over a division ring. In either case,  $(D, cl)$  is homogeneous.*

Suppose that  $D$  is  $\omega$ -categorical. Then  $D'$  is a sort in  $D^{eq}$ . Let  $R_n$  be an  $n+1$ -ary relation on  $D'$  such that  $(a_0, a_1, \dots, a_n) \in R_n$  if and only if  $a_0 \in cl(a_1, \dots, a_n)$ . Then  $R_n$  is  $\emptyset$ -definable in  $D'$  (so in  $D^{eq}$ ), and  $(D', R_n)_{n \in \omega}$  is a (definitional) reduct of  $D'$ . Hence, if  $(D, cl)$  is non-trivial and modular, then by the previous theorem,  $(D', R_n)_{n \in \omega}$  is strongly minimal and quantifier eliminable, with canonical pregeometry  $(D', cl)$ , that is projective over a finite field  $F$ . Then by the general result on strongly minimal structures, a vector space  $(V, +, \lambda)_{\lambda \in F}$  over  $F$  is definable in  $(D', R_n)_{n \in \omega}^{eq}$ , possibly over a finite parameter. We summarize this in the following theorem.

**Theorem 1.10.** *Let  $D$  be modular and  $\omega$ -categorical. Then the sort  $S_E$  in  $D^{eq}$ , where  $E(x, y) \equiv x \in cl(y)$ , has a (definitional) stable reduct preserving independence. Hence, in a non-trivial  $D$ , a vector space over a finite field is interpretable (i.e. definable in  $D^{eq}$  over some finite set.) In particular, the infinite additive group is definable in  $D^{eq}$ .*

If, additionally,  $D$  has  $SU$ -rank 1 and the usual algebraic closure relation, then  $D$  itself has a generic stable reduct.

Now we investigate the locally modular case. We fix some notation. Choose  $p \in D$ , and let  $D_0$  denote the pregeometry  $D \setminus cl(p)$  with closure operation  $cl_p$ .  $D'_0$  denotes the geometry of  $D_0$ . Recall that we use the same closure notation for the pregeometry and its geometry. Note that  $x \in cl_p(y)$  is an equivalence relation on  $D_0$ . We obtain the following results (1.11,12) analogous to the previous ones (1.8,9). However, (1.11,12) are not automatic consequences of (1.8,9), since in the simple context  $D_0$  need not be the solution set of a unique type over  $p$ .

For the rest of this section,  $D$  is locally modular.

**Lemma 1.11.** *Suppose that there are distinct  $a', b' \in D'_0$  such that  $cl_p(a'b') = \{a'b'\}$ . Then  $cl(c'd') = \{c', d'\}$  for any two points  $c', d'$  in  $D'_0$ . Hence,  $D'_0$  is trivial.*

*Proof.* Triviality will follow from the main assertion by Remark 1.4.2. Fix points  $a, b$  in  $D$  from  $a', b'$ , respectively. Due to the limitation on type amalgamation, we need more

complicated arguments to prove the lemma. We shall calculate dimension in  $D$ , unless stated otherwise.

Suppose that the lemma is not true. Then there is  $u' \in D'_0$  such that  $u' \in cl_p(c'd') \setminus \{c', d'\}$ . Now choose (in  $D$ ) points  $c, d, u$  from  $c', d', u'$  respectively. Then clearly  $dim(pcd) = 3$ , and  $dim(pX) = 3$  for each 2-subset  $X$  of  $\{c, d, u\}$  (\*).

Case I)  $u \in cl(cd)$ : Then the closure of any two of the  $u, c, d$  contains the third point (\*\*). Now, we claim the following.

Claim 1) There are  $v_i \in D$  with  $v_i \in v'_i \in D'_0$  (for  $i = 1, 2, 3$ ) such that,  $tp(cdu) = tp(v_1bv_2) = tp(v_1av_3)$ , and  $dim(pabv_1) = 4$ .

Proof of claim: Let  $q(x, d) = tp(c/d)$ . Then we can amalgamate  $q(x, b)$  and  $q(x, a)$ . Let  $v_1$  be a realization of a common nonforking extension. Then clearly there are  $v_2, v_3$  such that  $tp(cdu) = tp(v_1bv_2) = tp(v_1av_3)$ . Moreover, we can assume  $v_1 \downarrow_{ab}^p$ , so that  $dim(pabv_1) = 4$ . In particular,  $p \notin cl(v_1b)$ . Now by (\*\*),  $v_2 \in cl(v_1b)$ . Hence  $cl(p) \neq cl(v_2)$ , and  $v'_2 \in D_0$ . Similarly  $v'_3 \in D_0$ . Therefore the claim is proved.

As  $u \in cl(cd) \setminus \{c, d\}$ , we have  $v_3 \in cl(v, a) \setminus \{v, a\}$  and  $v_2 \in cl(v, b) \setminus \{v, b\}$ . If  $v_3 \in cl(ap)$ , then  $p \in cl(av_3) \subseteq cl(av_1)$ , contradicting  $dim(pabv_1) = 4$ . Similarly,  $v_3 \notin cl(v_1p)$  and  $v_2 \notin cl(bp) \cup cl(v_1p)$ . Thus  $(v'_1a'b')$  forms a non-modular triangle in  $D'_0$ , contradicting modularity. Hence case I does not happen.

We consider the remaining case.

Case II)  $u \notin cl(cd)$ : Then by (\*), each 3 subset of  $\{p, c, d, u\}$  has dimension 3 (\*\*\*). This time, let  $q(x, cd) = tp(p/cd)$ . There are  $u_2, u_4$  such that  $tp(cd) = tp(au_2) = tp(bu_4)$  and  $dim(abu_2u_4) = 4$ . Amalgamate  $q(x, au_2), q(x, bu_4)$  and let  $u_1 \in D$  be a realization of a common nonforking extension, independent of  $p_0$ . Then  $dim(pu_1abu_2u_4) = 6$ . Now there are  $u_3 \in u'_3 \in D'_0$  and  $u_5 \in u'_5 \in D'_0$  such that  $tp(pcd) = tp(u_1au_2u_3) = tp(u_1bu_4u_5)$ . By use of (\*\*\*), one can see that  $dim(pu_2u_3u_4u_5) = 5$  and  $a', b' \notin cl_p(u'_2u'_3u'_4u'_5)$ . Thus  $cl_p(a'b') \cap cl_p(u'_2u'_3u'_4u'_5) = \emptyset$ . This again violates the modularity of  $D'_0$ . Hence Case II does not happen either.

Therefore Lemma 1.11 is proved. □

**Theorem 1.12.**  $D'_0$  is either trivial or projective over a division ring.

**Lemma 1.13.**  $D$  is trivial if and only if  $D_0$  is trivial.

*Proof.* We can safely assume that  $D$  forms a geometry. It is enough to show that triviality of  $D_0$  implies that of  $D$ . Let  $D_0$  be trivial.

Claim) The closure of any two distinct points  $\{p, q\}$  in  $D$  is itself

Proof of claim: Suppose not, say  $r \in cl(pq) \setminus \{p, q\}$ . Choose  $s \notin cl(pq) = cl(pqr)$  (\*). Since  $D_0$  is trivial,  $cl(pqs) = cl(ps) \cup cl(pq)$  and  $dim(pqs) = 3$ . Hence  $q, r \notin cl(ps)$ . Now, since the localized pregeometry of  $D$  at  $s$  is equivalent to  $D_0$  and  $D_0$  is trivial,  $cl(qs) \cup cl(ps) = cl(pqs)$ . In particular,  $r \in cl(qs) \cup cl(ps)$ . Hence, as  $r \notin cl(ps)$ ,  $r \in cl(qs)$ . As  $dim(qs) = 2$ ,  $s \in cl(qr)$ , this contradicts (\*). Thus the claim is verified.

Now, by the claim, for  $n + 1$  points  $p, q_1, \dots, q_n$ , since  $D_0$  is trivial,  $cl(pq_1 \dots q_n) = cl(pq_1) \cup \dots \cup cl(pq_n) = \{p, q_1, \dots, q_n\}$ . Hence  $D$  is also trivial. □

By the same argument as for Theorem 1.10 above, if  $D$  is locally modular  $\omega$ -categorical, after naming a point  $p$  in  $D$ , a sort in  $(D_p)^{eq}$  representing  $D'_0$  has a generic (independence-preserving) stable reduct (over  $p$ ). Hence, if  $D$  is non-trivial, a vector space over a finite field is again interpretable over a finite parameter. We shall in fact prove the interpretability of a vector space in the more general 1-based context (3.23).

By now, one would naturally expect that, as in stable theories, a (non-trivial, non-modular) locally modular  $D$  should be affine. However, the example in the following section shows that this is *not* the case.

## 2. NON AFFINE LOCALLY MODULAR TYPE

First, we fix an extended language  $\mathcal{L}' = \mathcal{L}_A \cup \{P\}$  where  $P$  is a unary predicate and  $\mathcal{L}_A$  is the language of the theory of affine space over a finite field  $F$  of cardinality at least 5. (We shall see below a number of reasons why this condition is imposed. In fact, one can do the same construction with a projective space over a finite field of cardinality  $\geq 4$ .) Now, suppose  $V_1$  is a finite affine space over  $F$  generated by independent elements  $\{x_0, \dots, x_n\}$ . Let  $P$  colour some of the elements in  $V_1$ . We call the colouring *good* if for every line  $l$  in  $V_1$ , we can find at least 2 points in  $l$  coloured by  $P$ , and let  $gc(n) > 0$  be the finite number of good colourings, for  $n \geq 1$ , described by  $\mathcal{L}'$ -formulas  $Col_{r,n}(x_0, x_1, \dots, x_n)$ . Now suppose  $V_2 \supset V_1$  is an affine space with basis  $\{x_0, \dots, x_{n+1}\}$ . Then, given a good colouring  $Col_{r,n}(x_0, \dots, x_n)$  of  $V_1$ , there exists a good colouring extending it to  $V_2$ . (For example, colour every point in  $V_2 \setminus V_1$  with  $P$  while leaving the colouring on  $V_1$ . Since every line in  $V_2$  is either already in  $V_1$  or hits  $V_1$  once, the condition  $q \geq 5$  implies that this is a good colouring.) We denote the finite number of all possible such colourings by  $gc(r, n)$  and let them be described by  $\mathcal{L}'$ -formulas  $Col_{r,n}^{s,n+1}(x_0, x_1, \dots, x_n, x_{n+1})$ , for  $1 \leq s \leq gc(r, n)$ . For  $n = 0$ , we interpret  $Col_{0,0}^{s,1}(x_0, x)$  and  $Col_{1,0}^{s,1}(x_0, x)$  to mean that  $P$  gives a good colouring on the line generated by  $\{x_0, x\}$  extending the colourings  $\neg P$  and  $P$  respectively on  $x_0$ , and we let  $gc(0, 0)$  and  $gc(1, 0)$  denote the number of such colourings.

Now let  $T_A$  denote the theory of affine space over  $F$ . Then the  $\mathcal{L}_A$ -formulas  $x \in cl(y_1, \dots, y_n)$  and  $dim(y_1 \dots y_n) = m (\leq n)$  make sense. (We note here that the dimension defined throughout the paper is one more than the usual affine subspace dimension.) The theory  $T_{LM}$  is given in the language  $\mathcal{L}'$  by the following axioms:

$$T_{LM} \equiv T_A \cup \{LM\} \cup \{F_n : n \geq 0\},$$

where  $LM$  (local modularity) is the following sentence in  $\mathcal{L}'$ :

$$LM \equiv \forall x \forall y (x \neq y \rightarrow \exists z_1 \exists z_2 (z_1 \neq z_2 \wedge P(z_1) \wedge P(z_2) \wedge z_1, z_2 \in cl(x, y))),$$

$F_n$  (colour extension), for  $n \geq 1$ , is the  $\mathcal{L}'$ -sentence:

$$F_n \equiv \bigwedge_{1 \leq r \leq gc(n)} \forall y_0 \dots y_n (dim(y_0 \dots y_n) = n + 1 \wedge Col_{r,n}(y_0, y_1, \dots, y_n) \rightarrow \bigwedge_{1 \leq s \leq gc(r,n)} \exists x (dim(y_0 \dots y_n x) = n + 2 \wedge Col_{r,n}^{s,n+1}(y_0, y_1, \dots, y_n, x))),$$

and  $F_0$  is the sentence:

$$F_0 \equiv \forall y (\neg P(y) \rightarrow \bigwedge_{1 \leq s \leq gc(0,0)} \exists x (\dim(yx) = 2 \wedge Col_{0,0}^{s,1}(y, x))) \wedge \\ \forall y (P(y) \rightarrow \bigwedge_{1 \leq s \leq gc(1,0)} \exists x (\dim cl(yx) = 2 \wedge Col_{1,0}^{s,1}(y, x))).$$

We first show that the theory  $T_{LM}$  is consistent. This can be done by a union of chains argument inside a model  $V$  of  $T_A$ . We first point out the following on *any* geometry.

**Remark 2.1.** *Suppose that  $(S, cl)$  is a geometry. Let  $\{A_i \in S \mid i \in I\}$  be independent over  $B \subseteq S$ , i.e. for finite  $\bar{a}_i \in A_i$ ,  $\dim(\bar{a}_i/B) = \dim(\bar{a}_i/B \cup \bigcup_{j \neq i} A_j)$ .*

- (1)  $cl(B) = cl(BA_{i_1} \dots A_{i_n}) \cap cl(BA_{j_1} \dots A_{j_m})$  for  $i_1, \dots, i_n, j_1, \dots, j_m$  distinct.
- (2) *Each line in  $cl(B \cup \bigcup_i A_i)$  either already lies in one of  $cl(BA_i)$ , or hits  $\bigcup_i cl(BA_i)$  at most twice.*

*Proof.* The first statement clearly comes from the fact that

$$\dim(\bar{a}_{i_1} \dots \bar{a}_{i_n}/B) = \dim(\bar{a}_{i_1} \dots \bar{a}_{i_n}/B \cup \bigcup_{j \neq i_1, \dots, i_n} A_j).$$

For the second, assume that a line in  $cl(B \cup \bigcup_i A_i)$  hits  $\bigcup_i cl(BA_i)$  3 times, but does not lie in any  $cl(BA_i)$ . Let  $\{x, y, z\}$  be the hit points. Then clearly  $x \in cl(BA_k)$ ,  $y \in cl(BA_m) \setminus cl(BA_k)$ ,  $z \in cl(BA_n) \setminus (cl(BA_m) \cup cl(BA_k))$  for distinct  $k, m, n$ , whereas  $z \in cl(xy)$  and  $z \in cl(BA_n) \cap cl(BA_k A_m)$ , this contradicts (1).  $\square$

Now we proceed in a series of steps.

Step 1.

Choose a  $y_0 \in V$  and label  $y_0$  by  $P$ . Now add independent elements  $\{x_1, \dots, x_{gc(1,0)}\}$  over  $y_0$  and colour each  $cl(y_0 x_i)$  to witness  $F_0$  for  $y_0$ . Let  $W_1 = cl(y_0, x_1, \dots, x_{gc(1,0)})$ . Now colour all points in  $W_1 \setminus \cup_i cl(y_0 x_i)$  with  $P$ . Then  $W_1$  satisfies  $LM$ ; if a line in  $W_1$  already lies in one of  $cl(y_0 x_i)$ , then we are done by construction. Otherwise by the above remark (2), the assumption  $Card(F) \geq 5$  ensures that we can pick up at least 3 points on the line labelled with  $P$ .

Step 2.

We now want to satisfy  $F_0$  in  $W_1 \setminus \{y_0\}$ , so we repeat the method of Step 1 for each point in  $W_1 \setminus \{y_0\}$ . Namely, for  $a \in W_1 \setminus \{y_0\}$ , find independent  $\{z_1, \dots, z_{gc(k,0)}\}$  which are also independent from  $W_1$  ( $k = 0, 1$  depends on  $a \in \neg P$  or  $a \in P$ ). Then colour all the points in  $cl(W_1 \cup \{z_i\}_i)$  with  $P$ , except  $\cup_i cl(az_i)$  and  $W_1$ .  $W_1$  is already coloured. Colour each  $cl(az_i)$  to witness  $F_0$  for  $a$ . Then for the same reason as in step 1, with remark 2.1,  $cl(W_1 \cup \{z_i\}_i)$  satisfies  $LM$ . Now for  $b \in W_1 \setminus \{y_0, a\}$ , do the same thing over  $cl(W_1 \cup \{z_i\}_i)$ , and so on. We then obtain a subspace  $U_0 \supset W_1$  such that  $U_0$  satisfies  $LM$  and for all  $x \in W_1$ ,  $F_0$  is witnessed inside  $U_0$ .



Step 3.

Now by modifying Step 1 and 2, find  $U_1(\supset U_0)$  to make  $F_1$  hold for  $W_1$  inside  $U_1$ . Namely, for given distinct  $x, y \in W_1$ , choose independent points  $\{w_k\}_k$  independent from  $U_0$  to extend the good colouring on  $cl(xy)$  to all possible good colourings on distinct affine planes  $cl(xyw_k)$ . (These colourings are compatible due to the fact that  $cl(xy) = U_0 \cap cl(xyw_k) = cl(xyw_i) \cap cl(xyw_j)$  (for  $i \neq j$  by Remark 2.1.1.) Then colour all the points in  $cl(U_0 \cup \{w_k\}_k)$  other than  $U_0$  and  $\bigcup_k cl(xyw_k)$  with  $P$ . Again, as in step 1, the fact that  $Card(F) \geq 5$  and Remark 2.1.2 guarantees that  $cl(U_0 \cup \{w_k\}_k)$  satisfies  $LM$ . For the next line from  $W_1$  other than  $cl(xy)$ , repeat the process over  $cl(U_0 \cup \{w_k\}_k)$ . By iterating this argument, we obtain a chain of spaces  $W_1 \subset U_0 \subset \dots \subset U_n$  so that  $U_n$  witnesses  $\{F_0, \dots, F_n\}$  for  $W_1$ , and  $U_n$  satisfies  $LM$ . We can choose  $n$  large enough (depending on the dimension of  $W_1$ ), so that in fact  $U_n$  witnesses all  $F_i$  (for  $i \geq 0$ ) for  $W_1$ .

Rename  $U_n$  as  $W_2$ , and repeat the previous steps for  $W_2$ . Continuing in this way we obtain a chain of spaces  $y_0 \subset W_1 \subset W_2 \subset \dots \subset W_i \subset \dots$  such that each  $W_i$  satisfies  $LM$  (by 2.1.2), and all  $F_i$  ( $i \geq 0$ ) are witnessed for  $W_i$  inside  $W_{i+1}$ . Let  $W = \bigcup_i W_i$ , then by construction  $W \models T_A \cup \{LM\} \cup \{F_i : i \geq 0\}$ . Hence  $T_{LM}$  is consistent.

We now have the following theorem.

**Theorem 2.2.**  *$T_{LM}$  is  $\omega$ -categorical, complete and has quantifier elimination. Moreover,  $T_{LM}$  is a model companion of  $T_A \cup \{LM\}$ .*

The axioms  $\{F_n : n \geq 0\}$  of colour extensions allow a partial isomorphism to be continued. Hence quantifier elimination follows. In particular for an independent tuple  $(a_0, \dots, a_n)$ ,  $tp(a_0, \dots, a_n)$  is determined by the colouring on  $cl(a_0, a_1, \dots, a_n)$ .  $\omega$ -categoricity and hence completeness also follow by a straightforward back and forth partial isomorphism argument. Now given a model  $M$  of  $T_A \cup \{LM\}$ , by doing the above process starting with  $M$ , one can obtain an extension  $N$  of  $M$  which is a model of  $T_{LM}$ . Hence  $T_{LM}$  is a model companion of  $T_A \cup \{LM\}$ .

We now want to show that  $T_{LM}$  is a simple theory. As we observed, the condition  $Card(F) \geq 4$  is enough to obtain the previous theorem. However, we shall see that we need  $Card(F) \geq 5$  to ensure  $T_{LM}$  is simple, particularly for the Independence Theorem. Let us first define a notion of independence on subsets  $A, B, C$  of a saturated model  $\bar{V}$  of  $T_{LM}$  by  $A \downarrow_C B$  iff  $A$  is *independent* from  $B$  over  $C$  in the sense of affine space (i.e. in  $\mathcal{L}_A$ ). It follows easily that this notion is automorphism invariant and satisfies symmetry, transitivity, finite character and local character. We need to check the extension axiom and the Independence Theorem. Below we keep using  $cl$  to denote affine closure.

Nonforking extension:

So let  $\bar{a}$  and  $B, C$  be a tuple and two small subsets of  $\bar{V}$ . Then clearly there is  $\bar{a}'$  such that  $\bar{a}'$  is  $(\mathcal{L}_A)$ -independent from  $C$  over  $B$ . Hence, as in 2.1.1,  $cl(\bar{a}'B) \cap cl(BC) = cl(B)$ . Therefore the colouring of  $cl(\bar{a}'B)$  according to  $cl(\bar{a}B)$  is compatible with the colouring on  $cl(BC)$ . Again colour all the points in  $cl(\bar{a}'BC)$  with  $P$  except  $cl(\bar{a}'B)$  and  $cl(BC)$ . By a

similar argument to that in Remark 2.1.2,  $cl(\bar{a}'BC)$  with the new colouring satisfies  $LM$ . Hence by the axioms  $\{F_n\}$  we can embed  $\bar{a}'$ , while fixing  $BC$  with the new colouring, into  $\bar{V}$ , and then get the ( $\mathcal{L}'$ -)nonforking extension of  $tp(\bar{a}/B)$  over  $BC$ .

The Independence Theorem for types:

We will verify this over any closed set  $B$ . Here types are  $\mathcal{L}'$ -types. Let  $\bar{a}$  and  $\bar{b}$  be independent tuples over  $B$ , and  $\bar{c}, \bar{d}$  tuples such that  $\bar{c} \downarrow_B \bar{a}$ ,  $\bar{d} \downarrow_B \bar{b}$  and  $tp(\bar{c}/B) = tp(\bar{d}/B)$ .

By the extension property above, we can choose a tuple  $\bar{e}$  independent from  $\bar{b}$  over  $B\bar{a}$  such that  $tp(\bar{c}/B\bar{a}) = tp(\bar{e}/B\bar{a})$ . Then  $\{\bar{e}, \bar{a}, \bar{b}\}$  is independent over  $B$ . Consider  $cl(B\bar{e}\bar{a}\bar{b})$ . We now newly colour  $cl(B\bar{e}\bar{a}\bar{b})$  while leaving the (old) colouring on  $cl(B\bar{e}\bar{a}) \cup cl(B\bar{a}\bar{b})$ , as follows. First, colour  $cl(\bar{e}; B\bar{b})$  according to  $cl(\bar{d}; B\bar{b})$ . Since  $tp(\bar{e}/B) = tp(\bar{d}/B)$ , on  $cl(B\bar{e}\bar{b}) \cap (cl(B\bar{a}\bar{b}) \cup cl(B\bar{e}\bar{a})) = cl(B\bar{b}) \cup cl(B\bar{e})$  (Remark 2.1.1), the two colourings are the same. Hence the new colouring is compatible with the colouring on  $cl(B\bar{e}\bar{a}) \cup cl(B\bar{a}\bar{b})$ . Then colour the rest of the points in  $cl(B\bar{e}\bar{a}\bar{b}) \setminus (cl(B\bar{a}\bar{b}) \cup cl(B\bar{e}\bar{a}) \cup cl(B\bar{e}\bar{b}))$  with  $P$ . Now, since every line in  $cl(B\bar{e}\bar{a}\bar{b})$  either already lies in  $cl(B\bar{a}\bar{b}) \cup cl(B\bar{e}\bar{a}) \cup cl(B\bar{e}\bar{b})$  or hits the set at most 3 times, the condition  $Card(F) \geq 5$  ensures that this new colouring on  $cl(B\bar{e}\bar{a}\bar{b})$  is a good colouring. As we can embed  $\bar{e}$  with the new colouring on  $cl(B\bar{e}\bar{a}\bar{b})$  while fixing  $B\bar{a}\bar{b}$  the Independence Theorem for types holds. (Hence the notion of type coincides with that of Lascar strong type.)

So  $T_{LM}$  is simple. Moreover in  $\bar{V}(\models T_{LM})$ , as  $\mathcal{L}'$ -independence is the same as  $\mathcal{L}_A$ -independence,  $T_{LM}$  is a  $SU$ -rank 1 theory.

**Theorem 2.3.**  $T_{LM}$  is simple with  $SU$ -rank 1 having the same forking independence notion (and so algebraic relations) as  $T_A$ .

Now we take the solution set  $\bar{V}(P) = \{a \in \bar{V} \mid P(a)\}$ . As we checked the Independence Theorem over any closed set, in particular over  $\emptyset$ ,  $\bar{V}(P)$  is a 1-(Lascar strong) type. We also have that  $\bar{V}(P)$  is simple, rank-1 and  $\omega$ -categorical. It remains to show that  $\bar{V}(P)$  is locally modular but not affine.

Local modularity:

Choose  $d \in \bar{V}(P)$ . Let  $ef$  be a pair and  $B$  a set from  $\bar{V}(P)$  such that  $dim_d(ef) = 2$  and  $dim_d(ef/B) = 1$  inside  $\bar{V}(P)$ . Working in  $\bar{V}$ , we also have  $dim_d(ef) = 2$ ,  $dim_d(ef/B) = 1$ . As the geometry of  $\bar{V}$  is affine,  $acl(def) \cap acl(dB) = acl(dg)$ , where  $g \neq d$  is in  $\bar{V}$ . By the axiom  $LM$  for  $\bar{V}$  we can assume that  $g \in \bar{V}(P)$ . Hence,  $dim_d(cl_1(def) \cap cl_1(dB)) = 1$ , where  $cl_1(-) = acl(-) \cap \bar{V}(P)$  is the closure inside  $\bar{V}(P)$ . So  $\bar{V}(P)$  is a locally modular geometry.

Not affine:

For this it is sufficient to observe that a good colouring of a line in  $\bar{V}$  can label  $r$  points where  $2 \leq r \leq Card(F)$ .

So we have constructed a non-trivial, locally modular  $\omega$ -categorical rank-1 structure/type whose geometry is neither affine nor projective over a finite field. Another example of this kind can be constructed, as mentioned previously, by repeating the same process on any projective space over a finite field of cardinality  $\geq 4$ . (This is, in a sense, a necessary requirement. See 3.13.) As is well known, there is no such example in a stable structure, and also by 1.9 there is no example of a modular one.

### 3. 1-BASED TYPE

Let us recall some basic facts on 1-basedness. In this section, algebraic closure is taken over  $\bar{M}^{eq}$ . The following fact is folklore:

**Fact 3.1.** (*T simple.*) *The following are equivalent.*

- (1) *T is 1-based, i.e. any sets A and B are independent over  $acl(A) \cap acl(B)$ .*
- (2) *For any tuple  $\bar{c}$  and a set A,  $cb(\bar{c}/A) \subseteq acl(\bar{c})$ .*
- (3) 
$$A \downarrow_C B \quad \text{iff} \quad acl(AC) \cap acl(BC) = acl(C).$$
- (4) *For any indiscernible sequence  $I = \langle \bar{c}_i | i \in \omega \rangle$ ,  $I \setminus \{\bar{c}_0\}$  is a Morley sequence over  $\bar{c}_0$ .*

*Proof.* We only sketch the proof. The equivalence of 1 and 2 is well-known. In 3, the implication of the right hand side from the left hand side is true for *any* simple theory. Easily, the left implication of 3 is equivalent to 1.  $4 \Rightarrow 2$  follows from the fact that  $cb(\bar{c}/A)$  is in the algebraic (bounded) closure of a Morley sequence from  $Lstp(\bar{c}/A)$ . We supply a rather detailed proof for  $1 \Rightarrow 4$ . Note first that 1 implies  $\bar{c}_0 \downarrow_A \bar{c}_1 \bar{c}_2 \dots$  where  $A = acl(\bar{c}_0) \cap acl(\bar{c}_1 \bar{c}_2 \dots)$ .

Then since any indiscernible sequence over  $B$  is indiscernible over  $acl(B)$  (even over  $bdd(B)$ ) (\*),  $tp(\bar{c}_1 \bar{c}_2 \dots / acl(\bar{c}_0)) = tp(\bar{c}_2 \bar{c}_3 \dots / acl(\bar{c}_0))$ . Hence  $A = acl(\bar{c}_0) \cap acl(\bar{c}_2 \bar{c}_3 \dots)$ . Again by (\*),  $tp(\bar{c}_0 / acl(\bar{c}_2 \bar{c}_3 \dots)) = tp(\bar{c}_1 / acl(\bar{c}_2 \bar{c}_3 \dots))$ , and then  $A = acl(\bar{c}_1) \cap acl(\bar{c}_2 \bar{c}_3 \dots)$ . Then, using 1 again,  $\bar{c}_1 \downarrow_A \bar{c}_2 \bar{c}_3 \dots$ . From transitivity,  $\bar{c}_1 \downarrow_{\bar{c}_0 A} \bar{c}_2 \bar{c}_3 \dots$ , and  $\bar{c}_1 \downarrow_{\bar{c}_0} \bar{c}_2 \bar{c}_3 \dots$  since  $A \subseteq acl(\bar{c}_0)$ . Thus 4 follows.  $\square$

When, for example, a theory  $T$  is 1-based of finite  $SU$ -rank (e.g. any 1-based  $\omega$ -categorical  $T$ , see remark before 3.23), then as every tuple can be ‘built up’ (coordinatized) from rank-1 tuples, understanding the properties of rank-1 types is essential. This is the topic of the current section.

From now on, we further assume that  $D$  is the solution set of a *rank-1* Lascar strong type with the usual algebraic closure relation. We shall observe a strong connection between 1-basedness and modularity. We first note the following folklore and also supply a simple proof.

**Fact 3.2.** *If D is locally modular, then D is 1-based.*

*Proof.* (Remark) Suppose that  $W_0, W_1 \subseteq acl(X) \cap acl(Y)$  and  $X \downarrow_{W_0} Y$  and  $X \downarrow_{W_1} Y$ . Then  $acl(W_0) = acl(W_1)$ , since  $W_1 \downarrow_{W_0} Y$  and  $W_0 \downarrow_{W_1} Y$ . (This remark will be used occasionally.)

Claim 1) Choose  $p \in D$ . For any  $A \subseteq D^{eq}$ , there is  $B \subseteq D$  containing  $p$  such that  $A$  and  $B$  are interalgebraic over  $p$ : There is  $A_0 \subseteq D$  such that  $A \subseteq dcl(A_0)$ . Let  $A_1(\subseteq D)$  realizing  $tp(A_0/Ap)$  be independent from  $A_0$  over  $Ap$ . Now, since  $D$  is modular over  $p$ , if we let  $B = acl(A_0p) \cap acl(A_1p) \cap D$ , then  $A_0, A_1$  are independent over  $B(= Bp)$ . By the remark above,  $acl(Ap) = acl(Bp)$ . Hence the claim is proved.

Claim 2) Given  $B_0, B_1 \subseteq D^{eq}$ ,  $B_0$  and  $B_1$  are independent over  $B_2 = acl(B_0p) \cap acl(B_1p)$  for given  $p \in D$ . (i.e.  $D$  is 1-based over  $p$ ): Suppose not, say  $B_0$  forks with  $B_1$  over  $B_2$ . By Claim 1, there are  $C_0, C_1, C_2 \subseteq D$  containing  $p$  such that  $C_i$  is interalgebraic with  $B_i$  over  $p$ . In particular,  $C_0$  forks with  $C_1$  over  $C_2$ . (Note that  $p \in B_2, C_2$ ). Now as  $D$  is modular over  $p$ , there must exist  $d \in acl(C_0C_2) \cap acl(C_1C_2) \cap D$  such that  $d \notin acl(C_2) = acl(B_2)$ . However, since  $acl(C_0C_2) = acl(acl(C_0) \cup acl(C_2)) = acl(B_0pB_2) = acl(B_0B_2)$  and similarly  $acl(C_1C_2) = acl(B_1B_2)$ ,  $d \in acl(C_0C_2) \cap acl(C_1C_2) = acl(B_0B_2) \cap acl(B_1B_2) \subseteq B_2$ , this is a contradiction. Hence Claim 2 is proved.

Now we can finish the proof of the fact. Let  $I = \langle \bar{c}_i | i \in \omega \rangle$  be an indiscernible sequence from  $D^{eq}$ . If  $q \in D$  is independent from  $I$ , then after a  $c_0$  conjugation we may assume that  $I$  is indiscernible over  $q$ . Now by claim 2 (with a similar proof to 3.1, 1 $\Rightarrow$ 4),  $I \setminus \{\bar{c}_0\}$  is a Morley sequence over  $q\bar{c}_0$ . As  $q$  is independent from  $I$ , by transitivity  $I \setminus \{\bar{c}_0\}$  is Morley over  $\bar{c}_0$ . Hence,  $D$  is 1-based by 3.1.  $\square$

The converse of 3.2 is false in general. As in [21], one can easily construct a counterexample by adding a generic predicate  $P(x)$  to a vector space over a finite field. In Remark 3.7, we will deal with this example again.

**Definition 3.3.** We use  $G(D)$  to denote the collection of all rank-1 elements (over  $dom(D)$ ) in  $D^{eq}$ .

Clearly  $D \subseteq G(D)$ , and for any small set  $A$  containing  $dom(D)$ ,  $G(D)_A$  forms a pregeometry with the usual algebraic closure relation over  $A$ . We recall the definition of *linearity*.

**Definition 3.4.** For a set  $A$  containing  $dom(D)$ , we say that  $D_A$  ( $G(D)_A$ , resp.) is linear if for any two points  $a, b \in D$  ( $G(D)$  resp.) and parameters  $B$  with  $SU(ab/AB) = 1$ ,  $SU(C/A) \leq 1$  where  $C = cb(ab/AB)$ .

**Remark 3.5.** (1) When defining linearity above, it is safe to assume that the parameters  $B$  form a subset of  $D$  (or  $G(D)$ ) since we can preserve the rank by taking a Morley sequence as the parameter.

(2) If a canonical base  $Cb(p)$  of  $p(= Lstp$  of some tuple from  $D^{eq}$  over parameters containing  $dom(D))$  has rank 1, then we can find a singleton  $c$  in  $D^{eq}$  which can be considered as the canonical base up to interalgebraicity. Namely, as  $T$  has elimination of hyperimaginaries, any hyperimaginary  $e$  of ordinal  $SU$ -rank is interdefinable with a sequence  $(e_i : i < \omega)$  of imaginaries and there must be a subsequence such that  $e \subset acl(e_1, \dots, e_n)$ . In this case, we can take  $e = Cb(p)$  and find an element  $c$  in the sequence such that  $SU(c) = 1$  and  $Cb(p) \subseteq acl(c)$ . From now on, for this case ( $SU(Cb(p)) = 1$ ), when we refer to a canonical case of  $p$ , we actually mean any such  $c \in D^{eq}$ . Since we are mainly interested in the geometry of  $D$ , as we shall see, this abuse of notation will not cause any problems. (If  $T$  does not have elimination of hyperimaginaries, then  $G(D)$  is the set of rank-1 hyperimaginaries living in  $D^{heq}$ .)

(3) We now single out an easy but useful argument in the proof of Claim 1 in 3.2. Let  $D \subseteq F \subseteq G(D)$ , and let  $(F, \text{cl}(-) \cap F)$  form a modular subpregeometry. Then for any  $A \subseteq D^{\text{eq}}$ , there is  $B \subseteq F$  such that  $A$  and  $B$  are interalgebraic: For each  $c \in D^{\text{eq}}$ , there is a finite tuple  $\bar{d} \in D$  such that  $c \in \text{dcl}(\bar{d})$ . Hence, there is  $A_0 \subseteq D \subseteq F$  such that  $A \subseteq \text{dcl}(A_0)$ . Let  $A_1(\subseteq F)$  realizing  $\text{tp}(A_0/A)$  be independent from  $A_0$  over  $A$ . Let  $B = \text{acl}(A_0) \cap \text{acl}(A_1) \cap F$ . Since  $F$  is modular,  $A_0$  and  $A_1$  are independent over  $B$ . Then by the remark in the proof of 3.2,  $\text{acl}(B) = \text{acl}(A)$ .

Hence, if  $a \in G(D)$ , then there is a tuple  $\bar{b} = (b_1, \dots, b_n) \in F$  such that  $a, \bar{b}$  are interalgebraic. In particular,  $b_1 \in \text{acl}(a)$ , so  $a$  forks with  $b_1$ . Since  $\text{rk}(a) = 1$ ,  $a \in \text{acl}(b_1)$ , i.e.  $a, b_1$  are interalgebraic. This says that the canonical geometries of  $F$  and  $G$  are identical.

**Theorem 3.6.** *The following are all equivalent.*

- (1)  $D (D^{\text{eq}})$  is 1-based.
- (2)  $D$  is linear.
- (3)  $G(D)$  is linear.
- (4)  $G(D)$  is modular.
- (5)  $G(D)_A$  is linear, for any (some) small  $A$ .
- (6)  $G(D)_A$  is modular, for any (some) small  $A$ .

*Proof.* The proof of  $1 \Rightarrow 2$  is the same as the stable case. For completeness, we supply a proof. Let  $D$  be 1-based. Suppose that  $SU(L\text{stp}(ab/B)) = 1$  ( $a, b, B$  from  $D$ ). By 1-basedness,  $C = \text{cb}(ab/B)$  is in  $\text{acl}(ab)$ . Now,

$$SU(abC) = SU(ab/C) + SU(C) = 1 + SU(C) = SU(C/ab) + SU(ab) \leq 2.$$

Hence  $SU(C) \leq 1$ . Thus  $D$  is linear.

$2 \Rightarrow 3$ . Assume that  $D$  is linear. Now let  $x, y \in G(D)$ ,  $B \subseteq G(D)$  and  $SU(xy/B) = 1$  while  $SU(xy) = 2$  (1). Let  $C = \text{cb}(xy/B)$ . Then  $C \subseteq \text{acl}(B)$  and  $SU(xy/C) = 1$  ( $\begin{array}{c} xy \\ \downarrow \\ C \end{array} \begin{array}{c} B \\ \\ \end{array}$ )

( $\star$ ). We will show that  $C \subseteq \text{acl}(xy)$ . Now  $\{x, y\}$  is in the definable closure of a finite independent subset of  $D$ . In fact, we can find an independent subset  $F \cup \{a, b\}$  of  $D$  such that,  $\{x, y\}$  and  $\{a, b\}$  are interalgebraic over  $F$ , and  $F, \{x, y\}$  are independent. Then there is  $F' \subseteq D$  such that  $F$  and  $F'$  realize the same type over  $xy$ , and  $\begin{array}{c} F' \\ \downarrow \\ xy \end{array} \begin{array}{c} abBF \\ \\ \end{array}$ , hence

$\begin{array}{c} F' \\ \downarrow \\ \emptyset \end{array} \begin{array}{c} xyabBF \\ \\ \end{array}$  (2). Now as  $F$  and  $F'$  realize the same type over  $xy$ , there is  $\{a'b'\}$  such

that  $abF$  and  $a'b'F'$  realize the same type over  $xy$ . Note that  $\text{acl}(xyF') = \text{acl}(a'b'F')$  (3).

Claim 1)  $SU(a'b'/F'B) = 1$ : First we note that  $\begin{array}{c} a'b' \\ \not\downarrow \\ F' \end{array} \begin{array}{c} B \\ \\ \end{array}$ . (Otherwise, by (2) and (3),  $\{x, y\}$  is independent from  $B$ , which contradicts (1).) Now if  $SU(a'b'/F'B) = 0$ , then by (3)  $x, y \in \text{acl}(F'B)$ , and then by (2)  $x, y \in \text{acl}(B)$ , again contradicting (1). Hence  $SU(a'b'/F'B) = 1$ . Claim 1 is proved.

Therefore by Claim 1 and the linearity of  $D$ ,  $SU(a'b'/\text{acl}(a'b') \cap \text{acl}(F'B)) = 1$ . Now

$$\text{acl}(a'b') \cap \text{acl}(F'B) \subseteq \text{acl}(a'b'F') \cap \text{acl}(F'B) \subseteq \text{acl}(F'B).$$

Thus by transitivity, Claim 1 and (3),

$$SU(a'b'/W) = 1, \text{ where } W = acl(a'b'F') \cap acl(F'B) = acl(xyF') \cap acl(F'B) \quad (4).$$

Claim 2)  $SU(xy/W) = SU(xy/WC) = 1 (= SU(xy/C))$ : Since  $F' \subseteq W$ , by (3) and (4),  $xy \not\perp_{F'} W$ . Hence  $SU(xy/W) = 0$  or  $1$ . But if  $SU(xy/W) = 0$  then by (2),  $xy \in acl(B)$  which is not the case. Thus  $SU(xy/W) = 1$ . Similarly, since  $SU(xy/C) = 1$ , if  $SU(xy/WC) \neq 1$ , then by  $(\star)$  and (4),  $xy \in acl(WC) \subseteq acl(CF'B) \subseteq acl(F'B)$ . Then by (2),  $xy \in acl(B)$ . This contradicts (1) again. Hence  $SU(xy/WC) = 1$  and Claim 2 is proved.

Now since  $Lstp(xy/C)$  is a canonical type, by Claim 2,  $C \subseteq acl(W)$ . Therefore by (4),  $C \subseteq acl(xyF')$ . Then  $C \subseteq acl(xy)$ , since (2) with  $(\star)$  says that  $C \downarrow_{xy} F'$ . We have proved that  $G(D)$  is linear.

3 $\Rightarrow$ 6 and 4. Suppose that  $G(D)$  is linear. Let  $a, b \in G(D)$ , and  $SU(ab/A) = 2$ , whereas  $SU(ab/AB) = 1$  for  $B \subseteq G(D)$ . Now, there is a Morley sequence  $I$  of  $tp(ab/AB)$  in  $G(D)$  such that  $SU(ab/AB) = SU(ab/ABI) = SU(ab/I) = 1$ . By linearity of  $G(D)$ ,  $c \in acl(ab) \cap acl(I) \cap acl(AB)$  where  $c = cb(ab/I) = cb(ab/AB)$ , and  $SU(c) = 1$ . Hence  $c \in G(D)$  by the previous remark 3.5.2. Moreover, by the rank calculation of  $SU(cab/A)$ ,  $SU(c/A) = 1$ , i.e.  $c \notin acl(A)$ . Thus  $G(D)_A$  is modular by 1.3.

4 $\Rightarrow$ 1. Proof of this will be similar to the proof of fact 3.2. Suppose not, say there are  $a, b \in D^{eq}$  which are dependent over  $U = acl(a) \cap acl(b)$ . Now since  $G(D)$  is modular, by Remark 3.5.3, there are  $a_0, b_0, U_0 \subseteq G(D)$  which are interalgebraic with  $a, b, U$  respectively. In particular,  $a_0$  forks with  $b_0$  over  $U_0$ . Then again by modularity, there must exist  $c_0 \in acl(a_0U_0) \cap acl(b_0U_0) \cap G(D)$  such that  $c_0 \notin acl(U_0) = acl(U)$ . However, since  $acl(a_0U_0) = acl(acl(a_0) \cup acl(U_0)) = acl(aU)$  and similarly  $acl(b_0U_0) = acl(bU)$ ,  $c_0 \in acl(aU) \cap acl(bU) \subseteq U$ , a contradiction. Therefore, 4 $\Rightarrow$ 1 is proved.

3 $\Rightarrow$ 5. Suppose that  $G(D)$  is linear. Let us show that  $G(D)_A$  is linear. Let  $a, b \in G(D)$ , and  $SU(ab/B) = 1$  for  $B$  containing  $A$ . Now there is a Morley sequence  $I$  of  $tp(ab/B)$  in  $G(D)$  such that  $SU(ab/B) = SU(ab/BI) = SU(ab/I) = 1$ . Then by linearity of  $G(D)$ ,  $SU(c/A) \leq SU(c) \leq 1$  where  $c = cb(ab/I) = cb(ab/B)$ . Thus  $G(D)_A$  is linear.

5 or 6 $\Rightarrow$ 3. The proof will be similar to [17, 2.2.6(i) $\Rightarrow$ (ii)]. Let  $a, b \in G(D)$ ,  $B \subseteq G(D)$  and  $SU(ab/B) = 1$ . Let  $c = cb(ab/B)$ . If  $SU(ab) = 1$ , i.e.  $\{a, b\}$  is independent from  $B$ , then as  $c \in acl(\emptyset)$ , there is nothing to prove. Thus we can assume that  $SU(ab) = 2$ . Now we can further assume that  $abB$  is independent from  $A$ . Hence,  $SU(ab/BA) = 1$ , whereas  $SU(ab/A) = 2$ . Then by modularity (or linearity) of  $G(D)_A$ , there is  $d \in acl(abA) \cap acl(AB)$  and  $SU(d/A) = 1$ . Hence  $d$  and  $ab$  are dependent over  $A$ . Thus  $SU(ab/Ad) = 1 = SU(ab/AB)$ . Therefore  $c = cb(ab/AB) \in acl(Ad)$ , and as  $SU(d/A) = 1$ , by a rank calculation  $SU(c/A) \leq 1$ . Since  $c$  and  $A$  are independent,  $SU(c) \leq 1$ .  $\square$

**Remark 3.7.** (1) *Similarly,  $D$  is linear iff  $D_A$  is linear for any (some)  $A$ . We note that the geometry of a nonforking extension of  $D$  over  $A$  is a mere subgeometry of  $D_A$ . However one can still see the fact that linearity (1-basedness) is invariant via parallelism: Let  $q$  be a nonforking extension of  $D$  over  $A$ . It suffices to check that  $D$  is linear when  $q$  is linear. The proof is the same as 3.6(5) $\Rightarrow$ (3) with the trick that when an independent pair  $(a_0, a_1)$  is chosen from  $D$ , by amalgamation there is  $(b_0, b_1)$  such that  $tp(a_0, a_1) = tp(b_0, b_1)$  and each  $b_i$  realizes  $q$ .*

However, as is known, (local) modularity is not invariant via parallelism in general. (Frank O. Wagner was the first person to point this out.) Affine or projective space equipped with a binary random graph relation serves as an example, by a similar reason to that described in (2).

- (2) Vassiliev introduced a notion of generic substructure  $\bar{M}_1$  of  $\bar{M}$  for a supersimple  $SU$ -rank 1 theory (the submodel  $\bar{M}_1$  of  $\bar{M}$  does not have to be a small submodel). From [21], it follows that when we consider  $D$  as the universe of a model  $\bar{M}$ ,  $D$  is 1-based iff  $D_{\bar{M}_1}$  is modular. (He also proved the equivalence of 1 and 2 in the previous theorem 3.6 using a generic pair argument.)

However, in general, even if  $D$  is 1-based,  $D_A$  need not be modular over any (small) set  $A$ : Let  $V$  be a (saturated) affine space over a field, and let  $P$  be a generic predicate on  $V$ , as in [5]. Then there are two 1-types over  $\emptyset$  isolated by  $P(x)$  and  $\neg P(x)$ , both of which have  $SU$ -rank 1. By the genericity of  $P$ , both 1-types are 1-based.  $V$  is already non-modular over  $\emptyset$ . Now let  $W$  be a non-empty small subspace of  $V$  and consider a line  $l$ , spanned by  $\{a, b\}$  in  $V$ , independent from  $W$ , so  $\dim(l/W) = 2$ . Now let  $l'$ , spanned by  $\{a', b'\}$ , be parallel to  $l$  such that  $\dim(l/Wl') = 1$ . Then there is  $c \in V \setminus W$  such that  $\text{acl}(cW) = \text{acl}(lW) \cap \text{acl}(l'W)$ . However, as  $W$  is small, by genericity of  $P$ , we can find  $l, l'$  such that  $\{a, b, a', b'\} \subseteq P$  whereas  $P \cap \text{acl}(cW) = P \cap W$ . Then clearly  $P$  cannot be modular over  $W$  and similarly for  $\neg P$ . (We noticed later that a similar example is described in [21].)

To conclude, there are examples of non-trivial, 1-based  $D$  which are either non locally modular or locally modular but are neither affine nor projective (section 2) and examples of 1-based  $D$  which are affine/projective but not stable (an affine space with a binary random graph relation).

Since triviality of a (pre)geometry implies modularity, we have the following lemma by Remark 3.5.3.

**Lemma 3.8.**  *$D$  is trivial iff  $G(D)$  is trivial.*

**Theorem 3.9.** *Suppose  $D$  is 1-based. Then  $D$  is trivial or the geometry of  $G(D)$  is projective geometry over a division ring.*

*Proof.* Assume that  $D$  is non-trivial. By the above lemma and Fact 1.5, it suffices to show that any independent pair  $(a, b) \in G(D)$  (that is,  $\dim(ab) = 2$ ) is non-trivial in  $G(D)$  (i.e. there is  $c \in \text{cl}(ab) \setminus (\text{cl}(a) \cup \text{cl}(b))$  where  $\text{cl}$  is the closure of  $G(D)$ ).

Claim 1) Any independent pair  $(c, d) \in D$  is non-trivial in  $G(D)$ : Since  $G(D)$  is modular, if there is an independent pair from  $D$  trivial in  $G(D)$ , then any independent pair from  $D$  is trivial in  $G(D)$ . (Otherwise, by type amalgamation one can find a non-modular triangle which violates modularity of  $G(D)$ . See the proof of 1.8.) Then any independent triple  $(a, b, c) \in D$  is also trivial in  $G(D)$  (otherwise, say  $d \in \text{cl}(abc) \setminus (\text{cl}(a) \cup \text{cl}(b) \cup \text{cl}(c))$ , then  $ab$  and  $cd$  violate modularity of  $G(D)$ ). Iterating this argument shows that  $D$  is trivial, which is not the case.

Claim 2) Any independent pair  $(a, b) \in G(D)$  with  $a \in D$  is non-trivial in  $G(D)$ : Suppose not, say the independent pair  $(a, b) \in G(D)$  with  $a \in D$  is trivial. Now since  $b \in G(D)$ , there is an independent tuple  $\bar{c} = (c_1, \dots, c_n) \in D$  such that  $b \in \text{cl}(c_1, \dots, c_n)$ . By claim 1, we can assume  $n > 1$ . We can further assume that any  $n - 1$  subtuple of  $\bar{c}$  is independent from

$b$ , and  $\{a, b, c_1, \dots, c_{n-1}\}$  is independent. Hence  $\dim(a, b, \bar{c}) = n + 1$  (\*). By claim 1, there is  $d \in cl(ac_1) \setminus cl(a) \cup cl(c_1)$ . Then it follows that  $d \notin cl(c_2, \dots, c_n)$ , (otherwise  $a \in cl(\bar{c})$  violating (\*)). Hence the dimension of the set  $B = \{d, c_2, \dots, c_n\}$  is  $n$ , whereas by (\*) and triviality of  $(a, b)$ , it can easily be seen that  $cl(B) \cap cl(ab) = \emptyset$ . However, by modularity of  $G(D)$ , this contradicts (\*). Hence claim 2 follows.

Claim 3) Any independent pair  $(d, e) \in G(D)$  is non-trivial: Suppose not, say the independent pair  $(d, e) \in G(D)$  is trivial. There is a point  $c \in D$  which is independent from  $(d, e)$ . By claim 2,  $(c, d)$  and  $(c, e)$  are non-trivial. Again this violates modularity of  $G(D)$ , as  $(c, d, e)$  forms a non-modular triangle. This finishes the proof of claim 3 and the theorem.  $\square$

The theorem shows Vassiliev's result cheaply. Namely, the geometry of 1-based  $D$  is a subgeometry of projective geometry over a division ring.

From now on  $D$  will be 1-based.

The (pre)geometry of  $G(D)$  is not definable (it is a union of definable sets). However, with the reducibility condition we will discuss below,  $G(D)$  can be dealt with as a definable object, at least for the  $\omega$ -categorical case.

**Definition 3.10.** We say  $G(D)$  is reducible into (or has a reduction in)  $D^k$  if for any point  $a \in G(D)$  there is a  $k$ -tuple  $\bar{b} \in D$  such that  $a \in acl(\bar{b})$ .

When  $D$  is locally modular,  $G(D)$  has a reduction in  $D^2$ . In fact, more is true.

**Lemma 3.11.** ( $D$  locally modular.) For  $c \in G(D)$  and any  $d \in D$ , there is  $e \in D$  such that  $c \in acl(de)$ .

*Proof.* By Claim 1 in the proof of 3.2, there is  $\bar{a} = (a_1, \dots, a_n) \in D$  such that  $acl(\bar{a}d) = acl(dc)$ . If  $\bar{a} \in acl(d)$ , then  $c \in acl(d)$  and we are done. Hence we may assume that some  $a_i$ , say  $a_1$ , is not in  $acl(d)$ . Then, since  $rk(a_1) = 1$ ,  $a_1$  forks with  $c$  over  $d$ . By transitivity,  $c$  forks with  $a_1d$ . Since  $rk(c) = 1$ ,  $c \in acl(da_1)$ .  $\square$

Now, as by-products, we have the following lemmas.

**Lemma 3.12.** ( $D$  is non-trivial locally modular,  $p \in D$ .) The geometry of  $D_p$  and the geometry of  $G(D)$  are isomorphic as projective geometries.

*Proof.* Since  $G(D)'$ , the geometry of  $G(D)$ , is that of infinite dimensional projective space over some ring,  $G(D)'_p \cong G(D)'$ . Now as  $D \subseteq G(D)$ , clearly  $D_p$  is a subpregeometry of  $G(D)_p$ . Also, by the previous lemma, given  $x \in G(D) \setminus cl(p)$ , there is  $e \in D$  such that  $cl(px) = cl(ex)$  in  $G(D)$ . Hence  $D'_p \cong G(D)'_p \cong G(D)'$ , and the lemma holds.  $\square$

**Lemma 3.13.** ( $D$  locally modular.) If  $G(D)'$  is projective over a field with  $Card(F) \leq 3$ , then  $D'$ , the geometry of  $D$ , is affine or projective.

*Proof.* This can be shown by checking the possible cases, using amalgamation and 3.11, working in a projective plane. Let us first prove this when  $Card(F) = 3$ :

Claim 1) There is no independent  $\{a, b\} \subseteq D'$  such that  $cl(ab) \cap D' = \{a, b\}$ : if there exist such  $a, b$  then, by amalgamation, we can find  $c \in D'$  such that  $\dim(abc) = 3$  and any closed set in  $D'$  generated by a 2-subset of  $\{a, b, c\}$  is itself. Let  $b' \in (cl(bc) \cap G(D)') \setminus D'$ . Then by 3.11, we can find  $d \in D'$  such that  $b' \in cl(ad)$ . By modularity and the fact that any



line in  $G(D)'$  has size 4, there are  $c' \in cl(ac) \cap cl(bd) \cap G(D)$  and  $c'' \in cl(ac) \setminus D'$  such that  $c'' \neq a, c, c'$ . Then it is easy to see that  $a, b, c \notin cl(dc'')$ . Hence  $\emptyset \neq cl(dc'') \cap cl(bc) \in G(D) \setminus D$ , and also,  $cl(dc'') \cap cl(ab) \in G(D) \setminus D$ . Therefore  $cl(dc'') \cap D' = \{d\}$ , which contradicts 3.11. Hence claim 1 is proved.

Claim 2) There cannot exist independent pairs  $\{a_1, a_2\}, \{b_1, b_2\} \subseteq D'$  such that  $Card(cl(a_1a_2) \cap D') = 3$  and  $Card(cl(b_1b_2) \cap D') = 4$ : suppose that there are such points  $a_i$  and  $b_i$  in  $D'$ . By amalgamation, there are  $\{u, v, w\} \subseteq D'$  such that  $Card(cl(uv) \cap D') = Card(cl(vw) \cap D') = 3$ ,  $Card(cl(uw) \cap D') = 4$  and  $dim(uvw) = 3$ . Choose  $v' \in cl(uv) \setminus D'$  and  $w' \in cl(vw) \setminus D'v'$ . Then, as  $Card(cl(uw) \cap D') = 4$ , by modularity there is  $z \in cl(uw) \cap cl(v'w') \cap D'$ . Then  $1 \leq Card(cl(v'w') \cap D') \leq 2$ , contradicting 3.11 and Claim 1. Hence claim 2 follows and the lemma is established for  $Card(F) = 3$  by Fact 1.6.

When  $Card(F) = 2$ , by a similar argument, it can be seen that one cannot amalgamate independent  $(c_1c_2)$ ,  $(c_1c_2)$  and  $(d_1d_2)$  from  $D'$  such that  $Card(cl(c_1c_2) \cap D') = 2$  and  $Card(cl(d_1d_2) \cap D') = 3$ . Then 1.6 says the lemma holds for this case as well.  $\square$

**Remark 3.14.** *If  $D$  is a generic subset of locally modular  $U$ , then  $G(D)$  also has a reduction in  $D^2$ : Let  $c \in G(D)$ . Then by 3.11, there are  $d \in D, e \in U$  such that  $c \in acl(de)$ . Now there exists  $(a, b)$  realizing  $tp(de/c)$  such that  $\begin{matrix} ab \\ \downarrow \\ c \end{matrix} \begin{matrix} de \\ \\ \end{matrix}$ . Then by the remark in the proof of 3.2,  $acl(g) = acl(c)$  where  $g = cb(ab/de)$ . By the genericity of  $D$ , there exists  $(a', b') \in D$  such that  $tp^-(ab/de) = tp^-(a'b'/de)$  in the language of  $U$  without the predicate  $D$ . Then  $a'b'$  preserves the canonical base of  $tp(ab/de)$ . Hence  $acl(c) = acl(g) \subseteq acl(a'b')$ . We are done. (In this case, of course, strong reducibility in  $D^2$  as in 3.11 does not follow.)*

In general, the following holds. Hence any  $G(D)$  is reducible into  $D^3$  for 1-based  $D$ .

**Theorem 3.15.** *Let  $u \in G(D)$ .*

- (1) *Suppose that  $(x, y)$  is a fixed independent pair from  $D$ . Then  $u \in acl(x'y'z)$  where  $x', y', z \in D$  and  $tp(xy) = tp(x'y')$ .*
- (2) *Suppose that there are independent  $\{x, y\}$  from  $D$  such that for any  $z \in acl(xy) \cap G(D)$ ,  $z \in acl(z')$  for some  $z' \in (acl(xy) \cap D)$ , i.e. a full line is already in  $D$ . Then  $G(D)$  is reducible into  $D^2$ .*
- (3) *For given  $d_1 \in D$ , there is  $d_2d_3 \in D$  such that  $u \in acl(d_1d_2d_3)$ .*

*Proof.* We first prove (1) using induction on  $n$  where  $u \in G(D)$  has a reduction in  $D^n$ . The case  $n = 0$ , that is  $u \in acl(\emptyset)$ , is trivial. Now suppose that  $u \in acl(a_1 \dots a_{n+1})$  ( $a_i \in D$ ) with the induction hypothesis for  $n$ . We will verify (1) for  $n + 1$ . We can assume that  $\{a_1, \dots, a_{n+1}\}$  is independent. Now let  $g = cb(ua_1/a_2 \dots a_{n+1})$ . Thus  $SU(g) = 1$  and  $g \in G(D)$ . Since  $a_1 \notin acl(g) \subseteq acl(a_2 \dots a_{n+1})$ ,  $u$  is in the line generated by  $\{g, a_1\}$  (\*). Then, since  $g \in acl(a_2 \dots a_{n+1})$ , by the induction hypothesis,  $g \in acl(bcd)$  for some  $c, d \in D$  such that  $tp(cd) = tp(xy)$ . If  $g$  is already in  $acl(cd)$ , then from (\*),  $u \in acl(ga_1) \subseteq acl(cda_1)$ . Thus, in this case, the theorem holds. Therefore we only need to consider the case when  $g \notin acl(cd)$ . We can also clearly assume that  $g \notin acl(b)$  (otherwise we are done). Then, since  $G(D)$  is modular, we can find  $v \in acl(cd) \cap acl(bg) \cap G(D)$  such that  $g \in acl(bv)$  (†), and  $dim(vg) = 2$ . Also, at least one of  $\{v, c\}$  or  $\{v, d\}$  (say  $\{v, d\}$ ) is independent (\*\*). Now, as  $d, a_1 \in D$ ,  $Lstp(d) = Lstp(a_1)$ . Hence, we can find  $v'$  such that  $Lstp(vd) = Lstp(v'a_1)$  (\*). In particular,  $Lstp(v) = Lstp(v')$ . Then by (\*), (\*\*), (\*), we can amalgamate types

$tp(v'/a_1)$  and  $tp(v/g)$ , so that we obtain  $v'' \models tp(v'/a_1) \cup tp(v/g)$ . Hence, there are  $b', c'$  such that  $tp(vgb) = tp(v''gb')$ ,  $tp(vdc) = tp(v''a_1c')$  (by  $(\star)$ ), and so by  $(\dagger)$ ,  $tp(c'a_1) = tp(xy)$ ,  $v'' \in acl(c'a_1)$  and  $g \in acl(b'v'')$  ( $\ddagger$ ). Then by  $(*)$ ,  $(\ddagger)$ ,

$$u \in acl(ga_1) \subseteq acl(ga_1v'') \subseteq acl(a_1b'v'') \subseteq acl(c'a_1b').$$

Since  $tp(c'a_1) = tp(xy)$ , the  $(n+1)$ th induction hypothesis for (1) is deduced. Therefore we have proved (1). (2) comes from (1) and the modularity of  $G(D)$ .

For (3), use the proof of (1), replacing  $g$  with  $u$  and  $a_1$  with  $d_1$ .  $\square$

**Remark 3.16.** *From the above result 3.15, we can identify any element  $u$  in  $G(D)$  as  $e = \bar{d}/E \in G(D)$  (i.e.  $acl(u) = acl(e)$ ) where  $\bar{d} \in D^6$  and  $E$  is a definable equivalence relation on  $D^6$  (over  $dom(D)$ ), as follows: As usual we take  $dom(D) = \emptyset$  (by naming  $dom(D)$ ) for notational simplicity. By 3.15, there are triples  $(a, b, c), (a', b', c') \in D^3$  such that  $u \in acl(abc)$ ,  $(a'b'c') \models tp(abc/u)$  and  $abc \downarrow_u a'b'c'$ . Then by the remark in 3.2,  $acl(u) = acl(g)$  where  $g = cb(abc/a'b'c')$ . Now if an automorphism fixes  $(abca'b'c')$  then as the parallelism class of  $Lstp(abc/a'b'c')$  is setwise fixed, so is  $cb(abc/a'b'c')$ , i.e.  $g \in dcl(abca'b'c')$ . Hence we can identify  $g = (abca'b'c')/\bar{F}$  where  $\bar{F}$  is a type-definable equivalence relation on  $D^6$ . Therefore, since  $SU(g) = 1$ , as in 3.5.2, there is a definable equivalence relation  $E \in \bar{F}$  such that  $(abca'b'c')/E$  in  $G(D)$  and  $g$  are interalgebraic. As  $u$  and  $g$  are interalgebraic, the result follows.*

In particular, if  $D$  is  $\omega$ -categorical, then as there are only finitely many types and equivalence relations over  $dom(D)$  on  $D^6$ , by the above remark, the geometry of  $G(D)$  is isomorphic to that of some union of finitely many definable sets in finitely many sorts in  $G(D)$ . Since the union itself is  $\omega$ -categorical even as a 1-sorted structure, we have the following results.

**Corollary 3.17.**  *$D$   $\omega$ -categorical. Then,  $D$  is trivial or the geometry of  $G(D)$  is projective over a finite field.*

**Definition 3.18.** *Let  $G(D)$  have a reduction in  $D^k$  where  $k$  is minimal such. (Hence  $k \leq 3$ .) By a hypersurface of  $D^k$ , we mean a  $Lstp(a_1 \dots a_k/A)$  of rank  $k-1$  ( $a_i \in D$ ). We call a hypersurface of  $D^k$  with rank-0 canonical base rigid.*

*The Projective closure of  $D$ , denoted  $P(D)$ , is defined as:*

$$P(D) = D \cup \{e \mid e \text{ is the canonical base of a non-rigid hypersurface of } D^k\}.$$

By a rank calculation, the rank of the canonical base of a hypersurface of  $D^k$  is  $\leq 1$ . Hence  $P(D)$  is a collection of canonical bases  $e = cb(a_1 \dots a_k/e)$  where  $SU(a_1 \dots a_k/e) = k-1$  for independent  $(a_1, \dots, a_k) \in D^k$ . Since  $SU(e) = 1$ , by Remark 3.5.2 we can assume that  $P(D) \subseteq G(D)$  by identifying  $e$  with some interalgebraic  $g \in G(D)$ . If  $D$  is  $\omega$ -categorical, we can *moreover* assume, as in 3.16, that  $P(D)$  is a  $dom(D)$ -definable subset of  $G(D)$ .

**Lemma 3.19.** *(Notation as in 3.16) For any point  $u \in G(D)$ , there is  $g \in P(D)$  such that  $g$  and  $u$  are interalgebraic. (Hence the geometries of  $P(D)$  and  $G(D)$  coincide.)*

*Proof.* This is immediate from Remark 3.16, using the fact that there  $SU(abc/a'b'c') = 2$ .  $\square$

Then we get the following theorem immediately. This answers the main conjecture in [21] positively.

**Theorem 3.20.** *Suppose  $D$  is  $\omega$ -categorical. Then  $P(D)$  is  $\text{dom}(D)$ -definable (in  $D^{\text{eq}}$ ), as is its geometry  $P(D)'$ . Hence, if  $D$  is non-trivial,  $P(D)'$  has a strongly minimal stable reduct (over  $\text{dom}(D)$ ) preserving the projective geometry over a finite field  $F$ . Hence, over a finite parameter in  $D^{\text{eq}}$ ,  $D$  interprets a vector space (as projective and affine space do) over  $F$ .*

(Recently Ben-Yaacov, Tomasic and Wagner obtained an almost hyperimaginary group configuration in [2]. Using this, they also proved the interpretability of a vector space in an  $\omega$ -categorical non-trivial  $D$  [20].)

Now we study the relationship between the whole theory and rank-1 types. As in the stable case, the notion of triviality can be imposed on the underlying theory.

**Definition 3.21.** *We say that  $T$  is trivial if for  $a, b, c, A \subseteq \bar{M}^{\text{eq}}$ , whenever  $\{a, b, c\}$  is pairwise independent over  $A$ , then  $\{a, b, c\}$  is independent over  $A$ .*

By mimicking the proofs in the stable case found in [17, 4.2.5], with some novel argument, we can show 3.22 below. The following easy fact will be used. In any simple  $T$ , if a finite tuple  $\bar{a}$  is not in  $\text{acl}(\emptyset)$ , then there is a complete extension  $p$  of  $\text{tp}(\bar{a})$  such that  $SU(p) = 1$ : Let  $\text{tp}(\bar{a}) = p_0 \subseteq p_1 \dots \subseteq p_i \dots$  be a maximal non-algebraic forking chain. (The length is at most  $|T|$ .) Then  $p = \cup_i p_i$  is non-algebraic as well and, by maximality,  $SU(p) = 1$ .

**Lemma 3.22.** *Suppose that  $T$  is 1-based. Then  $T$  is trivial if and only if all rank-1 types are trivial.*

*Proof.* We only need to prove the right to left direction, as the other direction is obvious. Suppose that there exists a non-trivial triple  $(a, b, c)$  over  $A$ , i.e.  $(a, b, c)$  is pairwise independent over  $A$ , whereas  $\{a, b, c\}$  is dependent over  $A$ . We will find a non-trivial rank-1 type.

Claim 1) We may assume that  $a \in \text{acl}(bcA)$ ,  $b \in \text{acl}(acA)$  and  $c \in \text{acl}(baA)$ , (but  $\{a, b, c\}$  is still pairwise independent over  $A$ ): if we choose  $a_0 \in cb(a/bcA) \setminus \text{acl}(A)$ ,  $b_0 \in cb(b/a_0cA) \setminus \text{acl}(A)$ ,  $c_0 \in cb(c/a_0b_0A) \setminus \text{acl}(A)$  such that  $cb(a/bcA) \subset \text{acl}(a_0)$  and similarly for  $b_0, c_0$ , then, by 1-basedness, these elements will suffice.

Now, for notational convenience, let  $A = \emptyset$ .

Claim 2) There is  $F$  such that  $SU(a/F) = SU(b/F) = SU(c/F) = 1$  and  $(a, b, c)$  is still a non-trivial triple over  $F$ : By the remark before 3.22, there is  $d$  such that  $SU(a/d) = 1$ .

We assume  $d = cb(a/d)$ . Then by 1-basedness,  $d \in \text{acl}(a)$  and  $\begin{matrix} bd \\ \downarrow \\ F \end{matrix} \quad \begin{matrix} cd \\ \downarrow \\ F \end{matrix}$  where  $F =$

$\text{acl}(bd) \cap \text{acl}(cd)$ . Then  $\begin{matrix} b \\ \downarrow \\ F \end{matrix} \quad \begin{matrix} c \\ \downarrow \\ F \end{matrix}$ . Now we have  $SU(a/F) = 1$ , since otherwise  $a \in \text{acl}(bd)$ ,

so  $a \in \text{acl}(d)$  (as  $\begin{matrix} ad \\ \downarrow \\ \emptyset \end{matrix} \quad \begin{matrix} b \\ \downarrow \\ \emptyset \end{matrix}$ ) contradicting the fact that  $SU(a/d) = 1$ . Similarly  $\begin{matrix} a \\ \downarrow \\ F \end{matrix} \quad \begin{matrix} b \\ \downarrow \\ F \end{matrix}$

and  $\begin{matrix} a \\ \downarrow \\ F \end{matrix} \quad \begin{matrix} c \\ \downarrow \\ F \end{matrix}$ . Now by the rank calculation of  $SU(ab/cF)$ ,  $SU(b/cF) = SU(b/F) = 1$  since  $SU(a/cF) = 1$ . Similarly by the rank calculation of  $SU(ac/bF)$ ,  $SU(c/bF) = SU(c/F) = 1$ .

Hence, claim 2 is established.

Claim 3)  $tp(a/F)$  is non-trivial. (This finishes the proof.): First, there are  $a_1, a_2, b', c'$  such that  $tp(a_1c'/Fb) = tp(ac/Fb)$ ,  $c' \downarrow_F abc$ ,  $tp(a_2b'/Fc) = tp(ab/Fc)$  (\*) and  $b' \downarrow_F abcc'$ .

Hence  $b'c' \downarrow_F abc$  (\*\*). Now we subclaim that  $a, a_1, a_2$ , all realizing the rank-1 type  $tp(a/F)$ , form a non-trivial triple over  $Fb'c'$ . For pairwise independence, we only show  $a_1 \downarrow_{Fb'c'} a_2$ , the other cases follow similarly. We have  $b \downarrow_F c$ , hence  $b \downarrow_{Fb'c'} c$  by (\*\*). Thus  $Fbc' \downarrow_{Fb'c'} Fcb'$ . Then it follows  $a_1 \downarrow_{Fb'c'} a_2$  by Claim 2 and (\*). Now as  $a \in acl(bc)$ , again by (\*) and Claim 2, each of  $a, a_1, a_2$  is algebraic over  $Fb'c'$  together with the other two. Hence the subclaim is verified.

Secondly, by 1-basedness, there is a tuple  $a_3a_4a_5$  realizing  $tp(aa_1a_2/Fb'c')$  such that  $aa_1a_2 \downarrow_{a_3a_4a_5} Fb'c'$  and  $aa_1a_2 \downarrow_{Fb'c'} a_3a_4a_5$ . Then, it can be easily seen that  $(a, a_1, a_2)$  forms a non-trivial triple over  $Fa_3a_4a_5$ . This means that the localized geometry of  $tp(a/F)$  at  $\{a_3, a_4, a_5\}$  is non-trivial since all the  $a$  and  $a_i$  realize the rank-1 type  $tp(a/F)$ . Hence, the original type  $tp(a/F)$  cannot be trivial either. We have proved claim 3 and the lemma.  $\square$

Recall that if  $T$  is 1-based, then for any  $A$  and tuple  $\bar{a}$ , there is  $\bar{a}'$  with  $|\bar{a}| = |\bar{a}'|$  such that  $D(tp(\bar{a}/A), \varphi, k) = D(tp(\bar{a}'/A), \varphi, k)$  for each  $\varphi, k$  (by taking an element in a Morley sequence for example). Thus, if additionally  $T$  is  $\omega$ -categorical, then  $T$  is supersimple since every forking chain must have finite length. Moreover, since  $SU(\bar{a}/A) = SU(\bar{a}'/A)$ , any  $n$ -type in  $T$  has finite  $SU$ -rank ([9, 4.7]). (Evans and Wagner [8] showed that any supersimple,  $\omega$ -categorical group or any CM-trivial theory also has finite  $SU$ -rank.)

By previous results in this section, we obtain the following.

**Corollary 3.23.** *Let  $T$  be non-trivial, 1-based and  $\omega$ -categorical. Then an infinite dimensional vector space over a finite field, in particular the infinite additive group, is definable in  $\bar{M}^{eq}$  over a finite parameter.*

## REFERENCES

- [1] E. Artin, *Geometric algebra*, Interscience Publishers, New York (1957).
- [2] I. Ben-Yaacov, I. Tomasic and F. O. Wagner, ‘Constructing an almost hyperdefinable group’, preprint 2001.
- [3] S. Buechler, ‘The geometry of weakly minimal types’, *J. of Symbolic Logic* 50 (1985) 1044-1053.
- [4] S. Buechler, A. Pillay and F. O. Wagner ‘Supersimple theories’, *J. of AMS* 14 (1998) 109-124.
- [5] Z. Chatzidakis and A. Pillay, ‘Generic structures and simple theories’, *Ann. of Pure and Applied Logic* 95 (1998) 71-92.
- [6] G. Cherlin, L. Harrington, and A. H. Lachlan, ‘ $\aleph_0$ -categorical,  $\aleph_0$ -stable structures’, *Ann. Pure and Applied Logic* 28 (1985) 103-135.
- [7] J. Doyen and X. Hubaut, ‘Finite regular locally projective spaces’, *Math. Z.* 119 (1971) 83-88.
- [8] D. Evans and F. O. Wagner, ‘Supersimple  $\omega$ -categorical groups’, *J. of Symbolic Logic* 65 (2000) 767-776.
- [9] B. Hart, B. Kim and A. Pillay, ‘Coordinatisation and canonical bases in simple theories’, *Journal of Symbolic Logic*, 65 (2000) 293-309.
- [10] E. Hrushovski, ‘Unimodular minimal theories’, *Journal of London Math. Soc.*, 46 (1992) 385-396.
- [11] E. Hrushovski, ‘Locally modular regular types’, in *Classification theory*, Lecture Notes in Math. 1292, Springer-Verlag, Berlin (1987) 132-164.

- [12] E. Hrushovski, ‘The Mordell-Lang conjecture for function fields’, *J. of AMS* 9 (1996) 667-690.
- [13] E. Hrushovski, ‘Simplicity and the Lascar group’, preprint (1998).
- [14] E. Hrushovski and S. Shelah, ‘A dichotomy theorem for regular types’, *Ann. of Pure and Applied Logic* 45 (1989) 157-169.
- [15] B. Kim, ‘Forking in simple unstable theories’, *Journal of London Math. Soc.* (2) 57 (1998) 257-267.
- [16] B. Kim and A. Pillay, ‘Simple theories’, *Ann. of Pure and Applied Logic* 88 (1997) 149-164.
- [17] A. Pillay, *Geometric stability theory*, Oxford University Press, Oxford (1996).
- [18] S. Shelah, ‘Simple unstable theories’, *Annals of Math. Logic* 19 (1980), 177-203.
- [19] S. Shelah, *Classification theory*, 2nd edn, North-Holland, Amsterdam (1990).
- [20] I. Tomasic and F. O. Wagner, ‘Applications of the group configuration theorems in simple theories’, preprint 2001.
- [21] E. Vassiliev, ‘Expansions of rank 1 structures’, Ph. D. Thesis, Univ. of Notre Dame (2001).
- [22] F. O. Wagner, *Simple theories*, Kluwer Academic Publishers, Dordrecht (2000).
- [23] B. I. Zilber, *Uncountably categorical theories*, AMS Translations of Mathematical Monographs, Vol. 117 (1993).

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