SEVERI'S CONJECTURE AND SINGLE NODE CURVES

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Abstract. I will give a brief discussion of Severi's conjecture, concerning the existence of degenerations of a curve of degree n, to n lines in general position. Severi's original argument uses a "cone construction", which, in the context of proving the conjecture, is flawed, as was observed by Zariski. I will discuss some alternative approaches, based on Severi's projection method and the use of "asymptotic degenerations", which, at least, partially solve the deficiencies in his argument. In particular, we show, how these approaches can be combined, to solve the conjecture in the case of a curve, having at most one node as a singularity. I will also consider a class of "harmonic variations", which, potentially, provides a further method of resolving the problems involved.

Severi's conjecture states that;

Any plane nodal curve, of algebraic degree m, can be "degenerated" to a sequence of lines $\{l_1, \ldots, l_m\}$, with the property that no three of the lines intersect in a point, ([1]),that is, there does not exist x, with $x \in (l_i \cap l_j \cap l_k)$, for $i \neq j \neq k$.

Definition 0.1. *Nodal Degenerations*

By "degenerates", we mean that there exists a 1*-parameter family of curves* $\{C_t : t \in P^1\}$*, ([2])*

with;

(i). $C_0 = C$*.*

[1] This configuration is usually referred to as "lines in general position".

[2] The notation of P^1 to index the family of curves, is a convenient shorthand. More generally, the parameters should be defined by an irreducible curve $S \subset Par_m$, where Par_m is the parameter space for curves of degree m.

(ii). $C_t \in V_{m,g}$, $t \neq \infty$ *([3])*

Definition 0.2. *Nodal Degenerations to Lines in General Position*

A nodal degeneration with the additional axiom;

(iii). $C_\infty = l_1 \cup \ldots \cup l_m$.

The degree-genus formula states that;

$g(C) = \frac{(m-1)(m-2)}{2} - \sum_{v \in C} e(v)$

where v enumerates the singularities of C, and $e(v)$ is an integer, depending on the type of singularity. It follows that, during the degeneration, a node of C cannot "desingularize". If this were to occur, it would, generically, introduce a new singularity of higher complexity. This condition can easily shown to be closed, which leads to a contradiction. An alternative way of formulating this argument, is to show that the set $W_{m,k}$, of curves with fixed degree m and exactly k ordinary nodes, forms an open subset of $V_{m,g}$, where $g = \frac{(m-1)(m-2)}{2} - k$.

We now consider Severi's original argument, in support of this conjecture.

Argument (A). Starting with a plane curve $C \subset P^2$, of degree m, choose a point $Q \in P^3$, and let $Cone_Q(C)$ be the union of lines;

$\{\bigcup_{x \in C} l_{Qx}\}$

Let $\{H_t : t \in P^1\}$ be a family of planes with $C \subseteq H_0$, and $Q \in H_\infty$. Then;

$(Cone_Q(C) \cap H_\infty) = \bigcup_{x \in T} l_{Qx}$, where, by Bezout's Theorem, we can assume that $T = (H_\infty \cap C)$ consists of finitely many distinct points $\{x_1, \ldots, x_m\}$. This " intuitively" produces a degeneration of C to a sequence of lines passing through a point Q, see Severi's paper [3].

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[3] $V_{m,g}$ denotes the space of curves of fixed degree m and genus g.

Severi's argument (A) may be made more precise by the following construction;

Definition 0.3. *Projective Degenerations;*

Choose $\overline{C} \subset P^3$, *with the property that* $pr_P(\overline{C}) = C$, *relative to a hyperplane* $\omega \subset P^3$. *Let* l *be a line, passing through* P, *and intersecting* ω *in* Q. *The set of projections;*

$pr_x(\overline{C}) : x \in (l \setminus Q)\}$

together with the limit curve "$pr_Q(C)$*", form a family, which we refer to as a projective degeneration.*

The proof that this gives a degeneration, satisfying condition (ii) in Definition 0.1 above, is given in my paper [1], using Lemma 8.1 to show that degree is preserved, and Lemma 8.3, to show that the genus may be preserved for a suitable choice of P (a birational projection will preserve the genus of the original curve). The limit curve "$pr_Q(C)$" is not formally defined in terms of projections, however, is easily shown to be a union $\bigcup_{y\in(\overline{C}\cap\omega)} l_{Qy}$, where $y \in (\overline{C} \cap \omega)$. Again, the reader should look at the paper [1], Lemma 8.2.

Severi's subsequent argument is the following;

Argument(B). Having found a degeneration of C to a union of lines passing through a point Q, vary the lines to "general position", by moving them in parallel. Severi shows, in [3], that the closure $\overline{V_{m,g}}$ is invariant under this construction. However, it is possible that the degeneration may pass to another irreducible component, in which case, there is no way of producing a single "degeneration", in the sense defined above, from the two previous "degenerations". The problem, with combining the two families, as observed by Zariski in [5], is that the nodes of the curves C_t, in the limit C_∞, may not specialise to nodes, formed by the intersection of the lines.

Zariski gives an explicit example, in [5], of a family of plane cubics, with a single node, that degenerates to 3 lines through the origin. In order to circumvent this problem, one needs to ensure that the nodes of the curve C, and generic curves C_t in the family, are specialised in the limit.

For this purpose, we introduce the notion of an "asymptotic degeneration", ([4])

Definition 0.4. *Asymptotic Degenerations*

Let $\{C_t\}_{t\in P^1}$ be a family of plane projective algebraic curves of degree m, with the following properties, relative to a fixed coordinate system $\{x,y\}$;

(i). The intersection of each curve C_t in the family with the axis $x = 0$ consists of a fixed set of m distinct points $\{(0,a_1),\ldots,(0,a_j),\ldots,(0,a_m)\}$ in finite position.

(ii). For generic $t \in P^1$, C_t is irreducible.

(iii). Each curve C_t has m distinct non-singular branches, centred at the points $\{(0,a_1),\ldots,(0,a_m)\}$, and, the tangent lines $\{l_{a_1},\ldots,l_{a_m}\}$ at these branches are all fixed and distinct from the axis $x = 0$.

We call such a family of curves an asymptotic degeneration.

The technical advantage of imposing the condition (iii) in the definition, is the possibility of applying a uniform version of Newton's Theorem. We have that;

$$C_t = (y-\eta_1(x,t))(y-\eta_2(x,t))\ldots(y-\eta_m(x,t))$$

where $\eta_j(x,t)$ are power series in $L[[x]]$. As far as I know, there is no direct algebraic proof of this result, but it can be shown, by interpreting the power series $\eta_j(x,t)$, generically, as irreducible curves that cover C_t in the projective space P^3. In this interpretation, the intersections $(C_{i,t}\cap C_{j,t})$ correspond to nodes of the plane curve C_t. If a degeneration is chosen, which preserves nodes, then these intersections are preserved. A further advantage of imposing the condition (iii), is that, in the limit, none of the components of $C_{i,\infty}$ and $C_{j,\infty}$ coincide. It follows that the intersection pattern of $(C_{i,t}\cap C_{j,t})$ varies continuously to the limit $(C_{i,\infty}\cap C_{j,\infty})$, and, the nodes of the generic curve C_t, specialise to nodes,

[4] The use of the term asymptotic derives from the interpretation of the axis $x = 0$, in the following definition, as the line at ∞.

formed by the limit curve C_∞ $(*)$. For non-asymptotic degenerations, the condition $(*)$, as in Zariski's example, fails. ([5]).

The existence, $(**)$, of such degenerations follows from calculations of the dimension of $V_{m,g}$ as $3m + g - 1$, which is done in the papers [5] and [2]. The intuitive idea may be found in Plucker's work on formulae relating invariants such as the class, degree and genus of a plane curve. For a variation of curves in which nodes are preserved, we have that;

$$\frac{\partial F}{\partial x} F(x(t), y(t); t) = \frac{\partial F}{\partial y} F(x(t), y(t); t) = F(x(t), y(t); t) = 0$$

where $(x(t), y(t))$ parametrises a given, varying node. Applying the chain rule, we obtain $F(x(t), y(t); dt) = 0$, hence, the derived curve $F(x, y; dt)$ passes through the set of nodes of the original curve C_0, $(***)$. One can readily calculate the dimension of the space of such curves, satisfying this condition, $(***)$.

The requirement that m asymptotes of the curve are fixed, imposes $2m$ conditions on the space $V_{m,g}$, as $2m < 3m + g - 1$, this shows $(**)$.

In the case of single node curves, that is curves possessing at most one node as a singularity, it should be clear, from the previous discussion, that it is only necessary to impose a weaker condition, in order to make Severi's argument work, ([6]). We make the following definition;

Definition 0.5. *Partially Asymptotic Degenerations*

A degeneration is partially asymptotic, if, with the notation of Definition 0.4, there exist 2 *fixed points* $\{a, b\}$ *along the axis* $x = 0$*, at which the branches of* C_t *are non-singular, and, such that the tangent lines* $\{l_a, l_b\}$ *to these branches, are fixed and distinct from the axis* $x = 0$.

The interesting property of such variations, is that they can be obtained as projective degeneration, Definition 0.3, and, in particular, can also be nodal degenerations, Definition 0.1. This is achieved by the following projective construction;

[5]The interested reader, who is skeptical about the use of asymptotic conditions, should try to find an example, such as Zariski's, which satisfies the condition (ii) of Definition 0.4

[6]Again, the interested reader is invited to find an example, such as Zariski's, for which this condition holds.

Consider C as embedded in a hyperplane $\omega \subset P^3$. We choose P and $\overline{C}$, for which $pr_P(\overline{C}) = C$, and $\overline{C} \cap \omega$ consists of a finite number of distinct points $\{P_1, \ldots, P_m\}$, where $m = deg(C)$. This may be achieved, for example, by taking an appropriate section of the cone $Cone_P(C)$. We choose two points $\{a, b\}$ from the set $\{P_1, \ldots, P_m\}$, and let $\{Pl_a, Pl_b\}$ be the planes $\{< l_a^C, l_a^{\overline{C}} >, < l_b^C, l_b^{\overline{C}} >\}$ respectively. By elementary properties of projections, we have that $pr_P(l_a^{\overline{C}}) = l_a^C$ and $pr_P(l_b^{\overline{C}}) = l_b^C$, hence, $P \in (Pl_a \cap Pl_b)$. Now, let l be the line, defined by the intersection of the planes $(Pl_a \cap Pl_b)$, and, consider the projective degeneration of C, Definition 0.3. It is easily checked that, for $x \in l$, $pr_x(l_a^{\overline{C}}) = l_a^C$, and $pr_x(l_b^{\overline{C}}) = l_b^C$, therefore, the tangent lines of $pr_P(C)$ at $\{A, B\}$ are fixed, throughout the variation. As the original set of points are distinct, it follows, by Bezout's theorem, that these tangent lines are transverse to the intersection line $(\omega_1 \cap \omega)$, where ω_1 is the section containing the curve $\overline{C}$. Hence, the conditions of Definition 0.5, a partially asymptotic degeneration are satisfied.

For a partially asymptotic, projective variation, one has the property that, a *given* node of C, corresponding to the intersection of two covers $C_{i,0}$ and $C_{j,0}$, is preserved, in the limit C_∞, as the intersection of two lines, passing through the fixed point Q, in Definition 0.3. For curves with a single node, it follows that we can combine Severi's arguments, explained above, and obtain a single nodal degeneration to lines in general position, Definition 0.2, $(*)$

In the general case of nodal curves, one may use the method of asymptotic degenerations, to reduce a given curve C into components, by forcing the original curve to have a higher contact with a given asymptote. By Bezout's Theorem, an irreducible curve of degree m, can have contact at most m, with a line l, but a sequence of $(m-1)$ degenerations, ([7]), satisfying the conditions of Definition 0.4, will force it to have contact $(m+1)$, with one of its asymptotes. This contradiction can only be resolved if the limit curve C_∞ splits into components, most probably a union $(l \cup C')$, where C' has degree $(m-1)$, but, possibly a union of curves $C_1 \cup C_2$, where $deg(C_1) + deg(C_2) = m$. The properties of asymptotic degenerations that we considered earlier allows us to effectively understand the intersection geometry of the component

[7]The fact these are degenerations uses technical arguments, given in [2], it is possible that these arguments may be supported by generalising the definition of a projective degeneration to higher dimensions

curves, and to, potentially resolve Severi's conjecture in full generality.

The existence of nodal generations to lines in general position, allows us to consider a further class of variations, which we define as follows.

Definition 0.6. *Harmonic Variations*

Let C be a (single) nodal curve, with a nodal degeneration to lines in general position, (by $(*)$), or Severi's conjecture. Without loss of generality, we can choose a set T of g points of C, where $g = genus(C)$, with the following properties;

(a). For $x \in T$, x is nonsingular and the tangent l_x to C, passes through a fixed point Ω.

(b). The points $x \in T$ specialise to distinct intersections $l_{ij} = (l_i \cap l_j)$, each belonging to distinct alcoves, formed by the lines in general position.

Then we define a harmonic variation of C to be a family of curves $\{C_t\}_{t \in P^1}$, with the properties;

(i). The family is a nodal degeneration, in the sense defined above.

(ii). The points of T are fixed, that is, belong to every curve C_t in the family.

The existence of such variations, follows from the observation that g conditions are imposed on the parameter space $V_{m,g}$, as defined above. One may also impose the following extra condition;

Definition 0.7. *Strongly Harmonic Variations*

A variation is strongly harmonic, if, in addition to being harmonic, satisfies the additional axiom;

(iii). The tangent lines l_x to C, for $x \in T$, are fixed, that is are common to every curve C_t.

This condition imposes $2g$ conditions on the parameter space $V_{m,g}$. It follows that the existence of such variations, can only be guaranteed for curves of low degree m.

The first advantage of imposing condition (ii), in Definition 0.6, is the possibility of forcing a limit curve C_∞ in the variation, to develop a node at one of the given points in T. This may be seen by considering the variation in $V_{\kappa,g}$, where $\kappa = class(C)$, obtained by applying the birational duality map $C_t \leftrightarrow C_t^*$. At the corresponding points x_t^* of C_t^*, the tangent lines are fixed, hence, the possibility of obtaining a bitangent in C_∞^* is guaranteed, by an explicit computation of resultant polynomials for the pair $F^*(x, y; t)$ and $\frac{\partial F^*}{\partial x}(x, y; t)$. Using the degree-genus formula, quoted above, a sequence of $g + 1$ such variations,([8]), will again force the original curve C to split into components C_1 and C_2, in order to avoid the paradox that $g(C_\infty) < 0$. The second advantage of imposing the condition (ii), is the possibility of understanding the intersection geometry of the component curves C_1 and C_2, through a comparison of the fundamental groups $\pi_1(C, z)$ and $\pi_1(C_1 \cup C_2, z)$, where z is a new fixed point of the variation. Some interesting results in this direction are obtained in the paper [4]. However, the condition (ii) simplifies some of the technical complexities of the specialisation theorem for fundamental groups, by an explicit presentation of $\pi_1(\overline{C}, \overline{z})$, where $\overline{C}$ is the family of curves C_t, and $\overline{z}$ is the corresponding "elongated" point. This provides an alternative approach to Severi's conjecture, although admittedly a circular one. It is hoped that new results in this area will, perhaps, rephrase Definition 0.6(b).

References

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[8] Again, similar technical or projective arguments are required.

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