A THEORY OF HARMONIC VARIATIONS

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Abstract. We consider a class of "harmonic variations" for nonsingular curves, obtained as asymptotic degenerations along bitangents. On a geometric level, we obtain an attractive relationship between the class and the genus of C. The distribution of class points in pairs across nonsingular curves with such variations, further suggests applications to understanding covalent bonding in terms of shared electrons.

1. Alcoves and Class Formulas

Let n be an odd number, and C a circle of radius 1, centred about the origin $(0,0)$, of a real coordinate system (x,y). Suppose that a regular n-sided polygon is inscribed inside the circle, with vertices $\{p_0,\ldots,p_j,\ldots,p_{n-1}\}$, with coordinates $e^{\frac{2\pi ij}{n}}$ and $\{l_0,\ldots,l_j,\ldots,l_{n-1}\}$ are the lines formed by the edges of the polygon, so that l_j passes through the pair of vertices $\{p_j,p_{j+1}\}$, mod(n). By construction, the n intersections $(l_j\cap l_{j+1})$, for $0\leq j\leq n-1$, mod(n), lie on the unit circle. We claim, more generally, that;

Lemma 1.1. *If $1\leq k\leq\frac{n-1}{2}$, the n intersections $(l_j\cap l_{j+k})$, mod(n), lie on a circle, centred about $(0,0)$, of radius $\frac{sin(\frac{\pi}{2}(1-\frac{2}{n}))}{sin(\frac{\pi}{2}(1-\frac{2k}{n}))}$, with equal angles subtended by consecutive pairs to the origin $(0,0)$.*

Proof. For convenience of notation, let O denote the origin $(0,0)$, and let C denote the intersection $(l_j\cap l_{j+k})$. Let $\{\alpha,\beta,\gamma\}$ denote the angles $\{Op_jC,p_jOC,p_jCO\}$ of the triangle with vertices $\{p_j,O,C\}$, let δ denote the angle p_jOp_{j+1} of the triangle with vertices $\{p_j,O,p_{j+1}\}$, let ϵ denote the angle between the lines l_j and l_{j+1} and r the length of the edge OC. We have that;

$\delta=\epsilon=\frac{2\pi}{n}$

Thanks to Julius Plucker.

$\alpha = \frac{\pi-\epsilon}{2} = \frac{\pi}{2}(1-\frac{2}{n})$

$\beta = \frac{\delta(k+1)}{2} = \frac{\pi(k+1)}{n}$

$\gamma = \pi - (\alpha+\beta) = \frac{\pi}{2}(1-\frac{2k}{n})$

By the sine rule, applied to the triangle p_jOC, we have that;

$r = \frac{sin(\alpha)}{sin(\gamma)} = \frac{sin(\frac{\pi}{2}(1-\frac{2}{n}))}{sin(\frac{\pi}{2}(1-\frac{2k}{n}))}$

as required. The last claim follows easily from calculating the angle $C_1OC_2 = \frac{2\pi}{n}$, for two consecutive intersections in the set $l_j \cap l_{j+k}$.

□

Remarks 1.2. *It follows that all of the C_2^n intersections between the n lines $\{l_0, \ldots, l_j, \ldots, l_{n-1}\}$ lie on concentric circles about the origin O. The pattern of lines and intersections forms an attractive radiating pattern, harmoniously arranged in the plane.*

If n is an even number, we can perform the same construction, but obtain a slightly modified version of the previous lemma;

Lemma 1.3. *Let hypotheses and notation be as above. If $1 \leq k \leq \frac{n-2}{2}$, the n intersections $(l_j \cap l_{j+k})$ are arranged as in Lemma 1.1. If $k = \frac{n}{2}$, there exist $\frac{n}{2}$ intersections in the set $l_j \cap l_{j+k}$, situated on the circle at ∞, in the real projective plane $\mathcal{R}P^2$.*

Proof. It is sufficient to observe that, when $k = \frac{n}{2}$, the lines l_j and l_{j+k}, mod(n), are parallel, in the plane with affine coordinates (x, y). Embedding the real affine plane in the projective plane $\mathcal{R}P^2$, we obtain $\frac{n}{2}$ intersections between the $\frac{n}{2}$ pairs of parallel lines in the set $\{l_0, \ldots, l_j, \ldots, l_{n-1}\}$, as the pair gradients are distinct.

□

Definition 1.4. *We say that n lines in the real projective plane $\mathcal{R}P^2$ are in general position, if no three of the lines intersect in a point. We say that the lines are in bounded position, if all of the intersections lie in the affine plane $\mathcal{R}^2$. To n lines $\{l_0, \ldots, l_{n-1}\}$ in bounded general position, we can associate a graph G, whose* vertices *consist of the C_2^n intersections of the lines in the affine plane. We say that two vertices $\{v_1, v_2\} \subset G$ are connected by an* edge *if;*

(i). There exists a line in the set $\{l_0, \ldots, l_{n-1}\}$ containing v_1 and v_2.

(ii). There does not exist a third vertex v_3, lying between v_1 and v_2, on the same line.

We define an edge to be the closed line segment, connecting two such vertices. We define an alcove *of the graph G by the following properties;*

(i). A compact convex connected subset V of $\mathcal{R}^2$.

(ii). The boundary δV is a union of edges, belonging to distinct lines.

(iii). V does not contain a proper subset W, satisfying properties (i) and (ii).

We now show that;

Lemma 1.5. *If $\{v_0, \ldots, v_j, \ldots, v_{n-1}\}$ are vertices, connected by edges $\{e_0, \ldots, e_j, \ldots, e_{n-1}\}$, lying on distinct lines, forming a convex n-polygon V, then V is an alcove.*

Proof. Suppose not, then clearly condition (iii) fails. We can, therefore, find a proper subset $W \subset V$, satisfying conditions (i) and (ii). Let e be one of the edges of W, belonging to a line l. Suppose the edges of V belong to lines $\{l_0, \ldots, l_j, \ldots, l_{n-1}\}$ respectively. If, l does not coincide with one of these lines, then, by the definition of general position, it must intersect one of them in a vertex, distinct from $\{v_1, \ldots, v_j, \ldots, v_{n-1}\}$, on δV. This contradicts the fact that $\{e_0, \ldots, e_j, \ldots, e_{n-1}\}$ are edges. It follows, that l must coincide with one of the lines $\{l_0, \ldots, l_j, \ldots, l_{n-1}\}$, say l_0. If e does not coincide with the edge e_0, then l_0 must contain a point in the interior $(V \setminus \delta V)$ of V. It is straightforward to show that this contradicts the assumption that V is convex, ([1]). It follows that the boundary $\delta W \subset \delta V$. As δW is connected, $\delta W = \delta V$, hence, W must coincide with V, showing the result. □

We make the following definition;

Definition 1.6. *If $\{v_0, \ldots, v_j, \ldots, v_{n-1}\}$ are vertices, lying on distinct lines, forming a convex n-polygon V, we define the vertex number of*

[1] Let x be the interior point, and let $\{v_0, v_1, v_2\}$ be the vertices, connecting the edges e_0 and e_1. By convexity, the triangles with vertices $\{x, v_1, v_2\}$ and $\{v_0, v_1, v_2\}$ both lie inside V. This implies that the interior of the edge e_1 is interior to V, contradicting the fact that e_1 is contained in δV.

V, to be the number of vertices of the lines in bounded general position, either interior to V or interior to the line segments forming the boundary δV of V.

Remarks 1.7. *By the previous lemma, a convex n-polygon, with vertex number 0, is an alcove.*

We then have that;

Lemma 1.8. *If an edge e forms one side of a convex n-polygon V, which is not an alcove, having vertex number $m > 0$, then e forms one side of a convex, at most $n+1$-sided polygon W, having vertex number $0 \leq r < m$.*

Proof. Let $\{l_0, \ldots, l_j, \ldots, l_{n-1}\}$ enumerate the line segments, forming the boundary of V, with e corresponding to l_0. As V is not an alcove, there exists a vertex v, interior to one of these lines, on the boundary δV of V. As e is an edge, it cannot be interior to l_0, say v is interior to l_1. As V is convex, there exists a new line l_n, passing through v. As the lines are in general position, and e is an edge, the line l_n must intersect the interior of one of the lines $\{l_2, \ldots, l_j, \ldots, l_{n-1}\}$, say l_j. It is easily checked that the polygon W, formed by the sides $\{e, l_1, l_n, l_j, \ldots, l_{n-1}\}$ is convex, with $(n-j+3)$ sides, having vertex number $r < m$. □

As a straightforward consequence, we have that;

Lemma 1.9. *Every edge e lies on the boundary of at least one alcove.*

Proof. Suppose that the edge e has vertices v_0 and v_1, belonging to a line l_0, let l_1 and l_2 be further lines containing these vertices, respectively, intersecting in a vertex v_2. If the triangle with vertices $\{v_0, v_1, v_2\}$ is an alcove, the result is shown. Otherwise, it satisfies the hypotheses of the previous lemma; one may then apply the result inductively, together with the fact that the number of vertices are finite, and the previous remark, to obtain the same result.

□

The following results use a different argument;

Lemma 1.10. *If $\{V_1, V_2\}$ are two distinct alcoves, then $(V_1 \cap V_2) \subset (\delta V_1 \cup \delta V_2)$ and, consists of an edge or a vertex.*

Proof. We clearly have that $(V_1 \cap V_2)$ satisfies condition (i) in the definition of an alcove, and the boundary $\delta(V_1 \cap V_2)$ is contained in $(\delta V_1 \cup \delta V_2)$.

Suppose that $(V_1 \cap V_2)$ contains an open subset of $\mathcal{R}^2$, $(*)$ then the boundary $\delta(V_1 \cap V_2)$ is connected, therefore, must consist either of a vertex, or a union of line segments. That the line segments form edges, follows from the fact that the lines forming the boundary δV_1 of V_1, intersect the lines forming the boundary δV_2 of V_2, in vertices. It follows that $(V_1 \cap V_2)$ also satisfies condition (ii) in the definition of an alcove. As the intersection is a proper subset of both V_1 and V_2, this contradicts the assumption that V_1 and V_2 are both alcoves. It follows that $(*)$ fails, that is $(V_1 \cap V_2) \subset (\delta V_1 \cup \delta V_2)$. As $(V_1 \cap V_2)$ is connected, being convex, the intersection consists of a vertex or an edge, as required. □

Lemma 1.11. *Every edge e lies on the boundary of at most two alcoves.*

Proof. Suppose that e lies on the boundary of three distinct alcoves $\{V_1, V_2, V_3\}$. It is easily checked, that, if x is an interior point of the edge e, then x is an interior point of the union of alcoves $(V_1 \cup V_2)$. It follows that the intersection of V_3 with either V_1 or V_2, must contain an open subset of $\mathcal{R}^2$. This contradicts the previous result. □

Lemma 1.12. *If $n \geq 3$, there exist $\frac{(n-1)(n-2)}{2}$ alcoves, associated to the graph of Definition 1.4.*

Proof. When $n = 3$, it is easily checked that there is 1 alcove, formed by the vertices $\{v_1, v_2, v_3\}$ of a triangle, obtained from the intersection of three lines $\{l_1, l_2, l_3\}$ in bounded general position. We assume, inductively, that the result is true for n lines in bounded general position. Let l_{n+1} be a new line, added to n lines $\{l_1, \ldots, l_j, \ldots, l_n\}$ in bounded general position. This introduces n new vertices $\{v_1, \ldots, v_n\}$ and $(n-1)$ new edges, corresponding to line segments e_j between the vertices v_j and v_{j+1}. We claim that an edge e_j is on the boundary of two alcoves in the graph G_{n+1} of $(n+1)$ lines, $(*)$, if and only if it passes through the interior of an alcove in the graph G_n of n lines, $(**)$. For assume that $(*)$ holds, and e_j lies on the boundary of two alcoves V_1 and V_2. By Lemma 0.10, $e_j = (V_1 \cap V_2)$. Let $\{e_j, f_1, \ldots, f_r\}$ and $\{e_j, g_1, \ldots, g_s\}$ enumerate the consecutive edges of the alcoves V_1 and V_2 respectively. By the definition of lines in general position, the edges $\{f_r, g_1\}$ and $\{f_1, g_s\}$ belong to the same lines l_1 and l_2 respectively, in particular, either the polygon defining the boundary δV_1 or the polygon defining the boundary δV_2 is inscribed within the triangles, having either edges $\{e_j, f_1, f_r\}$ or $\{e_j, g_1, g_s\}$. In either case, it follows that the union of alcoves $(V_1 \cup V_2)$ is convex. We now relabel the boundary $\delta(V_1 \cup V_2)$, after removing the edge e_j, consecutively, as $\{h_1, f_2, \ldots, f_{r-1}, h_2, g_2, \ldots, g_{s-1}\}$, where h_1 are h_2 are the new edges in the graph G_n, obtained by joining the edges $\{f_r, g_1\}$ and $\{f_1, g_s\}$ in the graph G_{n+1}. As all the faces are edges, belonging to distinct lines, from the above, it follows that $(V_1 \cup V_2)$ is an alcove, by Lemma 0.5, hence $(**)$ follows. Conversely, assume that $(**)$ holds, and e_j passes through the interior of an alcove V in G_n. By the definition of an alcove, the boundary δV consists of a union of edges $\{k_1, \ldots, k_t\}$, arranged in a convex polygon, belonging to distinct lines. By the definition of lines in general position, the edge e_j passes through the interior of two of these edges, say k_1 and k_l, where $1 < l \leq t$, and forms two new pairs of edges $\{k_{11}, k_{12}\}$ and $\{k_{l1}, k_{l2}\}$ in the graph G_{n+1}. We let V_1 be the region, bounded consecutively by the edges $\{k_{11}, \ldots, k_{l-1}, k_{l1}, e_j\}$, and V_2 the region, bounded consecutively by the edges $\{k_{l2}, k_{l+1}, \ldots, k_t, k_{12}, e_j\}$. One of the regions is bounded, by the triangle with edges $\{e_j, l_1, l_l\}$, where l_1 and l_l are the lines containing the edges k_1 and k_l respectively. It follows, that both regions are convex, and, therefore, alcoves, by Lemma 0.5. Hence, $(*)$ is shown. Now, if l_{n+1} is a new line, each of the $(n-1)$ new edges, either passes through the interior of an alcove in G_n, in which case, by the above $(*)(**)$, and Lemma 0.11, one extra alcove is introduced into the graph G_{n+1}, or, does not pass, through an interior, in which case, by the above $(*)(**)$, and Lemma 0.9, an extra alcove is also introduced into the graph G_{n+1}. In total, $(n-1)$ new alcoves are introduced, which implies that the total number of alcoves in G_{n+1} is;

$$\frac{(n-1)(n-2)}{2} + (n-1) = \frac{n(n-1)}{2} = \frac{([n+1]-1)([n+1]-2)}{2}$$

This implies the result, by induction.

□

Remarks 1.13. *We return to the notation of Lemma 1.1, and the following remark. For n odd, we can apply the previous lemma, to obtain that there exist C_2^{n-1} alcoves associated to a regular bounded arrangement of lines. One may also extend the above definition of an alcove to regions in the real projective plane $\mathcal{R}P^2$, by, for example, assuming that all the intersections are in finite position. This may always be achieved by an appropriate choice of the line at ∞, so as not to include any of the vertices. With the convention that any two such regions intersecting in a vertex, on the line at ∞, are counted as a single alcove, the reader is invited to check that there are again C_2^{n-1} alcoves, associated to a set of lines in general position. The reason for this*

We can give a convenient description of the alcoves associated to a regular bounded line arrangement;

Lemma 1.14. *In the situation of Lemma 1.1, and Lemma 1.3, the alcoves are defined by;*

(i). For $n \geq 3$, the central alcove, with boundary defined by the n-polygon, inscribed in the unit circle.

(ii). For $n \geq 5$, n peripheral alcoves of the first kind, inscribed between the unit and first concentric circle, with boundaries defined by the triangles, formed by the lines $\{l_i, l_{i+1}, l_{i+2}\}$, mod (n).

(iii). For $n \geq 7$, n peripheral alcoves of the second kind, inscribed between the $(j-1, j, j+1)$ concentric circles, with boundaries defined by the quadrilaterals, formed by the lines

$\{l_i, l_{i+1}, l_{i+2j}, l_{i+2j+1}\}$, *mod* (n), $1 \leq j \leq (\frac{n-5}{2})$

Proof. The proof is left to the reader, one should observe that the total number of alcoves is correct, as;

$$1 + n + n.\frac{(n-5)}{2} = 1 + n.\frac{(n-3)}{2} = \frac{(n-1)(n-2)}{2}$$

□

Remarks 1.15. *Observe that, for a nonsingular plane projective curve $C \subset P^2(\mathcal{C})$ of degree n, if m is the class of C, then;*

$$m = n(n-1) = 2n + 2n + 2(\tfrac{n(n-5)}{2}) \ (*)$$

Under certain further constraints on C, we can construct a 1-parameter family $\{C_t : t \in Par_t\}$, with $C_0 = C$, and C_∞ consisting of n lines $\{l_1, \ldots, l_n\}$ in general position, with intersections described by the configurations in Lemmas 0.1 and Lemma 0.3, such that for each of the intersections $l_i \cap l_j$, $1 \leq i < j \leq n$, there exist exactly 2 *vertical tangents specialising to $l_i \cap l_j$, $(**)$. Using $(*)$ and Lemma 0.14, this suggests that the class points are uniformly distributed in three parts, across the periphery of the central alcove, the n peripheral alcoves of the first kind, and the $\frac{n(n-5)}{2}$ peripheral alcoves of the second kind. The proof of $(**)$ will be the subject of the next section.*

2. Harmonic Variations

Remarks 2.1. *We observe some consequences of the degree-genus formula, Theorem 3.36 of* [3]*, assuming Severi's conjecture, (*[2]*, see* [4]*), that, for any plane projective algebraic curve C, of degree n, having at most nodes as singularities, there exists an asymptotic family, see* [4]*, $\{C_t : t \in P^1\}$, with the property that $C_0 = C$ and C_∞ is a union of n lines in general position.*

Definition 2.2. *Let $\{l_1, \ldots, l_n\} \subset P^2(\mathcal{R})$ be a sequence of n projective lines, with coordinates (x, y). We say that $\{l_1, \ldots, l_n\}$ forms a harmonic arrangement if they satisfy the conditions of Lemma 1.1, in the case that n is odd, and, if, the intersections are in finite position, and satisfy the conditions of Lemma 1.1, after a linear change of variables.*

Definition 2.3. *Let $\{l_1, \ldots, l_n\} \subset P^2(\mathcal{C})$ be a sequence of n projective lines, defined over $\mathcal{R}$ and let $i : P^2(\mathcal{R}) \rightarrow P^2(\mathcal{C})$ be the canonical inclusion. We say that $\{l_1, \ldots, l_n\}$, forms a harmonic arrangement, if the pullbacks $\{i^*(l_1), \ldots, i^*(l_n)\}$ form a harmonic arrangement in the sense of Definition 2.2.*

Definition 2.4. *Let C be a nonsingular plane projective curve of degree n. We say that C is harmonic if there exist n lines, $\{l_1, \ldots, l_n\}$, which are bitangent to C, (*[3]*), which form a harmonic arrangement, in the sense of Definition 2.3, and such that, there exists lines l_a and l_b, with $(l_a \cap C) = \{p_{1,j} : 1 \leq j \leq n\}$, and $(l_b \cap C) = \{p_{2,j} : 1 \leq j \leq n\}$.*

Definition 2.5. *Let C be a harmonic curve of degree n, in the sense of Definition . Let $\{C_t : t \in Par_t\}$ be a 1-dimensional family of nonsingular plane projective curves, (*[4]*). We say that the family is a harmonic variation, if, there exist $\{0, \infty\} \subset Par_t$, $C_0 = C$, C_∞ is a union of lines $\{l_1, \ldots, l_n\}$, forming a harmonic arrangement, and $Par_t \subset W^{4n}$, where, W^{4n} is defined as;*

$$\{C_{\bar{a}} : \bigwedge_{1 \leq j \leq n} C_{\bar{a}}(p_{1,j}), C_{\bar{a}}(p_{2,j}), I_{p_{1,j}}(C_{\bar{a}}, l_j) = 2, I_{p_{2,j}}(C_{\bar{a}}, l_j) = 2\}$$

Remarks 2.6. *For a given harmonic variation, we can choose a coordinate system (x', y') such that the lines $\{l_a, l_b\}$ correspond to $\{x = 0, x = 1\}$, the intersection $(l_a \cap l_b) = [0 : 1 : 0]$, and the intersections*

[2] With the extra condition that the degeneration is asymptotic

[3] In the sense that there exist exactly 2 points $\{p_{1,j}, p_{2,j}\}$, on each l_j, such that $I_{p_{i,j}}(C, l_j) = 2$, for $1 \leq i \leq 2$, and no further points of higher multiplicity

[4] In the sense that $Par_t \subset P^{\frac{(n+1)(n+2)}{2}}$ is a 1-dimensional irreducible algebraic variety, containing the nonsingular curve C

$l_i \cap l_j$ are in finite position, for $1 \leq i < j \leq n$. We can keep track of the original configuration of lines, in (x, y), from Definitions 2.3, 2.5, through a linear isomorphism $L : P^1(\mathcal{C}) \rightarrow P^1(\mathcal{C})$.

Lemma 2.7. *For any given plane nonsingular curve C of degree n, there exists a finite sequence $\{C_i : 0 \leq i \leq r\}$ of nonsingular plane curves of degree n, linear systems $\{L_{i,i+1} \subset P^{\frac{(n+1)(n+2)}{2}} : 0 \leq i \leq r-1\}$, and parameters $\{a_0, a_h\} \cup \{a_{1,i}, a_{2,i} : 1 \leq i \leq r-1\}$, with $C_i \sim a_{1,i}$, in $L_{i-1,i}$, and $C_i \sim a_{2,i}$, in $L_{i,i+1}$, $C \sim a_0$ in $L_{0,1}$, $C \sim a_h$ in $L_{r-1,r}$, such that C_r is harmonic.*

Proof. For a 1-dimensional (generically nonsingular) family of curves, let $V_k \subset Par_t \times P^2$ be defined by;

$$V_k = \overline{\{(t, l) : \exists_{\geq k, x} x \in (l \cap C_t) \wedge I_x(l, C_t) \geq 2\}}$$

We have that for $k \geq 2$, $V_{k+1} \subseteq V_k$, and, by $(***)$ in footnote 5, each V_k is a finite cover of Par_t, of degree at most $\frac{n^3(n-2)}{2}$, ([5]).

[5] Let C^* denote the dual of $C = C_0$, then $deg(C^*) = cl(C) = n(n-1)$, $cl(C^*) = deg(C) = n$, using Lemma 5.12 of [3] and (†) above. Using Theorem 4.3 of [3], if D is a plane curve of degree n, having at most nodes as singularities;

$$n + m + 2d = n^2$$

In general, an easy adaptation of Theorem 4.3 shows that, if $Sing(D)$ denotes the set of singular points, and, for $j \leq 1$, $S_j(D) \subset Sing(D)$, denotes the set of singular, which are the origins of j branches,then;

$$m + n + \sum_{p \in Sing(D), \gamma_p} \alpha(\gamma_p) = n^2$$

$$m + n + \sum_{j \geq 1} \sum_{p \in S_j(D), \gamma_p} = n^2$$

Applying this result to the dual curve C^*, we obtain;

$$\sum_{p \in Sing(C^*), \gamma_p} \alpha(\gamma_p) = n(n-1)^2 - (n + n(n-1)) = n^3(n-2)$$

In particularly, if d_j denotes the number of j-tangents to C, (by a j-tangent, we mean a line l, for which there exist exactly j points $\{x_1, \ldots, x_j\} \subset (C \cap l)$, with $I_{x_j}(C, l) \geq 2$) then;

$$2(\sum_{j \geq 2} d_j) \leq \sum_{j \geq 2} j d_j \leq \sum_{j \geq 2} \sum_{p \in S_j(C^*), \gamma_p} \alpha(\gamma_p) \leq n^3(n-2)$$

$$\sum_{j \geq 2} d_j \leq \frac{n^3(n-2)}{2} \quad (***)$$

In the generic case, that C is nonsingular and has no tritangents, we obtain;

Now, given C, a nonsingular curve of degree n, we use the following method to reduce the k-tangents, for $k\geq 3$, to bitangents. $(***\,*)$, Enumerate the k-tangent lines, for $k\geq 3$, as $\{l_{k,1},\ldots,l_{k,s(k)}\}$, and the bitangents as $\{l'_1,\ldots,l'_r\}$. Pick 3 points $\{p_1,p_2,p_3\}$ on $l_{3,1}$, centred at $\{(a,b),(a',b'),(a'',b'')\}$, with tangent lines $l_{p_1}=l_{p_2}=l_{p_3}=l_1$, defined by $cx+dy-(ca+db)=0$. Let $W^4,W^6\subset P^{\frac{(n+1)(n+2)}{2}}$ be the hyperplanes, defined by;

$$W^4=\{\overline{a}:g(\overline{a},a,b)=g(\overline{a},a',b')=0,d\frac{\partial g}{\partial x}|_{(\overline{a},a,b)}-c\frac{\partial g}{\partial y}|_{(\overline{a},a,b)}$$

$$=0,d\frac{\partial g}{\partial x}|_{(\overline{a},a',b')}-c\frac{\partial g}{\partial y}|_{(\overline{a},a',b')}=0\}$$

$$W^6=\{\overline{a}:g(\overline{a},a,b)=g(\overline{a},a',b')=g(\overline{a},a'',b'')=0,d\frac{\partial g}{\partial x}|_{(\overline{a},a,b)}-c\frac{\partial g}{\partial y}|_{(\overline{a},a,b)}$$

$$=0,d\frac{\partial g}{\partial x}|_{(\overline{a},a',b')}-c\frac{\partial g}{\partial y}|_{(\overline{a},a',b')}=0,d\frac{\partial g}{\partial x}|_{(\overline{a},a'',b'')}-c\frac{\partial g}{\partial y}|_{(\overline{a},a'',b'')}=0\}$$

where $g(\overline{a},x,y)=\sum_{i+j\leq n}a_{ij}x^iy^j$. We have that $codim(W^4)=4$, $codim(W^6)=6$, $W^6\subset W^4$, and $\overline{a_0}\in W^6$, where $C=C_{\overline{a_0}}$. Choose a line $l''\subset W^4$, with $l''\cap W^6=\overline{a_0}$. Then l'' defines a 1-parameter family of (generically nonsingular) curves $\{C_t:t\in l''\}$, of degree n.

Let m denote the degree of the cover V_2/l'', and let $U\subset l''$ have the property that $Card(V_2(t))=m$, for $t\in U$. Let $D\subset U\times P^1\times P^2$ denote the family of curves defined by $D_{(t,s)}=\prod_{(l,t)\in V^2(t)}(y+l_1x+(l_2+s)$, with corresponding closure $\overline{D}\subset l''\times P^1\times P^2$. Let $W\subset l''\times P^1\times P^2$ be defined by;

$$W(t,s,p)\equiv p\in C_t\cap\overline{D}(t,s)$$

Then, using Bezout's theorem, W defines a finite cover of $l''\times P^1$ of degree mn.

$d_2=\frac{n^3(n-2)}{2}-\sum_{p\in S_1(C^*)}\alpha(\gamma_p)$

$=\frac{n^3(n-2)}{2}-i$

where i is the number of inflexions of C (including character). As $i=3(m-n)=3n(n-2)$, we obtain $d_2=\frac{n^3(n-2)}{2}-3n(n-2)=\frac{n(n-2)(n^2-6)}{2}$ bitangents, and $d_2\geq n$, if $n\geq 2$. For nonsingular quartic curves, we obtain 40 bitangents, which gives an interesting comparison to Plucker's result of 28 bitangents for real algebraic curves.

Using factoring multiplicity, see Lemma 2.6 of [7], for $(t_0, 0, p_0) \in W$, we have that;

$Mult_{(t_0,0,p_0)}(W/(l'' \times P^1))$

$= \sum_{s \in \mathcal{V}_0 \ generic, p_{0,i} \in (W(t_0,s) \cap \mathcal{V}_{p_0})} Mult_{(t_0,s,p_{0,i})}(W(s)/l'')$ (†)

$= \sum_{t \in \mathcal{V}_{t_0} \ generic, q_{0,j} \in (W(t,0) \cap \mathcal{V}_{p_0})} Mult_{(t,0,q_{0,j})}(W(t)/P^1)$ (††)

If $p_0 \in l_0$, we have that (†) $= I_{p_0}(C_{t_0}, l_0) Mult_{(t_0,l_0)}(V_2/l'')$

and (††) $= Mult_{(t_0,l_0)}(V_2/l'')(\sum_{q_{0,j} \in W(t,0)} I_{(q_{0,j},l_{0,j})}(C_t, l_{0,j}))$

where $q_{0,j} \in l_{0,j}$. Hence;

$I_{p_0}(C_{t_0}, l_0) = \sum_{q_{0,j} \in W(t,0)} I_{(q_{0,j},l_{0,j})}(C_t, l_{0,j})$ (††††)

It follows that, for $\overline{a}_0 \in l''$, we have that the total multiplicity;

$K = \sum_{l \in V_2(\overline{a}_0), p \in (l \cap C_{\overline{a}_0})} I_p(C_{\overline{a}_0}, l) = \sum_{l \in V_2(\overline{a}), p \in (l \cap C_{\overline{a}})} I_p(C_{\overline{a}}, l)$

for generic $\overline{a} \in (\mathcal{V}_{\overline{a}_0} \cap l'')$, hence, for all $\overline{a} \in \{(\mathcal{V}_{\overline{a}_0} \cap l'') \setminus \overline{a}_0\}$.

Removing the points of contact 1, we obtain, for $l \in V_2(\overline{a}_0)$, $p \in (l \cap C_{\overline{a}_0})$, with $I_p(C_{\overline{a}_0}, l) \geq 2$, that;

$I_p(C_{\overline{a}_0}, l) = \sum_{q \in C_{\overline{a}} \cap l' \cap \mathcal{V}_p, l' \in V_2(\overline{a}) \cap \mathcal{V}_l} I_q(C_{\overline{a}}, l')$

$\geq \sum_{q \in C_{\overline{a}} \cap l' \cap \mathcal{V}_p, l' \in V_2(\overline{a}) \cap \mathcal{V}_l, I_q(C_{\overline{a}}, l') \geq 2} I_q(C_{\overline{a}}, l')$

$L = \sum_{l \in V_2(\overline{a}_0), p \in (l \cap C_{\overline{a}_0}), I_p(C_{\overline{a}_0}, l) \geq 2} I_p(C_{\overline{a}_0}, l)$

$\geq \sum_{l \in V_2(\overline{a}), p \in (l \cap C_{\overline{a}}), I_p(C_{\overline{a}_0}, l) \geq 2} I_p(C_{\overline{a}}, l)$ (†††)

([6]).

By footnote 6, we can assume that the fibre $V_2(\overline{a_0})$ is unramified in the sense of Zariski structures, $(****)$. As $V_{k+1}\subset V_k$ is relatively closed, for $k\geq 2$, we have that, for $\overline{a_0}'\in(\mathcal{V}_{\overline{a_0}}\cap l'')$, $V_2(\overline{a_0}')\cap\mathcal{V}_{l'_j}\subset V_2$, for $1\leq j\leq r$, $V_2(\overline{a_0}')\cap\mathcal{V}_{l_{k,j(k)}}\subset(V_{k+1})^c$, for $1\leq j(k)\leq s(k)$, $k\geq 4$, and $2\leq j(k)\leq s(k)$, $k=3$, $V_2(\overline{a_0}')\cap(\mathcal{V}_{l_{3,1}})\subset V_3^c$, using the fact that $((l''\setminus\{\overline{a_0}\})\cap W^6)=\emptyset$, which gives that $\overline{a_0}$ is not a base point of the g_n^1 defined by the l'', (intersecting with $l_{3,1}$), and Lemma 2.10 of [5]. It follows again that the statement $\forall t'\in(\mathcal{V}_{\overline{a}_0}\setminus\{\overline{a_0}\}\cap l'')P(t')$ holds, where;

$$P(t')\equiv\exists_{(\geq r+1,l'_j)}\exists_{(\geq s(3)-1,l_{3,j'},l_{3,j'}\neq l'_j)}\ldots\exists_{(\geq s(k),l_{k,j_k},l_{k,j_k}\neq l'_j\cup_{3\leq s\leq k-1}l_{s,j_s})}$$

---

[6] It follows that, if $V(\overline{a}_0,l)$ holds, and $p\in(C_{\overline{a}_0}\cap l)$, with $I_p(C_{\overline{a}_0},l)=w\geq 2$, and $Mult_{(\overline{a}_0,l)}(V_2/l'')=b$, then, if $\epsilon>0$ is standard, with $B(p,\epsilon)^c\cap(C_{\overline{a}_0}\cap l)=B(l,\epsilon)^c\cap(V_2(\overline{a}_0))=\emptyset$, the statement $\forall t'\in(\mathcal{V}_{\overline{a}_0}\setminus\{\overline{a_0}\}\cap l'')Q(t')$ holds, where;

$Q(t')\equiv\exists_{\bigwedge_{1\leq j\leq b}l_j}\exists_{\bigwedge_{1\leq j\leq b,1\leq a(j)\leq t(j)}p_{j,a(j)}}[l_j\in B(l,\epsilon)\wedge(t',l_j)\in V_2(t')\wedge(p_{j,a_j}\in C_{t'}\cap l_j\cap B(p,\epsilon))\wedge I_{p_{j,a_j}}(C_{t'},l_j)\geq 2\wedge\bigvee_{1\leq j\leq r,1\leq a(j)\leq t(j),\sum\theta_{(j,a_j)}\leq w}$
$I_{p_{j,a_j}}(C_{t'},l_j)=\theta_{(j,a_j)}]$

where;

$I_{p_{j,a_j}}(C_{t'},l_j)\geq 2\equiv\forall s\in B(0,\epsilon)\setminus\{0\}\exists_{w_1\neq w_2}\bigwedge_{1\leq i\leq 2}(w_i\in C_{t'}\cap(y+l_{1,j}x+(l_{2,j}+s))\cap B(p_{j,a_j},\epsilon))$

$I_{p_{j,a_j}}(C_{t'},l_j)=\theta_{(j,a_j)}\equiv\forall s\in B(0,\epsilon)\setminus\{0\}\exists^{=\theta_{j,a_j}}w_k\bigwedge_{1\leq k\leq\theta_{j,a_j}}(w_k\in C_{t'}\cap(y+l_{1,j}x+(l_{2,j}+s))\cap B(p_{j,a_j},\epsilon))$

Using Theorem 17.1 of [6], that a monad $\mu(p)$ coincides with an infinitesimal neighborhood $\mathcal{V}_p$, for $p\in P^k(\mathcal{C})$, and the fact that, for any infinite $n\in{}^*\mathcal{N}$, $B(p,\frac{1}{n})\subset{}^*U$, for any open set U in the complex topology, the property Q holds for all infinite $n\in{}^*\mathcal{N}$, with $t'\in B(\overline{a}_0,\frac{1}{n})\setminus\{\overline{a}_0\}\cap l''$. By the underflow principle and transfer, see [1], it holds in the standard model, for all $n\in\mathcal{N}$, $n\geq k$, for some $k\in\mathcal{N}$, with $t'\in B(\overline{a}_0,\frac{1}{n})\setminus\{\overline{a}_0\}\cap l''$, in particular, for $t'\in B(\overline{a}_0,\frac{1}{k})\setminus\{\overline{a}_0\}\cap l''$.

Repeating this argument, for each $p\in(C_{\overline{a}}\cap l)$, with $I_p(C_{\overline{a}},l)\geq 2$, and each bitangent line l, with corresponding $B(\overline{a}_0,\frac{1}{k_{p,l}})$, it follows that, taking the intersection $\bigcap_{l,p}B(\overline{a}_0,\frac{1}{k_{p,l}})$, the total multiplicity of the new bitangent points is lowered. We can then, wlog, move the initial curve C_0 to a point $\overline{b_0}\in l''$, for which the fibre $V_2(\overline{b_0})$ is (in the sense of Zariski structures) unramified.

$[(l'_j \in V_2(t')) \wedge l_{3,j'} \in V_4^c(t') \wedge \ldots \wedge l_{k,j_k} \in V_{k+1}^c(t')]$

Using the argument of footnote 6 again, it follows that the property P holds for $t' \in B(\overline{a}_0, \frac{1}{k}) \setminus \{\overline{a}_0\} \cap l''$, for some $k \in \mathcal{N}$. Choosing $\overline{a}_1 \in B(\overline{a}_0, \frac{1}{k}) \setminus \{\overline{a}_0\} \cap l''$, and, using the result of footnote 6, it follows that, for the new curve $C_{\overline{a}_1}$, the total weight $\sum_{k \geq 3} s(k)$ is *strictly* (compare $(\dagger\dagger\dagger)$) reduced, $(*****)$. Now repeating the argument from $(****)$, and using $(*****)$, we obtain, after a finite number c of steps, a nonsingular curve $C_{\overline{a}_c}$, with the property that it has no k-tangents for $k \geq 3$, and, exactly $\frac{n(n-2)(n^2-6)}{2}$ bitangents, $(******)$, see footnote 5.

Relabelling $C_{\overline{a}_c}$ as $C_{\overline{a}_0}$, we choose n bitangent lines $\{l_1, \ldots, l_n\}$. We now show how to obtain the condition that the lines intersect in exactly $\frac{n(n-1)}{2}$ points. $(\sharp)$, Suppose not, then, wlog, $\{l_1, l_2, l_3\}$ intersect in a point q. Let $\{p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}, p_{3,1}, p_{3,2}\}$ denote the 6 distinct tangent points on $\{l_1, l_2, l_3\}$. Let $\{W^{10}, W^{12}\} \subset P^{\frac{(n+1)(n+2)}{2}}$ be defined as above, with W^{10} defining curves of degree n, bitangent to $\{l_1, l_2\}$ at $\{p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}\}$, and W^{12} defining curves of degree n, bitangent to $\{l_1, l_2\}$ at $\{p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}\}$, and tangent to l_3 at $p_{3,1}$. Again, choose a line $l''' \subset W^{10}$, with $l''' \cap W^{12} = \overline{a}_0$. Using $(\dagger\dagger\dagger\dagger)$, the points $\{l_1, \ldots, l_n\}$ in the fibre $V_2(\overline{a}_0)$ are unramified, ([7]) Then, if $\overline{a}'_0 \in (\mathcal{V}_{\overline{a}_0} \setminus \overline{a}_0)$, the corresponding $\{l'_1, l'_2, l'_3\}$ intersect in 3 distinct new points $\{q, q_1, q_2\} \subset \mathcal{V}_q$. Again, using the argument in footnote 6, and, considering the cover T/l''', defined by $T(p,t) \equiv \exists (l_1, l_2)(l_1 \neq l_2 \wedge p \in l_1 \cap l_2 \wedge \bigwedge_{1 \leq i \leq 2} V_2(l_i, t)$, it follows that the total number of intersections between the n bitangent lines is increased, in $(\mathcal{V}_{\overline{a}_0} \setminus \overline{a}_0)$. Again, using the argument of footnote 6, this property holds on some $B(\overline{a}_0, \frac{1}{k}) \setminus \overline{a}_0$, $k \in \mathcal{N}$. Now, we can choose $\overline{a}_1 \in B(\overline{a}_0, \frac{1}{k}) \setminus \overline{a}_0$ and obtain a new curve $C_{\overline{a}_1}$, with this property. Again, using $(\dagger\dagger\dagger)$, and the argument of footnote 6, with $B(\overline{a}_0, \frac{1}{k'}) \subset B(\overline{a}_0, \frac{1}{k})$, $k' > k$, the condition $(* * * * **)$ is maintained. Repeating the argument, from $(\sharp)$, we obtain, after a finite number c' of steps, a curve $C_{\overline{a}_c}$, with n bitangent lines $\{l_1, \ldots, l_n\}$, intersecting in $\frac{n(n-1)}{2}$ points, $(\sharp\sharp)$.

Now, again relabelling $C_{\overline{a}_c}$ to $C_{\overline{a}_0}$, with bitangent points $B = \{p_{1,1}, \ldots, p_{1,n}, p_{2,1}, \ldots, p_{2,n}\}$, (wlog in finite position) we show how to preserve the condition $(\sharp\sharp)$ and find lines $\{l_a, l_b\}$, with $\{p_{1,1}, \ldots, p_{1,n}\} \subset$

[7]If one of the lines l_1 ramifies to $\{l'_1, l''_1\}$, considering the cover W, and using $(\dagger\dagger\dagger\dagger)$, we obtain 4 points $p_{1,i,j} \in l'_i \cap \mathcal{V}_{p_{1,i}}$, $1 \leq i, j \leq 2$, with $I_{p_{1,i,j}}(C'_{\overline{a}_0}, l_i) = 1$, and no further points $p \in l'_i$, with $I_p(C'_{\overline{a}_0}, l_i) \geq 2$. It follows that the lines $\{l'_1, l''_1\}$ can no longer even be tangent to the curve, let alone bitangent.

l_a and $\{p_{2,1},\ldots,p_{2,n}\}\subset l_b$, (!!!). Choose $l_a\neq l_1$, passing through $p_{1,1}$, intersecting $\{l_2,l_3,\ldots,l_n\}$ at the distinct points $\{q_2,\ldots,q_n\}$ in finite position, distinct from $\{p_{1,2},\ldots,p_{1,n},p_{2,2},\ldots,p_{2,n}\}$, any of the other transverse intersections between $C_{\overline{a}_0}$ and $\{l_2,l_3,\ldots,l_n\}$, and the intersections $\{p_{i,j}=(l_i\cap l_j):1\leq i<j\leq n\}$, ($\sharp\sharp\sharp$). We follow the argument in the following footnote 8. After $n-1$ steps, we obtain a curve $C_{\overline{a}_{n-1}}$, such that the new tangent points to $\{q_1=p_{1,1},q_2,\ldots,q_n\}$ to $\{l_1,l_2,\ldots,l_n\}$ intersect the line l_a transversely, and the bitangents $\{l_1,\ldots,l_n\}$, formed by $\{q_1,p_{2,1},\ldots,q_n,p_{2,n}\}$ are in general position, (!!). Now choose l_b, passing through $p_{2,1}$, such that the intersections with $\{l_2,l_3,\ldots,l_n\}$ at the distinct points $\{r_2,\ldots,r_n\}$ are in finite position, distinct from $\{q_1,\ldots,q_n,p_{2,2},\ldots,p_{2,n}\}$, and any of the other transverse intersections between $C_{\overline{a}_0}$ and $\{l_2,l_3,\ldots,l_n\}$, and the intersections $\{p_{i,j}=l_i\cap l_j:1\leq i<j\leq n\}$. Again, using the argument in footnote 8, and, repeating the $(n-1)$ steps from (!!), replacing $\{q_2,\ldots,q_n\}$ with $\{r_2,\ldots,r_n\}$, ($r_1=p_{2,1}$, we obtain a curve $C_{\overline{a}_{2(n-1)}}$, with the required property (!!!), that the bitangent lines $\{l_1,l_3,\ldots,l_n\}$ are in general position, and the tangent points $\{q_1,\ldots,q_n,r_1,\ldots,r_n\}$ lie on the lines l_a and l_b respectively, ([8]).

[8] Moving tangents on fixed bitangent lines; for a given bitangent line l_j, with tangents a,b, and target c, move b to c, keeping a fixed, while preserving bitangent conditions on the other lines $\{l_i:i\neq j\}$, with bitangents $\{p_{k,i}:1\leq k\leq 2,i\neq j\}$. Let the object curve C, of degree n, be denoted by $C_{\overline{c}}$, for $\overline{c}\in P^{\frac{(n+1)(n+2)}{2}}$.

Consider the irreducible dual curve $(C_{\overline{c}})^*$, with nodes $\{\nu_j:1\leq j\leq t\}$, $t=\frac{n(n-2)}{2}$, and cusps $\{\kappa_j:1\leq j\leq s\}$, corresponding to the bitangents (the first n nodes corresponding to the bitangent array considered above) and inflexions of $C_{\overline{c}}$, see Lemma 2.8 and Theorem 5.1 of [3]. Assuming that the cusps are ordinary, that is of character $(2,1)$, we have that $n=deg(C_{\overline{c}})=cl((C_{\overline{c}})^*)=\frac{3deg((C_{\overline{c}})^*)-s}{3}=n(n-1)-\frac{s}{3}$, using Theorem 6.4 of [3], (in particular $s=3n(n-2)$). We consider the Severi variety, $V_d^{t,s}=V_{n(n-1)}^{\frac{n(n-2)}{2},3n(n-2)}$, consisting of curves of degree $d=n(n-1)$, with $t=\frac{n(n-2)}{2}$ nodes and $s=3n(n-2)$ cusps. Using deformation theory, developed in [9], $dim(V_d^{t,s})=3d+g-1-s$, where $g=g(C^*_{\overline{c}})=g(C_{\overline{c}})=\frac{(n-1)(n-2)}{2}$, so $dim(V_d^{t,s})=3n(n-1)+\frac{(n-1)(n-2)}{2}-1-3n(n-2)=\frac{n(n+3)}{2}$. We let $B_{n,d}\subset P^{\frac{(d+1)(d+2)}{2}}$ be the linear space of codimension n, consisting of curves of degree $d=n(n-1)$, passing through $\{\nu_j:1\leq j\leq n\}$, and $Z_d^{t,s}=(V_d^{t,s}\cap B_{n,d})$. By presmoothness, we have that $dim_{comp}(Z_d^{t,s})\geq dim(V_d^{t,s})-n=\frac{n(n+3)}{2}-n=\frac{n(n+1)}{2}$. (As above, for $\overline{e}\in V_d^{t,s}$, we have that the tangent space $T_{\overline{e}}(V_d^{t,s})$, consists of curves of degree d passing through the t nodes and s cusps of $C_{\overline{e}}$, (see Severi's calculations on curves infinamente vicine, in [3]).)

We let $\Phi : V_d^{t,s} \rightarrow P^{\frac{(n+1)(n+2)}{2}}$ be the duality map, $C_{\Phi(\overline{e})} = (C_{\overline{e}})^*$, for $\overline{e} \in V_d^{t,s}$, and let $W_d^{t,s} \subset P^{\frac{(n+1)(n+2)}{2}}$, $W_d^{t,s} = Im(\Phi) \cong V_d^{t,s}$ be the corresponding variety of nonsingular curves of degree n, and $t = \frac{n(n-2)}{2}$ bitangents, s inflexions, and $Y_d^{t,s} \subset W_d^{t,s}$ be the corresponding variety to $Z_d^{t,s}$, $dim_{comp}(W_d^{t,s}) \geq \frac{n(n+1)}{2}$. Let $G_{1,j,c} = (\overline{Y_d^{t,s}} \cap A_{1,j,c}$, then, again, by presmoothness, we have that $dim_{comp}(G_{1,j,c}) \geq \frac{n(n+1)}{2} - (7n-4) = \frac{n^2-13n+8}{2}$. Suppose there exists an $\overline{e} \in G_{1,j,c}$, such that the corresponding curve $C_{\overline{e}}$ is irreducible, $(\sharp\sharp)$. We claim that $Sing(C_{\overline{e}}) \cap (\{p_{k,j} : 1 \leq k \leq 2, i \neq j\} \cup \{a,c\}) = \emptyset$, $(***)$. In order to see $(***)$, suppose that there exists a singularity of $C_{\overline{e}}$, centred at one of the bitangents $p_{1,i}$ or $p_{2,i}$, for $j \neq i$. As $\overline{e} \in Y_d^{t,s}$, we have either that, say Case 1, $I_{p_{1,i}}(C_{\overline{e}}, l_i) \geq 3$ (one of the nodal branches is tangent to l_i), or Case 2, there exists another tangent of $C_{\overline{e}}$, centred at $q \in (l_i \setminus \{p_{1,i}, p_{2,i}\})$; as the corresponding dual curve $(C_{\overline{e}})^*$, belonging to $\overline{Z_d^{t,s}}$, has either a cusp or node singularity at the point ν_i, corresponding to the bitangent line l_i. Choose an irreducible curve $C_i \subset Y_d^{t,s} \cap A_{2,j,x}$, containing $\{\overline{c}, \overline{e}\}$, with $C_i \cap A_{1,j,c} = \overline{c}$, so for $\overline{e}' \in (C_i \setminus \{\overline{e}\})$, the corresponding curve $C_{\overline{e}'}$ is nonsingular. Consider the cover $R_i \subset C_i \times l_i$, defined by $R_i(\overline{t}, x) \equiv x \in (l_i \cap C_{\overline{t}}) \wedge \overline{t} \in C_i$. In Case 1, we claim there exists an irreducible component X_1 of R_i, passing through $(\overline{e}, p_{1,i}$, such that $pr_2(X_1) = l_i$, $pr_1(X_1) = C_i$. This follows from the fact, that if $\overline{e}' \in (\mathcal{V}_{\overline{e}} \setminus \{\overline{e}\})$, there exists $p'_{1,i} \in (\mathcal{V}_{p_{1,i}} \setminus \{p_{1,i}\})$, with $(\overline{e}', p'_{1,i}) \in R_i$. In order to see this, choose a direction (x_0, y_0), not on l_i, and, for $s \in P^1$, let $C^s_{\overline{t}}(x,y) = C_{\overline{t}}(x + sx_0, y + sy_0)$. Considering the cover $T_i \subset P^1 \times C_i \times l_i$, defined by $T_i(s, \overline{t}, x) \equiv x \in C^s_{\overline{t}} \cap l_i$, we have that $Mult_{(p_{1,i},0,\overline{e})}(T_i/P^1 \times C_i) \geq 3$, $Mult_{(p_{1,i},0,\overline{e}')}(T_i/P^1 \times C_i) = 2$, hence, by summability of specialisation, there exists $p'_{1,i} \in (\mathcal{V}_{p_{1,i}} \setminus \{p_{1,i}\})$, with $Mult_{(p'_{1,i},0)}(T_i(\overline{e}')/P^1) \geq 1$. Taking an irreducible component of X_2 of R_i through $(\overline{e}, q)$ with $q \neq p'_{1,i}$, we either have that $X_1 = X_2$, in which case $deg(X_2/C_i) \geq 2$, so there exists $(\overline{e}'', q') \in X_1$, with $\overline{e}'' \in (C_i \setminus \{\overline{e}\})$ and $Mult_(\overline{e}'', q')(R_i/C_i) \geq 2$. As $C_{\overline{e}''}$ is nonsingular, we have that q' defines a tangent with l_i. It follows that l_i is a tritangent to $C_{\overline{e}''}$ and the corresponding dual curve $(C_{\overline{e}''})^*$, has a triple node at ν_i, contradicting the definition of $\{Y_d^{t,s}, Z_d^{t,s}\}$. Otherwise, $X_1 \neq X_2$, and, as we can assume now that $deg(X_1/C_i) = 1$, we can find again find an intersection $(\overline{e}'', q') \in (X_1 \cap X_2)$, with $Mult_(\overline{e}'', q')(R_i/C_i) \geq 2$, $\overline{e}'' \in (C_i \setminus \{\overline{e}\})$, $q \neq p'_{1,i}$, and we can use the same argument as before. (Case 2 is similar, and the line l_j). It follows that $(***)$ holds.

We let $B_{1,j,c}$ be the linear space of codimension $4(n-1)+4 = 4n$, consisting of curves of degree n, tangent to l_i, $i \neq j$, at remaining bitangents, tangent to l_j at a and c, and $B_{2,j,c}$, the linear space of codimension $4(n-1)+2 = 4n-2$, consisting of curves of degree n, tangent to l_i, $i \neq j$, at remaining bitangents, tangent to l_j at a, so $B_{1,j,c} \subset B_{2,j,c}$. We have that the curve $C_{lines} \in B_{1,j,c}$, where C_{lines} consists of the union $\bigcup_{1 \leq j \leq n} l_j$. We let $B_{h_{i,j},1,j,c} \subset B_{1,j,c} \subset B_{2,j,c}$ be the $\frac{n(n-1)}{2}$ linear spaces of codimension $4n+1$, consisting of curves $C_{\overline{l}}$ in $B_{1,j,c}$, with $h_{i,j} \in C_{\overline{l}}$, where $h_{i,j} = (l_i \cap l_j)$, for $i \neq j$. Choose $C_{\overline{a}}$, with $\overline{a} \in B_{1,j,c}$ generic, and let $l = span(\overline{a}, lines)$, so $l \subset B_{1,j,c}$. If $C_{\overline{a}}$ is irreducible, then using the result of $(++)$, applied to the linear system l, if p is a singularity of $C_{\overline{a}}$, then

p must be situated at an intersection point $h_{i,j}$ for some $i \neq j$. As $\overline{a}$ is generic, we have that $h_{i,j} \notin C_{\overline{a}}$, for $i \neq j$, hence $C_{\overline{a}}$ is nonsingular. If $C_{\overline{a}}$ is reducible, with, wlog, irreducible components $\{C_1, C_2\}$, then, using (!!), (!!!!!), we have that $C_1 = \bigcup_{j\in J} l_j$, for some $J \subset \{1, \ldots, n\}$, with $Card(J) = n_1$, and there exists an isomorphic linear system L_2 of curves $D_{i_2^{-1}(\overline{l})}$, with degree $n - n_1$, $i_2 : L \rightarrow L_2$, with $C_{\overline{l}} = (C_1 \cup D_{i_2^{-1}(\overline{l})}$, for $\overline{1} \in L$, with fixed singularities $\{p_1, \ldots, p_r\}$ on $D_{\overline{l}}$. As above, we can assume that $\{p_1, \ldots, p_r\} \cap \{h_{i,j} : 1 \leq i < j \leq n\} = \emptyset$. Moreover, $\{p_1, \ldots, p_r\} \cap ((C_1 \cup C_2) \setminus C_{lines}) = \emptyset$, as the singularities are fixed. We must, then have that $\{p_1, \ldots, p_r\} \subset (C_0 \cap C_1)$, but $C_0 \subset C_{lines}$, hence, $\{p_1, \ldots, p_r\}$ define singularities of C_{lines}, as $\{p_1, \ldots, p_r\} \subset C_1 \cap D_{i_2^{-1}(\overline{1})}$, for $\overline{1} \in L$. This contradiction gives that $C_{\overline{a}}$ is irreducible, and, hence, by the previous part, nonsingular.

Considering again the variety $Y_d^{t,s}$, let $K \subset P^{\frac{(n+1)(n+2)}{2}}$ be defined by;

$K(\bar{l}) \equiv \bigwedge_{i=1}^{n} \exists (x_i \neq y_i)(l_i(x_i) \wedge l_i(y_i) \wedge l_{x_i} = l_{y_i} = l_i)$

Then we have that $K(lines)$, and $dim(\overline{K \setminus (K \cap Y_d^{t,s})}) < dim(\overline{K})$, $dim(\overline{Y_d^{t,s} \setminus (Y_d^{t,s} \cap K)}) < dim(\overline{Y_d^{t,s}})$. It follows, as $Y_d^{t,s}$ is irreducible, that $\overline{Y_d^{t,s}}(lines)$. Now, choosing $\overline{e} \in (\overline{Y_d^{t,s}} \cap B_{1,j,c}) \setminus \bigcup_{i\neq j} B_{h_{i,j},1,j,c}$ (do this..) generic, we claim that, if $l = span(\overline{e}, lines)$, then $l \subset (\overline{Y_d^{t,s}} \cap B_{1,j,c})$, (+). Suppose that $C_{\overline{e}}$ is irreducible. By the above proof of $(***)$, the singularities are disjoint from the bitangent points $\{p_{k,i} : 1 \leq k \leq 2, i \neq j\} \cup \{a, c\}$. It follows that, as the bitangent points $\{p_{k,i} : 1 \leq k \leq 2, i \neq j\} \cup \{a, c\} \subset (C_{lines} \cap C_{\overline{e}})$, they belong to every $C_{\overline{l}}$, with $\overline{l} \in l$, and for $\overline{e}' \in ((\mathcal{V}_{\overline{e}} \setminus \{\overline{e}\}) \cap l)$, we have that, they define nonsingular points of $C_{\overline{e}'}$. As $\overline{e}'$ is generic, we have that $\overline{e}' \notin B_{p_{i,j},1,j,c}$, for $1 \leq i < j \leq n$, hence, by the above analysis $C_{\overline{e}'}$ is nonsingular. Moreover, using $(***)$, no third tangent can occur along any of the $\{l_j : 1 \leq j \leq n\}$. It follows that the dual curve $(C_{\overline{e}'})^* \in Z_d^{t,s}$, hence, $C_{\overline{e}} \in Y_d^{t,s}$, giving the result (+). If $C_{\overline{e}}$ is reducible, then, again by the above analysis $C_{\overline{e}}$ is irreducible, and we obtain the result. Taking $\overline{e}' \in ((\mathcal{V}_{\overline{e}} \setminus \{\overline{e}\}) \cap l)$, we obtain a nonsingular curve $C_{\overline{e}'} \in \overline{Y_d^{t,s}} \cap B_{1,j,c}$, which satisfies the required properties of the footnote.

(Fixed Singularities 1) Let L be a (generically) irreducible, 1-dimensional linear system of plane curves of degree n, defined by $f(x, y; t)$, Let $F \subset L \times P^2$ be defined by;

$F(t, \overline{z}) \equiv \overline{z} \in Sing(C_t) \equiv f(\overline{z}, t) = 0 \wedge \frac{\partial f}{\partial x}|_{\overline{z}} = 0 \wedge \frac{\partial f}{\partial y}|_{\overline{z}} = 0$

We claim that the irreducible components of F are of the form $L \times \{(t_0, \overline{z}_0)\}$ or $(t_0, \overline{z}_0)$, with $(t_0, \overline{z}_0) \in L \times P^2$, or $(\{(t_0\} \times C'_{t_0})$, with C'_{t_0} defining a non-reduced component of C_{t_0}, (++).

Clearly every irreducible component of F, has dimension at most 1, as $deg(\frac{\partial f}{\partial x} < deg(f)$. As the family, defined by L, is generically irreducible, we can remove the finitely many parameters $\{t_j : 1 \leq j \leq s\} \subset L$ defining curves C_{t_j}, with reduced components. Suppose there exists an irreducible component F_0, of dimension 1, with $pr_{P^2}(F_0)$ defining an irreducible curve $D \subset P^2$, of degree m, with

$\overline{F_0 \not\subseteq (\{(t_0\} \times P^2)}$, so $pr_L(F_0) = L$. The series $W_t(\overline{z}) \equiv \overline{z} \in (D \cap C_t)$ defines a g^1_{nm} on D with parameter space $U = (L \setminus \{t_j : 1 \leq j \leq s\})$, as if there exists $t' \in U$, with $D \subset C_{t'}$, $C_{t'}$ would contain a reduced component. Using the fact that $pr_L(F_0) = L$, we have, for generic $t \in U$, that there exists $p \in (D \setminus Base(L)) \cap Sing(C_t)$. We have that $I^L(p, D, C_t) = 1$, and, by definition of F, that $I(p, D, C_t) \geq 2$, see notation in [5]. Using the result of [5], Lemma 2.10, (with the slight modification, that we have removed finitely many points from L) we obtain a contradiction. Hence, F_0 is of the form $L \times \{(t_0, \overline{z}_0)\}$, with $(t_0, \overline{z}_0) \in L \times P^2$, giving the claim $(++)$.

(Fixed Singularities, 2) Let $L \subset Sing(C)$ be a generically irreducible linear system, then there exists an open set $U \subset L$, such that for each $\overline{a} \in U$, $C_{\overline{a}}$ has exactly r singularities, centred at $\{p_1, \ldots, p_r\}$, $(\sharp)$. To see this, suppose that a generic curve has r singularities, so the condition holds on an open set $U_r \subset L$. The condition B_{r+1}, that there exist at least $r+1$ singularities is closed, hence holds on U_r^c. Choose independent generic points $\{\overline{a}_1, \overline{a}_2\}$ from L, then the line $l \subset L$, connecting $\overline{a}_1$ and $\overline{a}_2$, intersects B_{r+1} in finitely many points, not including $\{\overline{a}_1, \overline{a}_2\}$. Suppose $Sing(C_{\overline{a}_1}) \neq Sing(C_{\overline{a}_2})$, and consider the linear system defined by $l \subset L$. Using the result of $(***)$, we have, if q is a singularity of $C_{\overline{a}_1}$, not of $C_{\overline{a}_2}$, then, for $\overline{a}_1{}' \in \mathcal{V}_{\overline{a}_1}$, we have that $Sing(C'_{\overline{a}_1}) = Sing(\overline{a}_1 \setminus q)$, contradicting the fact there are no curves in the family with $r-1$ singularities. It follows that $Sing(C_{\overline{a}_1}) \neq Sing(C_{\overline{a}_2})$, and, as this condition is definable, the result follows.

(!) Let L be a 1-dimensional linear system, consisting of curves of degree n, then if the generic curve $C_{\overline{a}}$ is reducible with irreducible components $\{C_1, C_2\}$, of degrees $\{n_1, n_2\}$, such that $n_1 + n_2 = n$, then every curve in the family is reducible with components of degrees $\{n_1, n_2\}$. To see this, let $V \subset L \times P^{n_1} \times P^{n_2}$ be defined by;

$$V(\bar{l}, \overline{t_1}, \overline{t_2}) \equiv C_{\bar{l}} = C_{\overline{t_1}} C_{\overline{t_2}}$$

then V is closed, and by completeness of closed projective varieties, so is the projection $W \subset L$;

$$W = \exists \overline{t_1} \exists \overline{t_2} V$$

As $\overline{a}$ is generic and $W(\overline{a})$, we have that $W = L$ as required.

(!!) Let L be a (generically) reducible, 1-dimensional linear system of plane curves of degree n. Then, if $\overline{a} \in L$ is generic, with $C_{\overline{a}}$, having irreducible components $\{C_1, C_2\}$, with degrees $\{n_1, n_2\}$, such that $n_1 + n_2 = n$, then, either there exists a (generically) irreducible 1-dimensional linear system L_1, of plane curves of degree n_1, and a linear isomorphism $i_1 : L_1 \rightarrow L$, such that, for $\overline{l_1} \in L_1$, $C_{i_1(\overline{l_1})} = C_{\overline{l_1}}.C_2$, or, there exists a (generically) irreducible 1-dimensional linear system L_2, of plane curves of degree n_2, and a linear isomorphism $i_2 : L_2 \rightarrow L$, such that, for $\overline{l_2} \in L_2$, $C_{i_2(\overline{l_2})} = C_1.C_{\overline{l_2}}$.

In order to see this, as in $(***)$, we let $F \subset L \times P^2$ be defined by;

$F(t,\bar{z}) \equiv \bar{z} \in Sing(C_t) \equiv f(\bar{z},t) = 0 \wedge \frac{\partial f}{\partial x}|_{\bar{z}} = 0 \wedge \frac{\partial f}{\partial y}|_{\bar{z}} = 0$

where f defines L. We claim that, if F_0 is an irreducible component of F, such that $dim(pr_{P^2}(F_0)) = 1$, and $F_0 \nsubseteq (\{t_0\} \times P^2)$, then $pr_{P^2}(F_0) = C_1$ or $pr_{P^2}F_0 = C_2$, (!!!). Suppose not, then, we have, that for generic $\bar{l}' \in L$, $C_{\bar{l}'} \cap pr_{P^2}F_0$ is finite, otherwise, $C_{\overline{l''}} \supset pr_{P^2}F_0$, for all $\overline{l''} \in L$, which is not the case, as C_1 and C_2 are irreducible. As in $(***)$, if $D = pr_{P^2}(F_0)$, with degree m, the series $W_t(\bar{z}) \equiv \bar{z} \in (D \cap C_t)$, defines a g^1_{nm} on D with parameter space V, where $V = (L \setminus \{t \in L, C_t \supset D\})$. As above, for generic $t \in V$, and using the fact that $pr_L(F_0) = L$, we can find $p \in (D \setminus Base(L)) \cap Sing(C_t)$. We have that $I^L(p,D,C_t) = 1$, and, by definition of F, that $I(p,D,C_t) \geq 2$, see notation in [5]. Again we obtain a contradiction.

Suppose that $n_1 \neq n_2$. Let $V_1 \subset L \times P^{\frac{(n_1+1)(n_1+2)}{2}}$, $V_2 \subset L \times P^{\frac{(n_2+1)(n_2+2)}{2}}$ be defined by;

$V_1(\bar{l},\overline{l_1}) \equiv L(\bar{l}) \wedge C_{\bar{l}} \supset C_{\bar{l}_1}$

$V_2(\bar{l},\overline{l_2}) \equiv L(\bar{l}) \wedge C_{\bar{l}} \supset C_{\bar{l}_2}$

We have that, for $1 \leq j \leq 2$, V_j consists of a unique 1-dimensional irreducible component $V_{j,0}$ with $deg(V_{j,0}/L) = 1$, together with finitely many points $\{p_{j,k} : 1 \leq k \leq t(j)\}$. For $1 \leq j \leq 2$, we let $i_j : L \rightarrow V_{j,0}$ be the unique isomorphisms such that $pr_1 \circ i_j = Id_L$. We let $V_{1,2} \subset L \times P^2$ be defined by;

$V_{1,2}(\bar{l},p) \equiv p \in C_{i_1(\bar{l})} \cap C_{i_2(\bar{l})}$

We have that $V_{1,2}$ is a closed generically finite cover of L. Let Z be an irreducible component of $V_{1,2}$, not contained in $\{t_0\} \times P^2$, for some $t_0 \in L$. By presmoothness, we have that $dim(Z) = 1$. Suppose that $pr_{P^2}(Z)$ defines an irreducible curve $D \subset P^2$. If $D \notin \{C_1, C_2\}$, then, as Z defines an irreducible component of F, defined above, we obtain, by (!!!), a contradiction. Hence, for any irreducible component Z of $V_{1,2}$, not contained in $\{t_0\} \times P^2$, either Case 1; $pr_{P^2}(Z) = C_i$, for $i = 1$ or $i = 2$, or Case 2; $Z = L \times \{p_0\}$, for some $p_0 \in C_1 \cap C_2$. In Case 1, wlog $pr_{P^2}(Z) = C_1$. Suppose that C_1 is not an irreducible component of every $C_{\bar{l}}$, with $\bar{l} \in L$, then, as the condition $(\forall l \in L)C_{\bar{l}} \supset C_1$, (!!!!), fails, and this condition is closed, it follows there exist finitely many parameters $\{t'_k : 1 \leq k \leq t\}$ such that, if $\bar{l} \in W = (L \setminus \{t'_k : 1 \leq k \leq s\})$, then $(C_{\bar{l}} \cap C_1)$ is finite. We then obtain a $g^1_{n_1 n}$ on C_1, with parameter space W, given by $N(z) \equiv z \in C_1 \cap C_{\bar{l}}$, $\bar{l} \in W$. Choosing $p \in (C_1 \setminus Base(W)) \cap pr_{P^2}(Z \cap pr_L^{-1}(\bar{a}))$, for some generic $\bar{a} \in W$, (if this fails then there exists an open $Q \subset W$, with $C_1 \cap pr_{P^2}(Z \cap pr_L^{-1}(Q)) = Base(W)$, which is not the case. We thus obtain $p \in C_1 \cap Sing(C_{\bar{a}})$, as $p \in C_{i_1(\bar{a})} \cap C_{i_2(\bar{a})}$. We have that $I^W(p,C_1,C_{\bar{a}}) = 1$, and that $I(p,C_1,C_t) \geq 2$, see notation in [5]. Again we obtain a contradiction. However, we claim that Case 2 cannot always occur. We make the further assumption that, for *any* $\{\bar{l},\overline{l'}\} \subset L$, we have that $C_{i_1(\bar{l})} \cap C_{i_2(\overline{l'})}$ is finite, (!!!!!). (this is slightly stronger than the requirement that the condition (!!!!) fails, we will weaken it later).

□

Fix $\overline{l_0} \in L$, and consider the $g^1_{n_1 n_2, \overline{l_0}}$ on $C_{\overline{l_0}}$, with parameter space L, obtained by intersecting $C_{\overline{l_0}}$ with $C_{i_2(\bar{l})}$, for $\bar{l} \in L$. We then claim that there exists a multiple point $p_{\overline{l_0}}$ for this $g^1_{n_1 n_2, \overline{l_0}}$. In order to see this, consider the variety $G_{\overline{l_0}} \subset L \times P^2$, defined by $G_{\overline{l_0}}(\bar{l}, p) \equiv (f_1(p; \overline{l_0}) = f_2(p; \bar{l}) = 0 \wedge det(\frac{\partial f_i}{\partial x_j})_{1 \leq i,j \leq 2}|_{p,\bar{l}} = 0$. By presmoothness, and assuming that $det(\frac{\partial f_i}{\partial x_j})_{1 \leq i,j \leq 2}|_{p,\bar{l}} \neq c$, for $c \neq 0$, $G_{\overline{l_0}} \neq \emptyset$, witnessed by $(\overline{l_{1,0}}, p_{\overline{l_0}})$. Using the result of [5], Lemma 2.10, (which generalises easily to reducible curves), as above, either $p_{\overline{l_0}} \in Base(g^1_{n_1 n_2, \overline{l_0}})$, for this system, or it ramifies, that is $I^L_{p_{\overline{l_0}}}(C_{i_1(\overline{l_0})}, C_{i_2(\overline{l_{1,0}})}) \geq 2$. In the former case, we have that $\{\bar{l} \in L : I_{p_{\overline{l_0}}}(C_{i_1(\overline{l_0})}, C_{i_2(\bar{l})}) \geq k\}$ is definable and linear, hence, if k_1 is the minimum multiplicity of the $g^1_{n_1 n_2, \overline{l_0}}$ at $p_{\overline{l_0}}$, there exists $\overline{l_{1,0'}} \in L$, with $I_{p_{\overline{l_0}}}(C_{i_1(\overline{l_0})}, C_{i_2(\overline{l_{1,0'}})}) \geq k_1 + 1$, and for generic $\bar{l} \in L$, $I_{p_{\overline{l_0}}}(C_{i_1(\overline{l_0})}, C_{i_2(\bar{l})}) = k_1$, (††), and again we obtain ramification in L, that is $I^L_{p_{\overline{l_0}}}(C_{i_1(\overline{l_0})}, C_{i_2(\overline{l_{1,0'}})}) \geq 2$. Wlog we use the notation $\overline{l_{1,0}}$ for $\overline{l_{1,0'}}$. Now consider the variety $S \subset L \times L$, given by;

$$S(\bar{l}, \bar{l}') \equiv (\exists p)(p \in (C_{i_l(\bar{l})} \cap C_{i_2(\bar{l}')}) \wedge I_p^{L, g^1_{n_1 n_2, \bar{l}}}(C_{i_1(\bar{l})}, C_{i_2(\bar{l}')}) \geq 2)$$

By the above analysis, we obtain that S is a closed finite cover of L, hence, intersecting with the diagonal $\Delta \subset (L \times L)$, we can find $(\overline{l_1}, \overline{l_1}) \in S$, and $p_{\overline{l_1}} \in P^2$ with $I^L_{p_{\overline{l_1}}}(C_{i_1(\overline{l_1})}, C_{i_2(\overline{l_1})}) \geq 2$. It follows that, taking $\overline{l_1}' \in (\mathcal{V}_{\overline{l_1}} \setminus \{\overline{l_1}\})$, we can find distinct point $\{p_{1,\overline{l_1}}, p_{2,\overline{l_1}}\} \subset (\mathcal{V}_{p_{\overline{l_1}}} \cap C_{i_1(\overline{l_1})} \cap C_{i_2(\overline{l_1}')})$, where $p_{\overline{l_1}} \in (C_{i_1(\overline{l_1})} \cap C_{i_2(\overline{l_1})})$. Now consider the cover $Y \subset L \times L \times P^2$, defined by $Y(\bar{l}, \bar{l}', p) \equiv p \in (C_{i_l(\bar{l})} \cap C_{i_2(\bar{l}')})$, then, using summability of specialisation, we obtain that $Mult_{(\overline{l_l}, \overline{l_l}, p_{\overline{l_1}})}(Y/\Delta) \geq 2$, hence, there exist 2 distinct irreducible components $\{Z_l, Z_2\}$ of $V_{1,2}$, projecting onto L, passing through $(\overline{l_1}, p_{\overline{l_1}})$. Clearly, such components cannot both be of the form required in Case 2. It follows that Case 1 holds, and we obtain the result, as required.

To complete the proof, with just the assumption that (!!!!) fails, we have to allow for the possibility, that, for any given $C_{i_1(\overline{l_0})}$, there exist finitely many parameters $P = \{\overline{l_{0,j}} : 1 \leq j \leq t(\overline{l_{0,j}})\}$ such that $C_{i_1(\overline{l_0})} \cap C_{i_2(\overline{l_{0,j}})}$ contains a component of dimension 1. In this case, we can remove the parameters P, setting $W = (L \setminus P)$, and obtain a $g^1_{n_1 n_2}$ on $C_{i_1(\overline{l_0})}$, with parameter space W. We can then complete the $g^1_{n_1 n_2}$ to a $g^1_{n_1 n_2, c}$, with parameter space L, by letting $F \subset (L \times C_{i_1(\overline{l_0})})$ be defined by $F_{\overline{l_0}} = \overline{H_{\overline{l_0}}}$, where $\overline{H_{\overline{l_0}}} \subset W \times C_{i_1(\overline{l_0})}$ is given by $\overline{H_{\overline{l_0}}}(\bar{l}, p) \equiv (p \in C_{i_1(\overline{l_0})} \cap C_{i_2(\bar{l})}) \wedge W(\bar{l})$, and letting the weighted set $B_{\overline{l_{0,j}}} = pr_{P^2}(F(\overline{l_{0,j}}))$, with weights $Mult_a(F_{\overline{l_0}}/L)$, for $a \in pr_{P^2}F(\overline{l_{0,j}})$. It is an easy exercise, left to the reader, to show that results above hold for this more abstract definition.

If $n_1 = n_2$, we let $V_3 \subset L \times P^{\frac{(n_1+1)(n_1+2)}{2}}$ be defined by;

Lemma 2.8. *Let C be a harmonic curve and let $\{C_t : t \in Par_t\}$ be a family given as in Definition 2.5. Let $\{(x_{j,j'}, y_{j,j'}) : 1 \leq j < j' \leq n\}$ enumerate the points of intersection $l_j \cap l_{j'}$, in the coordinate system (x, y). Then, for each $(x_{j,j'}, y_{j,j'})$, and $t'_\infty \in ((\mathcal{V}_\infty \cap Par_t) \setminus \{\infty\})$ there exist exactly 2 vertical tangents $\{(x_{1,t'_\infty,j,j'}, y_{1,t'_\infty,j,j'}), (x_{2,t'_\infty,j,j'}, y_{2,t'_\infty,j,j'})\}$, specialising to $(x_{j,j'}, y_{j,j'})$.*

Proof. The family $\{C_t : t \in Par_t\}$ is a particular form of asymptotic degeneration, for which the methods of [4] apply. By Lemmas 3.44 and 4.3(iv)(d) of [4], (see notation there), if $t'_\infty \in ((\mathcal{V}_\infty \cap Par_t) \setminus \{\infty\})$, and $(x_0, y_0) \in (l_{j'} \cap l_j)$, for some $1 \leq j < j' \leq n$, then we can find $C_{i,j,\infty}$ and $C_{i',j',\infty}$, for some $1 \leq i \leq i' \leq t$, such that $(x_0, y_0, z_0)) \in (C_{i,j,\infty} \cap C_{i',j',\infty'})$, for some $z_0 \in A^1$, and $(x_1, y_1, z_1)) \in (C_{j,t'_\infty} \cap C_{j',t'_\infty})$, specialising to (x_0, y_0, z_0), where $t'_\infty \in \mathcal{V}_\infty$. By Lemma 3.44 of [4], as $C_{t'_\infty}$ is nonsingular, the corresponding (x_1, y_1) defines a vertical tangent of $C_{t'_\infty}$. Conversely, if (x_1, y_1), defines a vertical tangent of $C_{t'_\infty}$, specialising to (x_0, y_0), then, by Lemmas 3.44 and 4.3(iii)(b) of [4], there

---

$$V_3(\bar{l}, \overline{l_3}) \equiv L(\bar{l}) \wedge C_{\bar{l}} \supset C_{\overline{l_3}}$$

Then, as $|V_3(a)| = 2$, for generic $\overline{a} \in L$, V_3 has either two irreducible components $V_{j,0}$, for $1 \leq j \leq 2$, with $deg(V_{j,0}/L) = 1$, or a single irreducible component R, with $deg(R/L) = 2$. In the first case, we repeat the argument above to obtain the result. In the second case, we let $M_{1,2} \subset L \times P^2$ be defined by $M_{1,2} = \overline{W_{1,2}}$, where;

$$W_{1,2}(\bar{l}, p) \equiv \exists(\overline{l_1 l_2})(R(\bar{l}, \overline{l_1}) \wedge R(\bar{l}, \overline{l_2}) \wedge (\overline{l_1} \neq \overline{l_2}) \wedge (p \in C_{\overline{l_1}} \cap C_{\overline{l_2}})$$

Arguing, as above, with $M_{1,2}$ replacing $V_{1,2}$, and observing that there exists an open set $U \subset L$, with $pr_{P^2}(M_{1,2}(\bar{l})) \subset Sing(C_{\bar{l}})$, for $\bar{l} \in U$, we obtain the result if Case 1 holds above, and, in fact V_3 has two irreducible components.

(!!!!!) (Fixed Singularities 3). Let $L \subset (Sing(C))$ be a generically reducible linear system, (with 2 irreducible components) of curves of degree n. Then, for $\bar{l} \in L$, $C_{\bar{l}} = C_1 \cup D_{\bar{l}}$, where C_1 and $D_{\bar{l}}$ are generically irreducible, and the singularities of the generic $D_{\bar{l}}$ are fixed everywhere, centred at $\{p_l, \ldots, p_r\}$.

To see (!!!!!), suppose the generic curve $C_{\overline{a}} = C_1 \cup C_2$, $\overline{a} \in U$, where both C_1 and C_2 are irreducible curves, of degrees $\{n_1, n_2\}$ with singularities centred at $A = \{q_l, \ldots, q_{r'}\}$, $B = \{p_1, \ldots, p_r\}$. Choose an independent generic curve $C_{\overline{a}'}$, and consider the 1-dimensional linear system $l = span(\overline{a}, \overline{a}')$. Using the result (!!), we can suppose that $C_{\overline{a}} = (C_1 \cup D_{\overline{a}})$, $C_{\overline{a}'} = (C_1 \cup D_{\overline{a}'})$, where $\{D_{\overline{a}}, D_{\overline{a}'}\}$ are irreducible of degree n_2, and belong to a new linear system L_2. By the result ($\sharp$), we obtain that the singularities $B = \{p_l, \ldots, p_r\}$ of C_2 are fixed, for $D_{\bar{l}}$, $\bar{l} \in L_2$. As $\overline{a}'$ is independent of $\overline{a}$, generic, the fixed singularities, $\{p_l, \ldots, p_r\}$, are defined over $acl(\overline{a})$, and the conditions that $C_{\bar{l}} \supset C_1$ and $\{p_l, \ldots, p_r\} \subset Sing(C_{\bar{l}})$ are closed, the result holds on L, as required.

exists a corresponding $(x_1, y_1, z_1) \in (C_{j,t'_\infty,s'_\infty} \cap C_{j',t'_\infty,s'_\infty})$, hence, using Lemma 4.3(iv)(d) of [4], $((x_0, y_0) \in (l_{j'} \cap l_j)$, for some $1 \leq j' \neq j \leq n$, $(*)$.

Let $W \subset Par_t \setminus \{t_\infty\} \times P^2$ be defined by;

$W(t, x', y') \equiv [f(t, x', y') \wedge \frac{\partial f_t}{\partial x}(x', y') = 0]$

and let $\overline{W} \subset Par_t \times P^2$ define the Zariski closure. By $(*)$, we have that the fibre $\overline{W}(t'_\infty)$ consists of exactly the points $\{(t_\infty, x_{j,j'}, y_{j,j'}) : 1 \leq j < j' \leq n\}$. Suppose, for contradiction, that, for some $(t_\infty, x_{j_0,j'_0}, y_{j_0,j'_0})$, $Mult(\overline{W}/Par_t)_{(t_\infty, x_{j_0,j'_0}, y_{j_0,j'_0})} \geq 3$. By the degree-genus formula, Theorem 3.36, and Severi's Definition 3.33 of genus g, see also Theorem 3.36(†), in [3], we have that, for $t' \in (Par_t \setminus t'_\infty)$, $g(C_{t'}) = \frac{(n-1)(n-2)}{2}$, and $class(C_{t'}) = 2(g - (1 - n))$, hence, $class(C_{t'}) = n(n-1)$, (†). It follows, using $(*)$, as $Card(\overline{W}(t'_\infty)) = \frac{n(n-1)}{2}$, that there must exist $(t_\infty, x_{j_1,j'_1}, y_{j_1,j'_1})$, with $Mult(\overline{W}/Par_t)_{(t_\infty, x_{j_1,j'_1}, y_{j_1,j'_1})} = 1$, (!!!!).

Without loss of generality we can assume that the intersection $l_a \cap l_b$ corresponds to the point $[0 : 1 : 0]$ in the coordinate system $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$, and l_a is given by $x = 0$, l_b is given by $x = 1$. The arguments in the paper [4], see especially Lemma 3.44 and Theorem 4.3, apply to the given asymptotic degeneration, with distinct, ([9]) flashes $\{\eta_{1,t}, \ldots, \eta_{n,t}\}$, and $\{\eta'_{1,t}, \ldots, \eta'_{n,t}\}$, obtained from applying Newton's theorem along the lines $x = 0$ and $x = 1$. It follows that, for all $t \in U \subset Par_t$, the flashes $\bigcup_{1\leq j\leq n} \eta_{j,t}$ and $\bigcup_{1\leq j\leq n} \eta'_{j,t}$ intersect in finitely many points. Now applying the argument $(*)$, we obtain that, for $\overline{W}$ as above, that, $Mult(\overline{W}/Par_t) \geq 2$, contradicting (!!!!).

□

Lemma 2.9. *Let C be a harmonic curve, then there exists a linear system L, with $C_{\bar{l}_0} = C$, and $C_{\bar{l}_\infty} = C_{lines}$, where C_{lines} is a harmonic arrangement. Then, if $\bar{l} \in \mathcal{V}_{\bar{l}_\infty} \setminus \{\bar{l}_\infty\}$, and $p_{i,j} = (l_i \cap l_j)$, $p_{i,j} = (x_{i,j}, y_{i,j})$, there exist $\{z^k_{i,j} : 1 \leq k \leq 2\} \subset \mathcal{V}_{p_{i,j}}$, with $pr_x(z^k_{i,j}) = x^k_{i,j}$ distinct, $z^k_{i,j} = (x^k_{i,j}, y^k_{i,j})$, such that $I_{z^k_{i,j}}(C_{\bar{l}}, x = x^k_{i,j}) = 2$, and, if $z \in C_{\bar{l}} \cap \mathcal{V}_{p_{i,j}}$, with $pr_x(z) = x^k_{i,j}$, then $z = z^k_{i,j}$, and, for all $x' \in \mathcal{V}_{x_{i,j}} \setminus \{x^k_{i,j}\}$, there exist exactly two $\{z^t_x : 1 \leq t \leq 2\} \subset C_{\bar{l}} \cap \mathcal{V}_{p_{i,j}}$, with $pr_x(z^t_x) = x'$. If $\bar{l} \in \mathcal{V}_{\bar{l}_\infty} \setminus \{\bar{l}_\infty\}$, and $p \in C_{lines} \setminus \{p_{i,j} : 1 \leq i < j \leq n\}$,*

[9]We can assume that $\eta'_{j,t}(x + 1) \neq \eta_{k,t}(x)$, for $1 \leq j, k \leq n$, by, wlog, obtaining, using the above argument, that the bitangent y-coordinates $\{y(p_{1,1}), \ldots, y(p_{1,n})\}$ are distinct from $\{y(p_{2,1}), \ldots, y(p_{2,n})\}$.

$p=(x_p,y_p)$, then, for all $x'\in\mathcal{V}_{x_p}$, there exists a unique $z'\in C_{\bar{l}}\cap\mathcal{V}_p$, with $pr_x(z')=x'$. Finally, if $\bar{l}\in\mathcal{V}_{\bar{l}_\infty}\setminus\{\bar{l}_\infty\}$, and $z\in C_{\bar{l}}$, there exists $p\in C_{lines}$, with $z\in\mathcal{V}_p$.

Proof. The existence of L follows from the proof of footnote 8. By Lemma 2.8, if $\bar{l}\in\mathcal{V}_{\bar{l}_\infty}\setminus\{\bar{l}_\infty\}$, there exist exactly two vertical tangents $\{z_{i,j}^k:1\leq k\leq 2\}\subset\mathcal{V}_{p_{i,j}}$. Consider the cover $F\subset P^2\times L\times P^1$, defined by;

$$F(p,\bar{l},x')\equiv(p\in C_{\bar{l}}\cap x=x')$$

We claim that $Mult_{(p_{i,j},lines,x_{i,j})}(F/L\times P^1)=2$, $(*)$. [Considering the g_n^1, on $x=x_{i,j}$, we have, if $p_{i,j}\notin Base(g_n^1)$, and, using Lemma 2.10 of [5], that, $pr_x(z_{i,j}^k)\cap\{x_{i,j}\}=\emptyset$, hence, we can assume that $\{x_{i,j}^k:1\leq k\leq 2\}\cap\{x_{i,j}\}=\emptyset$, or say $z_{i,j}^1=p_{i,j}$, $(**)$, $(\dagger)$]. Considering the g_n^1, on $x=x_{i,j}$, we have that $I_{p_{i,j}}(C_{lines},x=x_{i,j})=2$, hence, we have that there exist at most 2 points $\{p'_{i,j},p''_{i,j}\}\subset(\mathcal{V}_{p_{i,j}}\cap C_{\bar{l}})$, with $pr_x(p'_{i,j})=pr_x(p''_{i,j})=x_{i,j}$. If $Mult_{(p_{i,j},lines,x_{i,j})}(F/L\times P^1)\geq 3$, then, using summability of specialisation, we have that;

$$Mult_{(p'_{i,j},\bar{l},x_{i,j})}(F(\bar{l})/P^1)+Mult_{(p''_{i,j},\bar{l},x_{i,j})}(F(\bar{l})/P^1)\geq 3$$

Hence, wlog $I_{p'_{i,j}}(C_{\bar{l}},x=x_{i,j})\geq 2$, implying that $p'_{i,j}$, (?), is a vertical tangent. This contradicts the assumption $(\dagger)$, if $(**)$ fails, as $pr_x(p'_{i,j})=x_{i,j}$. If $(**)$ holds, then we must have that $p'_{i,j}=p''_{i,j}=p_{i,j}$, and $I_{p_{i,j}}(C_{\bar{l}},x=x_{i,j})\geq 3$, giving $I_{p_{i,j}}(C_{lines},x=x_{i,j})\geq 3$, which is not the case. If $Mult_{(p_{i,j},lines,x_{i,j})}(F/L\times P^1)=1$, then clearly, again by summability, $Mult_{(p_{i,j},lines,x_{i,j})}(F(lines)/P^1)=1$, contradicting the fact that $I_{p_{i,j}}(C_{lines},x=x_{i,j})=2$, giving $(*)$. Suppose that $x'\in(\mathcal{V}_{x_{i,j}}\setminus\{x_{i,j}^k\})$, and there exists a single $z\in(\mathcal{V}_{p_{i,j}}\cap C_{\bar{l}})$, with $pr_x(z)=x'$. As $I_z(C_{\bar{l}},x=x')=1$, we have, for generic $(\bar{l},x'')\in\mathcal{V}_{(\bar{l},x)}$, that there exists a single $z\in(\mathcal{V}_{p_{i,j}}\cap C_{\bar{l}})$, with $pr_x(z)=x''$, contradicting $(*)$. Similarly, we can exclude $Card(\mathcal{V}_{p_{i,j}}\cap C_{\bar{l}}\cap pr_x^{-1}(x'))\geq 3$, in which case, we obtain that $Mult_{(p_{i,j},lines,x_{i,j})}(F/L\times P^1)\geq 3$, contradicting $(*)$. If, $Card(C_{\bar{l}}\cap pr^{-1}(x_{i,j}^k)\cap\mathcal{V}_{p_{i,j}})\geq 2$, then, as $I_{z_{i,j}^k}(C_{\bar{1}},x=x_{i,j}^k)=2$, we have that $Mult_{(p_{i,j},lines,x_{i,j})}(F/L\times P^1)\geq 2+1=3$, contradicting $(*)$, in particular it follows that $x_{i,j}^1\neq x_{i,j}^2$. We have that $Mult_{p,lines,x_p}(F/L\times P^1)=1$, as considering the g_n^1 on $x=x_p$, and, using the fact that $\{x=x_p,l_j\}$ intersect transversely, where $x_p\in l_j$, we have that there exists a unique $y''\in\mathcal{V}_{y_p}$, with $(x_p,y'')\in$

$C_{\bar{l}} \cap (x = x_p)$. As (x_p, y'') does not define a vertical tangent, we have that $I_{(x_p,y'')}(C_{\bar{l}}, x = x_p) = 1$, hence, for generic $(\bar{l}, x') \in \mathcal{V}_{(lines,x_p)}$, there exists a unique $y' \in \mathcal{V}_{y_p}$ with $(x', y'') \in C_{\bar{l}} \cap x = x'$. Taking $z' = (x', y')$ gives the required result. To see the final part, observe that the variety $Z = \{(\bar{l}', z') \in L \times P^2, z \in C_{\bar{l}}\}$ is closed, hence, if $(\bar{l}, z) \in Z$, with $\bar{l} \in \mathcal{V}_{\bar{l}_\infty}$, then its specialisation $(\bar{l}_\infty, p) \in Z$ as well. By definition, $p \in C_{lines}$.

□

Lemma 2.10. *Let hypotheses be as in Lemma 2.9, and assume that n is odd, then there exists an $\epsilon'' > 0$, and a ball $B(\bar{l}_\infty, \epsilon'')$ such that for all $\bar{l} \in (B(\bar{l}_\infty, \epsilon'') \cap L) \setminus \{\bar{l}_\infty\}$, $C_{\bar{l}}$ is topologically equivalent to a sphere with g attached handles, where $g = \frac{(n-1)(n-2)}{2}$. In particular, Severi's definition of genus g coincides with the topological definition, see* [3].

Proof. Using the result of Lemma 2.9, and Theorem 17.1 of [6], we have, for all infinite $n \in {}^*\mathcal{N}$ and $\delta' > 0$ standard, that the statements $D(n, p_{i,j}), E(n, \delta'), F(n)$ hold;

$$D(n, p_{i,j}) \equiv (\forall \bar{l} \in B(\bar{l}_\infty, \tfrac{1}{n}))(\forall x \in B(x_{i,j}, \tfrac{1}{n}))[x \notin S(\bar{l}) \rightarrow \exists^{=2} y((x, y) \in$$

$$C_{\bar{l}} \cap B(p_{i,j}, \tfrac{1}{n})) \wedge x \in S(\bar{l}) \rightarrow \exists^{=1} y((x, y) \in C_{\bar{l}} \cap B(p_{i,j}, \tfrac{1}{n}))]$$

$$F(n) \equiv (\forall \bar{l} \in B(\bar{l}_\infty, \tfrac{1}{n}))(\forall z \in C_{\bar{l}})(\exists = 1 w \in C_{lines})(z \in B(w, \tfrac{1}{n}))$$

$$E(n, \delta') \equiv (\forall \bar{l} \in B(\bar{l}_\infty, \tfrac{1}{n}))(\forall z \in C_{lines} \setminus \textstyle\bigcup_{1 \leq i < j \leq n'} B(p_{i,j}, \delta'))(\forall x' \in B(x_z, \tfrac{1}{n}))(\exists^{=1} y((x', y) \in C_{\bar{l}} \cap B(z, \tfrac{1}{n})))$$

where S is defined by $S(\bar{l}, x') \equiv (\exists y')(f(x', y', \bar{l}) = 0 \wedge \frac{\partial f}{\partial x}(x', y', , \bar{l}) = 0)$ By underflow, see [1], the statements hold for all $n \in \mathcal{N}$, with $n \geq n_0$. In particular, taking $\epsilon > 0$, such that $B(p_{i,j}, \epsilon) \cap \{p_{i',j'} : (i', j') \neq (i, j)\} = \emptyset$, we obtain that;

(i). For all $\bar{l} \in B'(\bar{l}_\infty, \epsilon''')$, $pr_x : (C_{\bar{l}} \cap B(p_{i,j}, \epsilon''')) \rightarrow B(x_{i,j}, \epsilon''')$ is a double cover, ramified at two distinct points $\{z^1_{i,j}(\bar{l}), z^2_{i,j}(\bar{l})\}$, $\epsilon''' \leq \epsilon$, ([10]).

---

[10] Observe that we can then find a closed loop $S_{i,j,\bar{l}} \subset (C_{\bar{l}} \cap B(p_{i,j}, \epsilon'''))$, passing through $\{z^1_{i,j}(\bar{l}), z^2_{i,j}(\bar{l})\}$, such that $(C_{\bar{l}} \cap B(p_{i,j}, \epsilon''')) \cong Ann^1 \sqcup_{S_{i,j,\bar{l}}} Ann^2$, where $\{Ann^1, Ann^2\}$ are complex annuli joined along the loop $S_{i,j,\bar{l}}$, see [8]

(ii). For all $\bar{l} \in B'(\bar{l}_\infty, \epsilon''')$, and $p \in C_{\bar{l}}$, there exists a unique $w \in C_{lines}$, with $p \in B(w, \epsilon''')$, $\epsilon''' \leq \epsilon$.

(iii). For $p \in (C_{lines} \setminus \bigcup_{1 \leq i < j \leq n} B(p_{i,j}, \epsilon))$, there exists $\epsilon' > 0$, such that, for all $\bar{l} \in B'(\bar{l}_\infty, \epsilon''')$, $\epsilon''' \leq \epsilon'$, $pr_x : (C_{\bar{l}} \cap B(p, \epsilon''')) \rightarrow B(x_p, \epsilon''')$ is an isomorphism.

Observing that $|p_{i,i+1} - p_{i+1,i+2}| = 2sin(\frac{2\pi}{n})$, we let $\epsilon'' = \frac{1}{3}min(\epsilon, \epsilon')$ and $m_{\epsilon''} = [\frac{2sin(\frac{2\pi}{n}) + \epsilon''}{\epsilon''}]$. For $0 \leq i \leq n-1$, $0 \leq s \leq m_{\epsilon''} - 1$ we let $p_{i,i+1,s} = p_{i,i+1} + s(\frac{p_{i+1,i+2} - p_{i,i+1}}{m_{\epsilon''}})$, and $S = \{s : 0 \leq s \leq m_{\epsilon''} : B(p_{i,i+1,s}, \epsilon'') \cap B(p_{i,i+1}, \epsilon) = \emptyset\}$. Clearly $S \subset [0, m_{\epsilon''} - 1]$ is an interval, and we let $s_1 = min(S) - 1$, $s_2 = max(S) + 1$. Then, for the real line segment $l^{\mathcal{R}}_{p_{i,i+1}, p_{i+1,i+2}} = \{tp_{i,i+1} + (1-t)p_{i+1,i+2} : 0 \leq t \leq 1\}$, we have $l^{\mathcal{R}}_{p_{i,i+1}, p_{i+1,i+2}} \subset \bigcup_{s_1 \leq s \leq s_2} B(p_{i,i+1,s}, \epsilon'') \cup B(p_{i,i+1}, \epsilon) \cup B(p_{i+1,i+2}, \epsilon)$. We then have that, for $\bar{l} \in B'(\bar{l}_\infty, \epsilon'')$;

$$(C_{\bar{l}} \cap (\bigcup_{0 \leq i \leq n-1} [B(p_{i,i+1}, \epsilon) \cup \bigcup_{s_1 \leq s \leq s_2} B(p_{i,i+1,s}, \epsilon'')] \cong T_1^n$$

where T_1^n is an n-holed torus. Observe that, using Lemma 1.1, for $0 \leq i \leq n-1$, $2 \leq k \leq \frac{n-1}{2}$;

$$|p_{i,i+1} - p_{i,i+k}| = \frac{sin(\frac{\pi}{2}(1-\frac{2}{n})) - sin(\frac{\pi}{2}(1-\frac{2k}{n}))}{cos(\frac{\pi}{2}(1-\frac{2}{n}))sin(\frac{\pi}{2}(1-\frac{2k}{n}))}$$

For $2 \leq k \leq \frac{n-1}{2}$, we let $m_{k,\epsilon''} = \frac{|p_{i,i+k} - p_{i,i+k-1}| + \epsilon''}{\epsilon''}$, and, for $0 \leq s \leq m_{k,\epsilon''} - 1$, we let $p'_{i,i+k-1,s} = p_{i,i+k-1} + s(\frac{p_{i,i+k} - p_{i,i+k-1}}{m_{k,\epsilon''}})$. We let $S_k = \{s : 0 \leq s \leq m_{k,\epsilon''} : B(p_{i,i+k-1,s}, \epsilon'') \cap B(p_{i,i+k-1}, \epsilon) = \emptyset\}$. Again $S_k \subset [0, m_{k,\epsilon''} - 1]$ is an interval, and we let $s_{k,1} = min(S_k) - 1$, $s_{k,2} = max(S_k) + 1$. Then, for the real line segment $l^{\mathcal{R}}_{p_{i,i+k-1}, p_{i,i+k}} = \{tp_{i,i+k-1} + (1-t)p_{i,i+k} : 0 \leq t \leq 1\}$, we have $l^{\mathcal{R}}_{p_{i,i+k-1}, p_{i,i+k}} \subset \bigcup_{s_{k,1} \leq s \leq s_{k,2}} B(p_{i,i+1,s}, \epsilon'') \cup B(p_{i,i+k-1}, \epsilon) \cup B(p_{i,i+k}, \epsilon)$. It is then clear that, for $\bar{l} \in B'(\bar{l}_\infty, \epsilon'')$;

$$(C_{\bar{l}} \cap (\bigcup_{0 \leq i \leq n-1} [B(p_{i,i+1}, \epsilon) \cup B(p_{i,i+2}, \epsilon) \cup \bigcup_{s_1 \leq s \leq s_2} B(p_{i,i+1,s}, \epsilon'') \cup \bigcup_{s_{2,1} \leq s \leq s_{2,2}} B(p'_{i,i+1,s}, \epsilon'')])) \cong T^n_{1,n}$$

where $T_{1,n}$ is a torus with n attached handles, and $T^n_{1,n}$ is a $T_{1,n}$ with n-holes. Repeating the process l times, we obtain that, for $\bar{l} \in B'(\bar{l}_\infty, \epsilon'')$;

$(C_{\bar{l}}\cap(\bigcup_{0\leq i\leq n-1}[B(p_{i,i+1},\epsilon)\cup\bigcup_{2\leq k\leq 2+(l-1)}B(p_{i,i+k},\epsilon)\cup\bigcup_{s_1\leq s\leq s_2}B(p_{i,i+1,s},\epsilon'')\cup \bigcup_{2\leq k\leq 2+(l-1),s_{k,1}\leq s\leq s_{k,2}}B(p'_{i,i+k,s},\epsilon'')]))\cong T^n_{1,nl}$, ([11]).

Repeating the process $(\frac{n-1}{2}-2)+1=\frac{n-3}{2}$ times, and, using Lemma 1.14, we obtain that, for $\bar{l}\in B'(\bar{l}_\infty,\epsilon'')$;

$(C_{\bar{l}}\cap(\bigcup_{0\leq i\leq n-1}[B(p_{i,i+1},\epsilon)\cup\bigcup_{2\leq k\leq \frac{n-1}{2}}B(p_{i,i+k},\epsilon)\cup\bigcup_{s_1\leq s\leq s_2}B(p_{i,i+1,s},\epsilon'')\cup \bigcup_{2\leq k\leq \frac{n-1}{2},s_{k,1}\leq s\leq s_{k,2}}B(p'_{i,i+k,s},\epsilon'')]))\cong T^n_{1,n(\frac{n-3}{2})}=T^n_{1,g-1}$

where $g=\frac{(n-1)(n-2)}{2}$.

Finally, let $\{p_{i,\infty}:1\leq i\leq n\}$ denote the points at ∞ of C_{lines}. Changing coordinates to (x',y') with $\{p_{i,\infty}:1\leq i\leq n\}$ in finite position, say at $\{(0,y'_i):0\leq i\leq n\}$, we can assume that for all $\bar{l}\in B'(\bar{l}_\infty,\epsilon_0)$, $1\leq i\leq n$, $pr_{x'}:C_{\bar{l}}\cap B((0,y'_i),\epsilon_0)\rightarrow B(0,\epsilon_0)$ is an isomorphism, ([12]). For $\bar{l}\in B(lines,\epsilon_0)$, we let $D'_{\bar{l}}=C_{\bar{l}}\cap(pr^{-1}_{x'})(B(0,\epsilon_0))$, $D_{i,\bar{l}}=l_i\cap(pr^{-1}_{x'})(B(0,\epsilon_0))$, $D'_{i,\bar{l}}=C_{\bar{l}}\cap B((0,y'_i),\epsilon_0)$, so $D_{i,\bar{l}}\cong D'_{i,\bar{l}}$. Choose a standard $\lambda>0$ such that, for $1\leq i\leq n$, $(D(\bar{0},\lambda)\cap D_{i,\bar{l}})\neq\emptyset$, in coordinates (x,y). Then it follows, taking $\epsilon<<\epsilon_0$, that, $C_{\bar{l}}=(D(\bar{0},\lambda)\cap C_{\bar{l}})\cup\bigcup_{1\leq i\leq n}D'_{i,\bar{l}}$, for $\bar{l}\in B'(lines,\epsilon)$, $(\dagger)$. We can obviously assume that $\bigcup_{1\leq i<j\leq n}B(p_{i,j},\epsilon'')\subset D(\bar{0},\lambda)\subset D(\bar{0},\lambda+\epsilon'')$, and, hence, that;

$$\bigcup_{1\leq i\leq n,s_1\leq s\leq s_2}B(p_{i,i+1,s},\epsilon'')\cup\bigcup_{1\leq i\leq n,2\leq k\leq\frac{n-1}{2},s_{k,1}\leq s\leq s_{k,2}}B(p'_{i,i+k,s},\epsilon'')$$

$$\subset D(\bar{0},\lambda)\subset D(\bar{0},\lambda+\epsilon'')$$

As

$(\overline{D(\bar{0},\lambda+\epsilon'')}\cap l_i)\setminus[\bigcup_{1\leq i\leq n,s_1\leq s\leq s_2}(B(p_{i,i+1,s},\epsilon'')\cap l_i)\cup$

---

[11] At each stage, the loop $S_{i,j,\bar{l}}$, corresponding to the attachment of the new handle around $p_{i,j}$, should be thought of as connecting annuli on the handles corresponding to $\{p_{i,j-1},p_{i+1,j}\}$. The number of holes n is unchanged, as the loop $S_{i,j,\bar{l}}$ blocks any new passages along the surface. We then obtain a $T^n_{1,n(l-1),n}$, where $T_{1,n(l-1),n}$ is a $T_{1,n(l-1)}$ with n attached handles. Sliding these attachments to the main body, $T_{1,n(l-1),n}\cong T_{1,nl}$, giving the required $T^n_{1,nl}$.

[12] Strictly speaking, we should include this coordinate change and the fixed points at infinity in the definitions of $D(n,p_{i,j})$

$\bigcup_{1\leq i\leq n,2\leq k\leq\frac{n-1}{2},s_{k,1}\leq s\leq s_{k,2}}(B(p'_{i,i+k,s},\epsilon'')\cap l_i)]$

is compact, for each $1\leq i\leq n$, we can find a finite set Q_i, $|Q_i|=P$, with $Q_i\subset l_i\cap\overline{D(\bar{0},\lambda+\epsilon'')}$, distinct from $W_i=\{p_{i,i+1,s},p'_{i,i+k,s},p_{i,j}: j\neq i,2\leq k\leq\frac{n-1}{2},s_1\leq s\leq s_2,s_{k,1}\leq s\leq s_{k,2}\}$, such that $(l_i\cap\overline{D(\bar{0},\lambda+\epsilon'')})=\bigcup_{p\in Q_i\cup V_i}B(p,\epsilon'')\cup\bigcup_{p\in W_i\setminus V_i}B(p,\epsilon)$, where $V_i=(W_i\setminus\{p_{i,j}:j\neq i\})$, $(*)$. Using (ii), we have, as $\epsilon''<\epsilon$, that if $w\in(D(\bar{0},\lambda)\cap C_{\bar{l}})$, there exists $w'\in C_{lines}$, with $w\in B(w',\epsilon'')$. By the triangle inequality, we have that $w'\in D(0,\lambda+\epsilon'')$, hence, $w'\in(l_i\cap\overline{D(\bar{0},\lambda+\epsilon'')})$, for some $1\leq i\leq n$, $(**)$. It follows, by $(*)$, $(**)$, that $(D(\bar{0},\lambda)\cap C_{\bar{l}})\subset\bigcup_{1\leq i\leq n,p\in Q_i}(C_{\bar{l}}\cap B(p,2\epsilon''))\cup 1\leq i\leq n,p\in V_i(C_{\bar{l}}\cap B(p,\epsilon''))\bigcup\bigcup_{1\leq i\leq n,p\in W_i\setminus V_i}(C_{\bar{l}}\cap B(p,\epsilon))$. By (iii), as $2\epsilon''<\epsilon'$, we have that, for $1\leq i\leq n$, $p\in Q_i$, $pr_x:(C_{\bar{l}}\cap B(p,2\epsilon''))\rightarrow B_{x_p,2\epsilon''}$ is an isomorphism, $(***)$. Moreover, for any such disc $(C_{\bar{l}}\cap B(p,2\epsilon''))$, there exists a finite chain $\{t_i:1\leq i\leq r(p)\leq P\}$, with the property that $t_1=p$, $(C_{\bar{l}}\cap B(t_i,2\epsilon'')\cap(C_{\bar{l}}\cap B(t_{i+1},2\epsilon'')))\neq\emptyset$, $(C_{\bar{l}}\cap B(t_{r(p)},2\epsilon'')\cap(C_{\bar{l}}\cap(B(p,\epsilon)\cup B(q,\epsilon'')))\neq\emptyset$, some $p\in W_i\setminus V_i$, $q\in V_i$, $(****)$. We let;

$K_1=\bigcup_{1\leq i\leq n}W_i$

$C_{1,\bar{l}}=\bigcup_{1\leq i\leq n,p\in V_i}(C_{\bar{l}}\cap B(p,\epsilon''))\cup\bigcup_{1\leq i\leq n,p\in W_i\setminus V_i}(C_{\bar{l}}\cap B(p,\epsilon))$

and inductively, define;

$K_{j+1}=K_j\cup\bigcup_{1\leq i\leq n}\{p\in Q_i:B(p,2\epsilon'')\cap C_{j,\bar{l}}\neq\emptyset\}$

$C_{j+1,\bar{l}}=C_{j,\bar{l}}\cup\bigcup_{p\in K_{j+1}\setminus K_j}(C_{\bar{l}}\cap B(p,\epsilon''))$

By $(****)$, we have that $C_{\bar{l}}\cap B(\bar{0},\lambda)=C_{B,\bar{l}}$, for some $B\leq P$, and, by $(\dagger)$, $C_{\bar{l}}=C_{B+1,\bar{l}}=C_{B,\bar{l}}\cup\bigcup_{1\leq i\leq n}D'_{i,\bar{l}}$.

We have that $C_{j,\bar{l}}\subset C_{j+1,\bar{l}}$, for $1\leq j\leq B$, and $C_{1,\bar{l}}\cong T^n_{1,g-1}$. It follows easily, as each $C_{j,\bar{l}}\subset P^2$ is open in the complex topology, for $1\leq j\leq B$, and $C_{B+1,\bar{l}}$ is closed, nonsingular, that $C_{\bar{l}}$ is isomorphic (topologically) to $T_{1,g-1}=S_g$, where S_g is a sphere with g attached handles.

The final claim follows from the proof of the degree-genus formula, with Severi's definition of genus, see [3].

□

Remarks 2.11. *The case when n is even, is left to the reader, the idea being simply to change coordinates, so that there are no intersections* $p_{i,j} = (l_i \cap l_j)$ *at infinity, and apply the methods of Section 2.*

Remarks 2.12. *This gives an alternative proof of the (topological) degree-genus formula, see 4.1.1 of* [2]*, another proof can be found in 4.1.2 of* [2].

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