A SIMPLE PROOF OF THE FOURIER INVERSION THEOREM USING NONSTANDARD ANALYSIS

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Abstract. We give a proof of the Fourier Inversion Theorem, using the methods of nonstandard analysis.

We first make the following, which can be found in [10];

Definition 0.1. *We denote by the Schwartz space $S(\mathcal{R})$, the set of all functions $g : \mathcal{R} \rightarrow \mathcal{C}$, such that g and all its derivatives $\{g', g'', \ldots, g^{(n)}, \ldots, \}_{n\in\mathcal{N}}$ are rapidly decreasing, in the sense that;*

$sup_{x\in\mathcal{R}}|x|^k|g^{(n)}(x)| < \infty$. *(for all $k, n \geq 0$)*

For such a function g, we define its Fourier transform by;

$\hat{g}(t) = \int_{-\infty}^{\infty} g(x)e^{-\pi ixt}dx$

Remarks 0.2. *It is a well known fact that, if $g \in S(\mathcal{R})$, then its Fourier transform $\hat{g} \in S(\mathcal{R})$ as well, see* [10]*. However, this is, perhaps, not the usual definition of the Fourier transform. In* [10]*, it is given as;*

$\hat{g}(t) = \int_{-\infty}^{\infty} g(x)e^{-2\pi ixt}dx$

while, in [4]*, it is defined as;*

$\hat{g}(t) = \int_{-\infty}^{\infty} g(x)e^{-ixt}dx$

Of course, these definitions only differ by a scaling factor, but for each one you choose, you get a distinct rescaled statement of the Inversion Theorem. Once you have proved the Fourier Inversion theorem for one definition, you obtain the other statements by a simple change of variables. The reason for our choice of notation will become apparent later.

Theorem 0.3. *Fourier Inversion Theorem*

Let $g \in S(\mathcal{R})$*, then;*

$g(x) = \frac{1}{2}\int_{-\infty}^{\infty} \hat{g}(t)e^{\pi ixt}dt$ *for all* $x \in \mathcal{R}$.

Remarks 0.4. *There are many standard proofs of this result, for example in* [10]*. This is not the best statement possible. In* [4]*, the requirement that* $g \in S(\mathcal{R})$ *is weakened to* $g \in L^1(\mathcal{R}) \cap C$ *and* $\hat{g} \in L^1(\mathcal{R}) \cap C$*, where* C *denotes the space of complex valued continuous functions on* $\mathcal{R}$*. In our proof, we do not actually require that* $g \in S(\mathcal{R})$*, but we do need some assumptions about the differentiability of* g*, and also about its rate of decrease. We have chosen this assumption, mainly because the Schwartz space seems to be often used in the presentation of the Fourier Inversion Theorem.*

We now introduce the principal spaces which we are going to work with;

Definition 0.5. *Let* $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$*, and* $\omega \in {}^*\mathcal{N}$*, with* $\omega \geq n\eta$*, for* all $n \in \mathcal{N}$*. We define;*

$\overline{\mathcal{R}}_{\omega,\eta} = \{\tau \in {}^*\mathcal{R} : -\frac{\omega}{\eta} \leq \tau < \frac{\omega}{\eta}\}$

We let $\mathfrak{C}$ *be the* $*$*-finite algebra consisting of internal unions of intervals of the form* $[\frac{i}{\eta}, \frac{i+1}{\eta})$*, for* $-\omega \leq i < \omega$.

We define a counting measure on $\mathfrak{C}$ *by* $\lambda([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$.

Then $(\overline{\mathcal{R}}_{\omega,\eta}, \mathfrak{C}, \lambda)$ *is a hyperfinite measure space with* $\lambda(\overline{\mathcal{R}}_{\omega,\eta}) = \frac{2\omega}{\eta}$.

We denote by $(\overline{\mathcal{R}}_{\omega,\eta}, L(\mathfrak{C}), L(\lambda))$ *the associated Loeb space, ([1]).*

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[1] The existence of such a space follows from [5]. However, the uniqueness of the extension of $^\circ\lambda$ to $\sigma(\mathfrak{C})$ was only shown there in the case that λ is finite. Later, Ward Henson proved the uniqueness of the extended measure, even in the case that λ is infinite. After producing the extension, we are then passing to the completion, see [3].

We let $(\mathcal{R}, \mathfrak{B}, \mu)$ denote the completion of the Borel field $\mathfrak{D}$ on $\mathcal{R}$, with respect to Lebesgue measure μ, ([2]).

We let $\mathcal{R}^{+-\infty}$ denoted the extended real line $\mathcal{R} \cup \{+\infty, -\infty\}$, and let $\{g_\infty, \hat{g}_\infty\}$ be the extensions of functions in Definition 0.1, obtained by setting $g_\infty(+\infty) = g_\infty(-\infty) = 0$, and similarly for $\hat{g}_\infty$.

Lemma 0.6. *There exists a unique σ-algebra $\mathfrak{B}'$ on $\mathcal{R}^{+-\infty}$, which separates the points $+\infty$ and $-\infty$, and such that $\mathfrak{B}'|_{\mathcal{R}} = \mathfrak{B}$. Moreover, there is a unique extension of μ to a measure μ' on $\mathfrak{B}'$ with the property that $\mu(\infty) = \mu(-\infty) = \infty$. The same holds with $\mathfrak{D}$ and $\mathfrak{D}'$ replacing $\mathfrak{B}$ and $\mathfrak{B}'$. The resulting measure space $(\mathcal{R}^{+-\infty}, \mathfrak{B}', \mu')$ is the completion of $(\mathcal{R}^{+-\infty}, \mathfrak{D}', \mu')$.*

Proof. The construction of $\mathfrak{B}'$ is easy. We let $\mathfrak{B}_{+\infty}$ consist of all sets of the form $B \cup \{+\infty\}$, where $B \in \mathfrak{B}$, and, similarly, define $\mathfrak{B}_{-\infty}$ and $\mathfrak{B}_{+-\infty}$. Then, let $\mathfrak{B}' = \mathfrak{B} \cup \mathfrak{B}_{+\infty} \cup \mathfrak{B}_{-\infty} \cup \mathfrak{B}_{+-\infty}$. Clearly, $\mathfrak{B}'$ separates the points $+\infty$ and $-\infty$, moreover $\mathfrak{B}'|_{\mathcal{R}} = \mathfrak{B}$. It is a simple exercise to verify that $\mathfrak{B}$ is a σ-algebra. In order to see uniqueness, let $\mathfrak{B}''$ have these properties. As $\mathfrak{B}''|_{\mathcal{R}} = \mathfrak{B}$, we have $\mathfrak{B} \subset \mathfrak{B}''$. Choose a set B containing $+\infty$, but not $-\infty$, then $\{+\infty\} = B \cap \bigcap_{n \in \mathcal{N}} (-n, n)^c$ belongs to $\mathfrak{B}''$. Moreover $\{+\infty, -\infty\} = \mathcal{R}^{+-\infty} \setminus \mathcal{R}$ belongs to $\mathfrak{B}''$, so, $-\infty$ belongs to $\mathfrak{B}''$. Hence, $\mathfrak{B}' \subset \mathfrak{B}''$. If C belongs to $\mathfrak{B}''$, then clearly $C \cap \mathcal{R} \in \mathfrak{B}$, so it must be of the above form, that is $\mathfrak{B}' = \mathfrak{B}''$. Now define μ' by setting $\mu = \mu'$ on $\mathfrak{B}$, and letting $\mu'(C) = \infty$, for any $C \in \mathfrak{B}' \setminus \mathfrak{B}$. It is straightforward to see that μ' defines a measure, with $\mu'(\infty) = \mu'(-\infty) = \infty$, extending μ. If μ'' satisfies these properties, then as any set $C \in \mathfrak{B}' \setminus \mathfrak{B}$ contains at least one of $\{+\infty, -\infty\}$, it must be ∞ on these sets, so $\mu' = \mu''$. Exactly the same argument gives the result for $\mathfrak{D}$ and $\mathfrak{D}'$. The completeness statement follows directly as $(\mathcal{R}, \mathfrak{B}, \mu)$ is complete, and any set of measure 0, μ', in $\mathfrak{B}'$, belongs to $\mathfrak{B}$.

□

Theorem 0.7. *The standard part mapping;*

$$st : (\overline{\mathcal{R}}_{\omega,\eta}, L(\mathfrak{C}), L(\lambda)) \rightarrow (\mathcal{R}^{+-\infty}, \mathfrak{B}', \mu')$$

[2] Again, Caratheodory's Theorem provides the existence of Lebesgue measure μ on the σ-algebra $\mathfrak{D}$ generated by the open sets. Uniqueness of the extension follows easily by restricting to finite intervals.

is measurable and measure preserving. In particular, if $\{g_\infty, \hat{g}_\infty\}$ *are as in Definition 0.5, and* $\{st^*(g_\infty), st^*(\hat{g}_\infty)\}$ *are their pullbacks under st, then,* $\{st^*(g_\infty), st^*(\hat{g}_\infty)\}$ *are integrable with respect to* $L(\lambda)$, $\{g_\infty, \hat{g}_\infty\}$ *are integrable with respect to* μ', $\{g, \hat{g}\}$ *are integrable with respect to* μ, *and;*

$$\int_{\overline{\mathcal{R}}_{\omega,\eta}} st^*(g_\infty) dL(\lambda) = \int_{\mathcal{R}+-\infty} g_\infty d\mu' = \int_{\mathcal{R}} g d\mu$$

$$\int_{\overline{\mathcal{R}}_{\omega,\eta}} st^*(\hat{g}_\infty) dL(\lambda) = \int_{\mathcal{R}+-\infty} \hat{g}_\infty d\mu' = \int_{\mathcal{R}} \hat{g} d\mu$$

Proof. We let $\Sigma_0' \subset \mathfrak{B}'$ denote the sets consisting of finite unions of the form;

$$[-\infty, b_1) \cup [a_2, b_2) \cup \ldots \cup [a_r, b_r) \cup [b_{r+1}, \infty]$$

where $b_1 \leq a_2 \ldots \leq b_{r+1}$ belong to $\mathcal{R}$. It is an easy exercise to check that Σ_0 is an algebra. Let $\mathfrak{D}' \subset \mathfrak{B}'$ be the σ-algebra generated by Σ_0'. Then $\mathfrak{D}'|_{\mathcal{R}}$ is just the Borel field $\mathfrak{D}$ on $\mathcal{R}$, and by Lemma 0.6, $\mathfrak{D}'$ is obtained from $\mathfrak{D}$ by adjoining at least one of the points $\{+\infty, -\infty\}$. Then $\mathfrak{B}'$ is just the completion of $\mathfrak{D}'$ with respect to $\mu'|_{\mathfrak{D}'}$, using the definition of $\mathfrak{B}$ and the fact that $\mathfrak{B}'|_{\mathcal{R}} = \mathfrak{B}$. Now, if $a, b \in \mathcal{R}$;

$$st^{-1}([a, b)) = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} [\tfrac{[\eta(a-\frac{1}{n})]}{\eta}, \tfrac{[\eta(b-\frac{1}{m})]}{\eta})$$

$$st^{-1}([-\infty, a)) = \bigcup_{m=1}^{\infty} [\tfrac{-\omega}{\eta}, \tfrac{[\eta(a-\frac{1}{m})]}{\eta}) \ (*)$$

where $[\ ]$ denotes integer part. Observing that $\{i \in {}^*\mathcal{Z} : -\omega \leq i \leq [\eta(b - \frac{1}{m})] - 1\}$ is internal, these sets belong to $L(\mathfrak{C})$. Now consider $\{B \in \mathfrak{B}' : st^{-1}(B) \in L(\mathfrak{C})\}$. This is a σ-algebra containing $\mathfrak{D}'$ by $(*)$. In particular, $st^{-1}(-\infty)$ and $st^{-1}(+\infty)$ belong to $L(\mathfrak{C})$. Moreover;

$$L(\lambda)(st^{-1}([a, b))) = lim_{m\rightarrow\infty} lim_{n\rightarrow\infty} {}^\circ(b - a + \tfrac{1}{n} - \tfrac{1}{m}) = (b - a)$$

$$L(\lambda)(st^{-1}(+\infty)) = L(\lambda)(st^{-1}(-\infty)) = \infty \ (**)$$

In the first claim, we have used elementary properties of measures on σ-algebras and the definition of $\lambda|\mathfrak{C}$. In the second claim, we have used the fact that $st^{-1}(+\infty) \supset [\frac{\omega}{2\eta}, \frac{\omega}{\eta})$, and $L(\lambda)([\frac{\omega}{2\eta}, \frac{\omega}{\eta})) = {}^\circ(\frac{\omega}{2\eta}) = \infty$, by the choice of ω. Similarly, for $st^{-1}(-\infty)$. It follows that the push forward measure $st_*(L(\lambda))$ on $\mathfrak{D}'$, agrees with μ on the algebra $\Sigma_0|\mathcal{R}$, hence, by footnote 2, it agrees with μ on $\mathfrak{D} = \mathfrak{D}'|_{\mathcal{R}}$. By

Lemma 0.6, it agrees with μ' on $\mathfrak{D}'$. Now if $B \in \mathfrak{B}'$, we can find $C \subset B \subset D$, with C and D belonging to $\mathfrak{D}'$, such that $\mu'(D \setminus C) = 0$. Then $st^{-1}(C) \subset st^{-1}(B) \subset st^{-1}(D)$ and $L(\lambda)(st^{-1}(D \setminus C) = L(\lambda)(st^{-1}(D) \setminus st^{-1}(C)) = 0$. Hence, as $(\overline{\mathcal{R}}_{\omega,\eta}, L(\mathfrak{C}), L(\lambda))$ is complete, we have that $st^{-1}(B) \in L(\mathfrak{C})$ and $L(\lambda)(st^{-1}(B)) = L(\lambda)(st^{-1}C) = \mu'(C) = \mu'(B)$, as required. For the second part of the theorem, observe that $S(\mathcal{R}) \subset L^1(\mathcal{R})$ and use Remarks 0.2. Clearly, the extensions $\{g_\infty, \hat{g}_\infty\}$ are $\mathfrak{D}'$-measurable. Using [9](Definition 1.23), and Lemma 1.1;

$$\int_{\mathcal{R}+-\infty} g_\infty d\mu' = \int_{\mathcal{R}} g_\infty d\mu' + \int_{\{+\infty,-\infty\}} g_\infty d\mu' = \int_{\mathcal{R}} g d\mu$$

and, similarly, for $\hat{g}$. Then, it follows, using the first part of the Theorem, and Lemma 1.2, that, $\{st^*(g_\infty), st^*(\hat{g}_\infty)\}$ are integrable with respect to $L(\lambda)$, and;

$$\int_{\overline{\mathcal{R}}_{\omega,\eta}} st^*(g_\infty) dL(\lambda) = \int_{\mathcal{R}+-\infty} g_\infty d\mu'$$

and, similarly, for $st^*(\hat{g}_\infty)$.

□

We make the following;

Definition 0.8. *Let $(G, +, 0)$ be a finite commutative group, and let $(\mathcal{C}^*, \cdot, 1)$ denote the multiplicative group of complex numbers, with absolute value 1, then by a a character γ of G, we mean a homomorphism $\gamma : G \rightarrow \mathcal{C}^*$.*

Let $m, n \in \mathcal{N}_{>0}$. We let $(\mathcal{Z}_m, +, 0) = (\mathcal{Z}/m\mathcal{Z}, +, 0)$ denote the additive group of integers mod m. For $x, y \in \mathcal{Z}_m$, we let xy denote ordinary multiplication in $\mathcal{Z}$, where $\{x, y\}$ are uniquely represented in $\{0, \ldots, m-1\}$

$G_{2m} = \{-m, -(m-1), \ldots, m-1\}$ denotes the group of order $2m$, with addition given by $m_1 + m_2 = S^{m_1}(m_2)$, where S is the shift map $S(x) = x+1$ if $x \neq m-1$, $S(m-1) = -m$.

$G_{m,n} = \{\frac{-m}{n}, \frac{-(m-1)}{n}, \ldots, \frac{m-1}{n}\}$ denotes the group of order $2m$, with addition as defined for G_m. As before, for $x \in G_{m,n}$,$y \in G_{m,n}$ or $y \in \mathcal{Z}$, we let xy denote ordinary multiplication in $\mathcal{Z}$.

For a finite commutative group G, we let $\mathfrak{G}$ denote the finite σ-algebra consisting of all subsets of G, and μ_G the associated probability measure. $L^1(G)$ denotes the set of functions $g : G \rightarrow \mathcal{C}$. For $g, h \in L^1(G)$, we let $< g, h >= \int_G g\bar{h}d\mu_G$.

The following can be found in [6];

Theorem 0.9. *Let $(G, +, 0)$ be a finite commutative group of order m, then there exist exactly m characters on G, and they form an orthonormal basis of $L^1(G)$, with respect to $<, >$, ([3]). The characters on $\mathcal{Z}_m$ are given by;*

$\gamma_k(x) = exp(\frac{2\pi i}{m}kx)$ *for* $k \in \{0, 1, \ldots, m-1\}$

Definition 0.10. *Let $(G, +, 0)$ be a finite commutative group of order m, and let G_* denote its commutative group of characters, of order m, ([4]), then, if $g \in L^1(G)$, we define $\hat{g} : G_* \rightarrow \mathcal{C}$, by;*

$\hat{g}(\gamma) =< g, \gamma >= \int_G g\bar{\gamma}d\mu_G$

We then obtain;

Theorem 0.11. *Inversion Theorem for Finite Groups*

Let $\{G, G_*, g, \hat{g}\}$ be as in Definition 0.10, then;

$g(x) = \sum_{j=0}^{m-1} \hat{g}(\gamma_j)\gamma_j(x)$

where $x \in G$, and j enumerates G_*.

Proof. This is almost immediate. By Theorem 0.9;

$g = \sum_{j=0}^{m-1} < g, \gamma_j > \gamma_j$ in $L^1(G)$

Then, by Definition 0.10, and the fact that $\mu_G(x) > 0$, if $x \in G$;

[3]It is shown in [6] that the characters form an orthogonal basis with respect to the measure $m\mu_G$. However, it is then a simple computation, using the definition of a character, to see that they are an orthonormal basis with respect to the probability measure μ_G

[4] In fact, G and G_* are isomorphic, see [6].

$g(x) = \sum_{j=0}^{m-1} < g, \gamma_j > \gamma_j(x) = \sum_{j=0}^{m-1} \hat{g}(\gamma_j)\gamma_j(x)$

□

We now compute the character group on $G_{m,n}$;

Lemma 0.12. *Let $G_{m,n}$ be as in Definition 0.8, then the characters on $G_{m,n}$ are given by;*

$\gamma_y(x) = exp(\frac{\pi i n^2}{m} xy)$

where $x, y \in G_{m,n}$.

Proof. First observe that there exists an isomorphism $\phi : G_m \rightarrow \mathcal{Z}_{2m}$, defined by $\phi(x) = (x + 2m)_{mod2m}$. Hence, by Theorem 0.9, the characters on G_m are given by;

$exp(\frac{2\pi i}{2m}(x + 2m)_{mod2m} j) = exp(\frac{\pi i}{m}(x + 2m)_{mod2m} j) = exp(\frac{\pi i}{m} xj)$

where $x \in G_m$, $j \in \{0, 1, \ldots, 2m - 1\}$. Here, we have also used the facts that;

$\frac{[x+2m]_{mod2m}}{m} = \frac{x}{m}$, if $0 \leq x \leq m - 1$

$\frac{[x+2m]_{mod2m}}{m} = \frac{x}{m} + 2$, if $-m \leq x < 0$

and $exp(2\pi i) = 1$. Now writing $j = y + m$, for $y \in G_m$, we obtain that;

$exp(\frac{\pi i}{m} xj) = -exp(\frac{\pi i}{m} y) = exp(\frac{\pi i}{m}(y - m))$

Observe that the characters $exp(\frac{\pi i}{m}(y - m))$ correspond to $exp(\frac{\pi i}{m} y')$, where $y' = y - m$ belongs to $\{-m, \ldots, -1\}$ if $y \in \{0, \ldots, m - 1\}$, and correspond to $exp(\frac{\pi i}{m} y'')$, where $y'' = y + m$ belongs to $\{0, \ldots, m - 1\}$ if $y \in \{-m, \ldots, -1\}$. Hence, the characters in G_{m*} are given by;

$\gamma_y(x) = exp(\frac{\pi i}{m} xy)$ $(*)$

for $x, y \in G_m$. Now observe there exists an isomorphism $\psi : G_{m,n} \rightarrow G_m$ defined by $\psi(x) = nx$. Hence, by $(*)$, the characters in $G_{m,n*}$ are given by;

$\gamma_y(x) = exp(\frac{\pi i}{m}(nx)(ny)) = exp(\frac{\pi i n^2}{m}xy)$

for $x, y \in G_{m,n}$. $\square$

Definition 0.13. *Let $n \in \mathcal{N}_{>0}$, let $G_{n^2,n}$ be the group of order $2n^2$, as in Definition 0.8, and let $g \in L^1(G_{n^2,n})$. Let $\mathfrak{G}$ be as before, and let λ_G be the rescaled measure, given by $\lambda_G = 2n\mu_G$. Then, we define $\hat{g} \in L^1(G_{n^2,n})$ to be the function;*

$\hat{g}(t) = \int_{G_{n^2,n}} g(x)exp(-\pi ixt)d\lambda_G \ (t \in G_{n^2,n}, x \in G_{n^2,n})$

Theorem 0.14. *Inversion Theorem for $G_{n^2,n}$*

Let $\{G_{n^2,n}, \lambda_G, g, \hat{g}\}$ be as in Definition 0.13, then;

$g(x) = \frac{1}{2}\int_{G_{n^2,n}} \hat{g}(t)exp(\pi ixt)d\lambda_G \ (x \in G_{n^2,n})$

Proof. By Lemma 0.12, the characters on $G_{n^2,n}$ are given by;

$\gamma_y(x) = exp(\frac{\pi i n^2}{n^2}xy) = exp(\pi ixy) \ (*)$

for $x, y \in G_{n^2,n}$. Using Definition 0.10, and the fact that $\mu_G(x) = \frac{1}{2n^2}$, for $x \in G_{n^2,n}$, we have;

$\hat{g}(\gamma_y) = \frac{1}{2n^2}\sum_{k=-n^2}^{n^2-1} g(\frac{k}{n})exp(-\pi i\frac{k}{n}y) \ (**)$

where $y \in G_{n^2,n}$. By Theorem 0.11, $(*), (**)$ and the fact that $\lambda_G(x) = \frac{1}{n}$, for $x \in G_{n^2,n}$;

$g(x) = \sum_{l=-n^2}^{n^2-1} \hat{g}(\gamma_{\frac{l}{n}})\gamma_{\frac{l}{n}}(x)$

$= \sum_{l=-n^2}^{n^2-1} \hat{g}(\gamma_{\frac{l}{n}})exp(\pi i\frac{lx}{n})$

$= \sum_{l=-n^2}^{n^2-1}[\frac{1}{2n^2}\sum_{k=-n^2}^{n^2-1} g(\frac{k}{n})exp(-\pi i\frac{k}{n}\frac{l}{n})]exp(\pi i\frac{lx}{n})$

$= \frac{1}{2}\frac{1}{n}\sum_{l=-n^2}^{n^2-1}[\frac{1}{n}\sum_{k=-n^2}^{n^2-1} g(\frac{k}{n})exp(-\pi i\frac{k}{n}\frac{l}{n})]exp(\pi i\frac{lx}{n})$

$= \frac{1}{2}\frac{1}{n}\sum_{l=-n^2}^{n^2-1}[\int_{G_{n^2,n}} g(y)exp(-\pi iy\frac{l}{n})d\lambda_G]exp(\pi i\frac{lx}{n})$

$= \frac{1}{2}\frac{1}{n}\sum_{l=-n^2}^{n^2-1} \hat{g}(\frac{l}{n})exp(\pi i\frac{lx}{n})$

$= \frac{1}{2}\int_{G_{n^2,n}} \hat{g}(t)exp(\pi ixt)d\lambda_G$

$\square$

Definition 0.15. *We let $\overline{\mathcal{R}_\eta} = \overline{\mathcal{R}_{\eta^2,\eta}}$ and let $\{\mathfrak{C}_\eta, \lambda_\eta\}$ be as before. We let $\mathfrak{C}_\eta^2$ denote the *-finite algebra on $\overline{\mathcal{R}_\eta}^2$, consisting of internal unions of the form $[\frac{k}{\eta}, \frac{k+1}{\eta}) \times [\frac{j}{\eta}, \frac{j+1}{\eta})$, $-\eta^2 \leq k, j < \eta^2$, and λ_η^2 be the counting measure on $\mathfrak{C}_\eta^2$, defined by $\lambda_\eta^2([\frac{k}{\eta}, \frac{k+1}{\eta}) \times [\frac{j}{\eta}, \frac{j+1}{\eta})) = \frac{1}{\eta^2}$.*

We let $^*exp(\pi ixt), {^*exp}(-\pi ixt) : {^*\mathcal{R}^2} \rightarrow {^*\mathcal{C}}$ be the transfers of the functions $exp(\pi ixt), exp(-\pi ixt) : \mathcal{R}^2 \rightarrow \mathcal{C}$, and use the same notation to denote the restrictions of the transfers to $\overline{\mathcal{R}_\eta}^2$.

We let $exp_\eta(\pi ixt), exp_\eta(-\pi ixt) : \overline{\mathcal{R}_\eta}^2 \rightarrow {^*\mathcal{C}}$ denote their $\mathfrak{C}_\eta^2$-measurable counterparts, defined by;

$exp_\eta(\pi ixt) = {^*exp}(\pi i\frac{[\eta x]}{\eta}\frac{[\eta t]}{\eta})$, $(x, t) \in \overline{\mathcal{R}_\eta}^2$

and, similarly, for $exp_\eta(-\pi ixt)$. Given $f : \overline{\mathcal{R}_\eta} \rightarrow {^*\mathcal{C}}$, which is $\mathfrak{C}_\eta$-measurable, we define;

$\hat{f}_\eta(t) = \int_{\overline{\mathcal{R}_\eta}} f(x)exp_\eta(-\pi ixt)d\lambda_\eta$

so $\hat{f}_\eta : \overline{\mathcal{R}_\eta} \rightarrow {^*\mathcal{C}}$ is $\mathfrak{C}_\eta$-measurable. $(*)$

Given $g : \mathcal{R} \rightarrow \mathcal{C}$, we let $^*g : {^*\mathcal{R}} \rightarrow {^*\mathcal{C}}$ denote its transfer and its restriction to $\overline{\mathcal{R}_\eta}$. We let g_η denote its $\mathfrak{C}_\eta$-measurable counterpart, as above, and let $\hat{g}_\eta$ be as in $(*)$.

For $n \in \mathcal{N}$, we let $\mathcal{R}_n = \overline{\mathcal{R}_n} \cap \mathcal{R}$. We let $\mathfrak{C}_{n,st}$ consist of all finite unions of intervals of the form $[\frac{i}{n}, \frac{i+1}{n})$, for $-n^2 \leq i \leq n^2 - 1$. $\lambda_{n,st}$ is defined on $\mathfrak{C}_{n,st}$, by setting $\lambda_n([\frac{i}{n}, \frac{i+1}{n})) = \frac{1}{n}$.

$\{\mathfrak{C}_{n,st}^2, \lambda_{n,st}^2, exp_{n,st}(\pi ixt), exp_{n,st}(-\pi ixt)\}$ are all defined as above, restricting to $\mathcal{R}$. If $g : \mathcal{R} \rightarrow \mathcal{C}$, we similarly define, $\{g_{n,st}, \hat{g}_{n,st}\}$, (st is suggestive notation for standard). Observe that $\lambda_{n,st}$ is just the restriction of Lebesgue measure μ to $\mathfrak{C}_{n,st}$, and transfers to λ_n.

$\{exp_{n,st}(\pi ixt), exp_{n,st}(-\pi ixt), g_{n,st}, \hat{g}_{n,st}\}$ are all standard functions, which transfer to $\{exp_n(\pi ixt), exp_n(-\pi ixt), g_n, \hat{g}_n\}$.

Finally, we let $\mathfrak{C}_{n,ext}$ *denote the* σ*-algebra on* $\mathcal{R}$*, consisting of countable unions of intervals of the form* $[\frac{i}{n},\frac{i+1}{n})$*, for* $i\in\mathcal{Z}$*, and* $\lambda_{n,ext}$ *be the corresponding measure. We similarly define* $\{\mathfrak{C}^2_{n,ext},\lambda^2_{n,ext},exp_{n,ext}(\pi ixt), exp_{n,ext}(-\pi ixt)\}$

If $g:\mathcal{R}\rightarrow\mathcal{C}$*, we let* $g_{n,ext}:\mathcal{R}\rightarrow\mathcal{C}$ *be the* $\mathfrak{C}_{n,ext}$*-measurable function obtained by setting* $g_{n,ext}(x)=g(\frac{[nx]}{n})$*, so* $g_{n,ext}|_{\mathcal{R}_n}=g_{n,st}$.

We now have;

Lemma 0.16. *Inversion Theorem for* $\overline{\mathcal{R}_\eta}$

Let $\{\overline{\mathcal{R}_\eta},\lambda_\eta,f,\hat{f}_\eta\}$ *be as in Definition 0.15, then;*

$f(x)=\frac{1}{2}\int_{\overline{\mathcal{R}_\eta}}\hat{f}_\eta(t)exp_\eta(\pi ixt)d\lambda_\eta(t)\ (x\in\overline{\mathcal{R}_\eta})$

Proof. As $f(x)$ is $\mathfrak{C}_\eta$-measurable and $exp_\eta(\pi ixt)$ is $\mathfrak{C}^2_\eta$-measurable, both sides of the equation are unchanged if we replace x by $\frac{[\eta x]}{\eta}$. Now the result follows directly, by transfer, from the corresponding result for $G_{n^2,n}$, Theorem 0.14, and the definition of the internal integral $\int_{\overline{\mathcal{R}_\eta}}$ on $\overline{\mathcal{R}_\eta}$, see Definition 1.3,([5]). □

We now want to specialise the result of Lemma 0.16 to $(\overline{\mathcal{R}_\eta},L(\mathfrak{C}_\eta),L(\lambda_\eta))$, using Loeb integration theory. The problem now is to obtain the S-integrability conditions, see [3] for a definition of S-integrability.

Theorem 0.17. *Let* $g\in S(\mathcal{R})$*, then* g_η*, as given in Definition 0.15, is* S*-integrable on* $\overline{\mathcal{R}_\eta}$*. Moreover* ${}^\circ g_\eta=st^*(g_\infty)$*, everywhere* $L(\lambda_\eta)$*, and;*

${}^\circ\int_{\overline{\mathcal{R}_\eta}}g_\eta d\lambda_\eta=\int_{\overline{\mathcal{R}_\eta}}st^*(g_\infty)dL(\lambda_\eta)=\int_{\mathcal{R}}gd\mu$

Proof. We first claim that $g_{n,ext}$ is integrable μ, and $lim_{n\rightarrow\infty}||g-g_{n,ext}||_{L^1}=0$, $(*)$. In order to see this, let $\epsilon>0$ be standard, and choose $N\in\mathcal{N}\geq 2$, such that;

$\int_{-\infty}^{\infty}|g|d\mu-\int_{-N}^{N}|g|d\mu<\frac{\epsilon}{3}$

[5] If the reader is anxious about some ambiguity in transferring double sums or integrals, the important point to realise is that the * operator factors through any set of standard predicates or functions, so ${}^*\mathcal{R}\models(\forall n\in{}^*\mathcal{N})({}^*(S_{1,n}\circ S_{2,n})=({}^*S_{1,n}\circ{}^*S_{2,n}))$ if $\{{}^*S_{1,n},{}^*S_{2,n}\}$ are hyperfinite sums.

and $N > \frac{9C}{\epsilon}$. As $|g|$ is continuous on the interval $[-N, N]$, by Darboux's theorem, see [2], there exists $M \in \mathcal{N}$, such that for all $n \geq M$;

$$\int_{-N}^{N} (|g - g_{n,ext}|) d\mu < \frac{\epsilon}{3}$$

Now, for $n \in \mathcal{N}_{>0}$, using Definition 0.1;

$$\int_{|x|>N} |g_{n,ext}|(x) d\mu(x)$$

$$= \frac{1}{n}(\sum_{|j| \geq Nn+1} |g(\tfrac{j}{n})| + |g(N)|)$$

$$\leq |\frac{g(N)}{n}| + \frac{1}{n} \sum_{|j| \geq Nn+1} \frac{Cn^2}{j^2}$$

$$\leq \frac{C}{N} + Cn \int_{|x|>Nn} \frac{1}{x^2} dx$$

$$= \frac{C}{N} + \frac{2Cn}{Nn} = \frac{3C}{N} < \frac{\epsilon}{3}$$

Combining these estimates, it follows that, $g_{n,ext}$ is integrable μ, and for $n \geq M$;

$$\int_{-\infty}^{\infty} |g - g_{n,ext}| d\mu < \epsilon$$

As ϵ was arbitrary, we obtain the result $(*)$. Now, using $(*)$, choose $N_1 \in \mathcal{N}$, such that $||g\chi_{[L,N]}||_{L^1} < \frac{\epsilon}{2}$ and $||g - g_{n,ext}||_{L^1} < \frac{\epsilon}{2}$, for all $n \in \mathcal{N}_{>0}$, and $L, N \in \mathcal{Z}$, $LN \geq 0$,with $min(n, |L|, |N|) > N_1$. Then;

$$||g_{n,ext}\chi_{[L,N]}||_{L^1} \leq ||(g_{n,ext} - g)\chi_{[L,N]}||_{L^2} + ||g\chi_{L,N}||_{L^1} < \epsilon \ (**)$$

for all such $\{n, L, N\}$. We now transfer the result $(**)$. We have that;

$$\mathcal{R} \models (\forall n_{(n>N_1)})(\forall L, N_{(LN \geq 0, N_1 < |L|, |N| < n)}) \int_L^N |g_{n,st}| d\lambda_{n,st} < \epsilon$$

Hence, the corresponding statement is true in ${}^*\mathcal{R}$. In particular, if η is infinite, and $\{L, N\}$ are infinite, of the same sign, belonging to $\overline{\mathcal{R}}_\eta$, we have that;

$$\int_L^N |g_\eta| d\lambda_\eta < \epsilon$$

As ϵ was arbitrary we conclude that;

$\int_L^N |g_\eta| d\lambda_\eta \simeq 0 \ (***)$

for all infinite $\{L,N\}$, of the same sign, in $\overline{\mathcal{R}}_\eta$. Now consider the internal sequence;

$\{s_n\}_{1\leq n\leq \eta} = \{\int_{\overline{\mathcal{R}}_\eta}(|g_\eta - g_\eta \chi_{[-n,n)}|) d\lambda_\eta\}_{1\leq n\leq \eta}$

Then, by $(***)$, $s_{\omega'} \simeq 0$, for all infinite $\omega' \leq \eta$. Applying Theorem 1.5, we have that $lim_{n\rightarrow\infty}({}^{\circ}s_n) = 0$. That is;

$lim_{n\rightarrow\infty}{}^{\circ}(\int_{\overline{\mathcal{R}}_\eta} |g_\eta - g_\eta \chi_{[-n,n)}| d\lambda_\eta) = 0 \ (\dagger)$

As g is bounded by M, the same is true for g_η, hence, the functions $\{g_\eta \chi_{[-n,n)}\}$ are finite, in the sense of Definition 1.7. Applying Theorem 1.8 and $(\dagger)$, we obtain that g_η is S-integrable. Now, using the fact that $lim_{x\rightarrow\infty} g(x) = 0$, it is straightforward, using Theorem 1.4, to show that $g_\eta(x) \simeq 0$, for all infinite $x \in \overline{\mathcal{R}}_\eta$. As g is continuous, by Theorem 1.6, we have that $g_\eta(x) = {}^{*}g(\frac{[\eta x]}{\eta}) \simeq g({}^{\circ}x)$, for all finite $x \in \overline{\mathcal{R}}_\eta$. Hence, for all $x \in \overline{\mathcal{R}}_\eta$, ${}^{\circ}g_\eta(x) = st^{*}(g_\infty)(x)$. Finally, by Theorem 1.9 and Theorem 0.7;

${}^{\circ}\int_{\overline{\mathcal{R}_\eta}} g_\eta d\lambda_\eta = \int_{\overline{\mathcal{R}_\eta}} st^{*}(g_\infty) dL(\lambda_\eta) = \int_{\mathcal{R}} g d\mu$

□

The corresponding result for $\hat{g}_\eta$ is more difficult to show. We require the following;

Definition 0.18. *If $n \in \mathcal{N}$, and $g_{n,st}$ is $\mathfrak{C}_{n,st}$-measurable, we define the discrete derivative $g'_{n,st}$ by;*

$g'_{n,st}(\frac{j}{n}) = n(g_{n,st}(\frac{j+1}{n}) - g_{n,st}(\frac{j}{n})) \ (-n^2 \leq j < n^2 - 1)$

$g'_{n,st}(\frac{n^2-1}{n}) = 0$

$g'_{n,st}(x) = g'_{n,st}(\frac{[nx]}{n}) \ (x \in \mathcal{R}_n)$

and the shift $g^{sh}_{n,st}$ by;

$g^{sh}_{n,st}(\frac{j}{n}) = g_{n,st}(\frac{j+1}{n}) \ (-n^2 \leq j < n^2 - 1)$

$g^{sh}_{n,st}(\frac{n^2-1}{n}) = 0$

$g^{sh}_{n,st}(x) = g^{sh}_{n,st}(\frac{[nx]}{n})$ $(x \in \mathcal{R}_n)$

So both are $\mathfrak{C}_{n,st}$*-measurable.*

Lemma 0.19. *Discrete Calculus Lemmas*

Let $\{g_{n,st}, h_{n,st}\}$ *be* $\mathfrak{C}_{n,st}$*-measurable and let* $\{g'_{n,st}, h'_{n,st}, g^{sh}_{n,st}, h^{sh}_{n,st}\}$ *be as in Definition 0.18. Then;*

(i). $\int_{\mathcal{R}_n} g'_{n,st} d\lambda_{n,st} = g_{n,st}(\frac{n^2-1}{n}) - g_{n,st}(-n)$

(ii). $(g_{n,st}h_{n,st})' = g'_{n,st}h^{sh}_{n,st} + g_{n,st}h'_{n,st}$

(iii). $\int_{\mathcal{R}_n} g'_{n,st}h_{n,st} d\lambda_{n,st} = -\int_{\mathcal{R}_n} g^{sh}_{n,st}h'_{n,st} d\lambda_{n,st} + gh_{n,st}(\frac{n^2-1}{n}) - gh_{n,st}(-n)$

Proof. *(i).* We have, using Definition 0.18, see also Definition 1.3;

$\int_{\mathcal{R}_n} g'_{n,st} d\lambda_{n,st}$

$= \frac{1}{n}\sum_{j=-n^2}^{n^2-2} g'_{n,st}(\frac{j}{n})$

$= \frac{1}{n}\sum_{j=-n^2}^{n^2-2} n(g_{n,st}(\frac{j+1}{n}) - g_{n,st}(\frac{j}{n}))$

$= g_{n,st}(\frac{n^2-1}{n}) - g_{n,st}(-n)$

(ii). Again, by Definition 0.18;

$(gh_{n,st})'(\frac{j}{n})$

$= n(gh_{n,st}(\frac{j+1}{n}) - gh_{n,st}(\frac{j}{n}))$

$= n((g_{n,st}(\frac{j+1}{n}) - g_{n,st}(\frac{j}{n}))h_{n,st}(\frac{j+1}{n}) + g_{n,st}(\frac{j}{n})(h_{n,st}(\frac{j+1}{n}) - h_{n,st}(\frac{j}{n})))$

$= g'_{n,st}(\frac{j}{n})h^{sh}_{n,st}(\frac{j}{n}) + g_{n,st}(\frac{j}{n})h'_{n,st}(\frac{j}{n})$

$= (g'_{n,st}h^{sh}_{n,st} + g_{n,st}h'_{n,st})(\frac{j}{n})$ $(-n^2 \leq j \leq n^2 - 2)$

$(g'_{n,st}h^{sh}_{n,st} + g_{n,st}h'_{n,st})(\frac{n^2-1}{n}) = (g_{n,st}h_{n,st})'(\frac{n^2-1}{n}) = 0$

(iii). By $(i),(ii)$;

$\int_{\mathcal{R}_n}(h_{n,st}g_{n,st})'d\lambda_{n,st}$

$=\int_{\mathcal{R}_n}(h'_{n,st}g^{sh}_{n,st}+h_{n,st}g'_{n,st})d\lambda_{n,st}$

$=gh_{n,st}(\frac{n^2-1}{n})-gh_{n,st}(-n)$

□

Definition 0.20. *For $n\in\mathcal{N}$, we let $\theta_n:\mathcal{R}\rightarrow\mathcal{C}$ be defined by $\theta_n(t)=n(exp(\frac{-\pi it}{n})-1)$, and let $\beta_n:\mathcal{R}\rightarrow\mathcal{C}$ be defined by $\beta_n(t)=n(exp(\frac{\pi it}{n})-1)$. We let $\{\phi_n,\psi_n\}$ denote their $\mathfrak{C}_n$-measurable counterparts on $\mathcal{R}_n$. If $g_{n,st}$ is $\mathfrak{C}_{n,st}$-measurable, we let;*

$C_n(t)=g_{n,st}(\frac{n^2-1}{n})exp_{n,st}(-\pi i\frac{n^2-1}{n}t)-g_{n,st}(-n)exp_{n,st}(-\pi i(-n)t)$

$D_n(t)=-\frac{1}{n}g_{n,st}(-n)exp_{n,st}(\pi i\frac{t}{n})exp_{n,st}(-\pi i(-n)t).$

$C'_n(t)=-g'_{n,st}(-n)exp_{n,st}(-\pi i(-n)t)$

$D'_n(t)=-\frac{1}{n}g'_{n,st}(-n)exp_{n,st}(\pi i\frac{t}{n})exp_{n,st}(-\pi i(-n)t).$

$E_n(t)=\phi_n(t)D_n(t)-C_n(t)$

$E'_n(t)=\phi_n(t)D'_n(t)-C'_n(t)$

$F_n(t)=\psi_n(t)\phi_n(t)D_n(t)-\psi_n(t)C_n(t)+\phi_n(t)D'_n(t)-C'_n(t)$

considered as $\mathfrak{C}_{n,st}$-measurable functions.

Lemma 0.21. *Discrete Fourier transform*

Let $g_{n,st}$ be $\mathfrak{C}_{n,st}$-measurable. Then, for $t\neq 0$;

$\hat{g}_{n,st}(t)=\frac{\hat{g'}_{n,st}(t)+E_n(t)}{\psi_n(t)}=\frac{\hat{g''}_{n,st}(t)+F_n(t)}{\psi_n^2(t)}$

Proof. We have, using Lemma 0.19(iii), that;

$\hat{g'}_{n,st}(t)=\int_{\mathcal{R}_n}g'_{n,st}(x)exp_{n,st}(-\pi ixt)d\lambda_{n,st}(x)$

$=-\int_{\mathcal{R}_n}g^{sh}_{n,st}(x)exp'_{n,st}(-\pi ixt)d\lambda_{n,st}(x)+C_n(t)$

Moreover, for $-n^2 \leq j < n^2 - 1$;

$exp'_{n,st}(-\pi i\frac{j}{n}t) = n(exp_{n,st}(-\pi i\frac{j+1}{n}t) - exp_{n,st}(-\pi i\frac{j}{n}t))$

$= nexp_{n,st}(-\pi i\frac{j}{n}t)(exp_{n,st}(-\pi i\frac{t}{n}) - 1)$

$= exp_{n,st}(-\pi i\frac{j}{n}t)\phi_n(t)$.

Hence, noticing that $g^{sh}_{n,st}(\frac{n^2-1}{n}) = 0$, by Definition 0.18;

$\hat{g'}_{n,st}(t) = -\int_{\mathcal{R}_n} g^{sh}_{n,st}(x)exp_{n,st}(-\pi ixt)\phi_n(t)d\lambda_{n,st}(x) + C_n(t)$

$= -\phi_n(t)\hat{g^{sh}_{n,st}}(t) + C_n(t)$

We also have, using a change of variables, and Definition 0.18, that;

$\hat{g^{sh}_{n,st}}(t) = \int_{\mathcal{R}_n} g^{sh}_{n,st}(x)exp_{n,st}(-\pi ixt)d\lambda_{n,st}(x)$

$= \int_{\frac{1-n^2}{n}}^{n} g_{n,st}(u)exp_{n,st}(-\pi i(u - \frac{1}{n})t)d\lambda_{n,st}(u)$

$= exp_{n,st}(\pi i\frac{t}{n})(\hat{g}_{n,st}(t) - \frac{1}{n}g_{n,st}(-n)exp_{n,st}(-\pi i(-n)t))$

$= exp_{n,st}(\pi i\frac{t}{n})\hat{g}_{n,st}(t) + D_n(t)$

Therefore;

$\hat{g'}_{n,st}(t) = -\phi_n(t)exp_{n,st}(\pi i\frac{t}{n})\hat{g}_{n,st}(t) - \phi_n(t)D_n(t) + C_n(t)$

$= \psi_n(t)\hat{g}_{n,st}(t) - E_n(t)$

and by the same calculation;

$\hat{g''}_{n,st}(t) = \psi_n(t)\hat{g'}_{n,st}(t) - E'_n(t)$

$= \psi_n(t)(\psi_n(t)\hat{g}_{n,st}(t) - E_n(t)) - E'_n(t)$

$= \psi^2_n(t)\hat{g}_{n,st}(t) - F_n(t)$

Rearranging, we have that, for $t \neq 0$;

$\hat{g}_{n,st}(t) = \frac{\hat{g'}_{n,st}(t)+E_n(t)}{\psi_n(t)} = \frac{\hat{g''}_{n,st}(t)+F_n(t)}{\psi_n^2(t)}$

as required.

$\square$

Lemma 0.22. *If $g \in S(\mathcal{R})$, then the functions $\hat{g''}_{n,st}(t)$ and $F_n(t)$ are uniformly bounded, independently of n, for $n \geq 2$.*

Proof. Observing that;

$|D_n(t)| \leq \frac{1}{n}|g_{n,st}|(-n)$

$|D'_n(t)| \leq \frac{1}{n}|g'_{n,st}|(-n)$

$|\phi_n(t)| \leq 2n$, $|\psi_n(t)| \leq 2n$

$|C_n(t)| \leq |g_{n,st}|(\frac{n^2-1}{n}) + |g_{n,st}|(-n)$

$|C'_n(t)| \leq |g'_{n,st}|(-n)$

we obtain;

$|F_n(t)| \leq 6n|g_{n,st}|(-n) + 2n|g_{n,st}|(\frac{n^2-1}{n}) + 3|g'_{n,st}|(-n)$

$\leq 6n|g_{n,st}|(-n) + 2n|g_{n,st}|(\frac{n^2-1}{n}) + 3n|g_{n,st}|(\frac{1-n^2}{n}) + 3n|g_{n,st}|(-n)$

$= 9n|g_{n,st}|(-n) + 2n|g_{n,st}|(\frac{n^2-1}{n}) + 3n|g_{n,st}|(\frac{1-n^2}{n})$

As $g \in S(\mathcal{R})$, there exist a constant D_1, such that $|g(x)| \leq \frac{D_1}{|x|}$, $(x \neq 0)$. Then;

$|F_n(t)| \leq D_1(\frac{9n}{n} + 5\frac{n^2}{n^2-1}) \leq 16D_1$

We now calculate;

$|\hat{g''}_{n,st}|(t) = |\int_{\mathcal{R}_n} g''_{n,st}(x)exp_n(-\pi ixt)d\lambda_n(x)|$

$\leq \int_{\mathcal{R}_n} |g''_{n,st}|(x)d\lambda_n(x)$

$= \frac{1}{n}\sum_{j=-n^2}^{n^2-2} |g''_{n,st}|(\frac{j}{n})$

$= \frac{1}{n}(\sum_{j=-n^2}^{n^2-3} n|g'_{n,st}(\frac{j+1}{n}) - g'_{n,st}(\frac{j}{n})|) + D_n$

where $D_n = |g'_{n,st}|(\frac{n^2-2}{n})$.

$|\hat{g''}_{n,st}|(t) \leq (\sum_{j=-n^2}^{n^2-3} |g'_{n,st}(\frac{j+1}{n}) - g'_{n,st}(\frac{j}{n})|) + D_n$

Without loss of generality, we can assume that g is real valued, otherwise, take real and imaginary parts. Then, by the mean value theorem, for $-n^2 \leq j \leq n^2 - 3$;

$g'_{n,st}(\frac{j}{n}) = g'(\frac{j}{n} + c(j,n))$, where $0 < c(j,n) < \frac{1}{n}$

$|\hat{g''}_{n,st}|(t) \leq (\sum_{j=-n^2}^{n^2-3} |g'(\frac{j+1}{n} + c(j+1,n)) - g'(\frac{j}{n} + c(j,n))|) + D_n$

$= (\sum_{j=-n^2}^{n^2-3} |\int_{\frac{j}{n}+c(j,n)}^{\frac{j+1}{n}+c(j+1,n)} g''(x)dx|) + D_n$ (by the FTC)

$\leq (\sum_{j=-n^2}^{n^2-2} \int_{\frac{j}{n}+c(j,n)}^{\frac{j+1}{n}+c(j+1,n)} |g''|(x)dx) + D_n$

$= (\int_{-n+c(-n^2,n)}^{\frac{n^2-1}{n}+c(n^2-1,n)} |g''|(x)dx) + D_n$

$\leq (\int_{-n}^{n} |g''(x)|dx) + D_n \leq M + 2B$

where $M = ||g''||_{L^1(\mathcal{R})}$, and $B = ||g'||_{C(\mathcal{R})}$.

□

Lemma 0.23. *If $g \in S(\mathcal{R})$ and $\epsilon > 0$ is standard, there exists a constant $N(\epsilon) \in \mathcal{N}_{>0}$, such that for all $n > N(\epsilon)$, for all $L, L' \in \mathcal{N}$ with $N(\epsilon) < |L| \leq |L'| \leq n$, $LL' > 0$;*

$\int_L^{L'} |\hat{g}_{n,st}|(t)d\lambda_n(t) < \epsilon$

Proof. We first calculate;

$|\psi_n(t)| = n|exp(\frac{\pi it}{n}) - 1|$

$= n|cos(\frac{\pi t}{n}) + isin(\frac{\pi t}{n}) - 1|$

$= n((cos(\frac{\pi t}{n}) - 1)^2 + sin(\frac{\pi t}{n})^2)^{\frac{1}{2}}$

$= n((2 - 2cos(\frac{\pi t}{n}))^{\frac{1}{2}})$

$= \sqrt{2}n(2sin^2(\frac{\pi t}{2n}))^{\frac{1}{2}}$

$= 2n|sin(\frac{\pi t}{2n})| \geq 2n(\frac{|t|}{n}) = 2|t|\ (-n \leq t < n)$

$|\psi_n(t)|^2 \geq 4t^2\ (-n \leq t < n)\ (*)$

Letting W denote the bound obtained in Lemma 0.22, using Lemma 0.21, $(*)$, and, assuming, without loss of generality, that $0 \leq L \leq L'$;

$\int_L^{L'} |\hat{g}|_{n,st}(t)d\lambda_n(t)$

$\leq \int_L^n |\hat{g}|_{n,st}(t)d\lambda_n(t)$

$\leq \int_L^n \frac{W}{4t^2}d\lambda_n(t)$

$= \frac{1}{n}\sum_{j=Ln}^{n^2-1} \frac{W}{4(\frac{j}{n})^2}$

$= n\sum_{j=Ln}^{n^2-1} \frac{W}{4j^2}$

$\leq n\int_{Ln-1}^{n^2-1} \frac{W}{4x^2}dx$

$= n[\frac{-W}{4x}]_{Ln-1}^{n^2-1} = \frac{Wn}{4}(\frac{1}{Ln-1} - \frac{1}{n^2-1}) \leq \frac{W}{4}(\frac{1}{L-1} + \frac{1}{n-1}) < \epsilon$

if $min(n, L) > N(\epsilon) = \frac{W}{2\epsilon} + 1$

□

We can now show the analogous result to Theorem 0.17;

Theorem 0.24. *Let $g \in S(\mathcal{R})$, then $\hat{g}_\eta$, as given in Definition 0.15, is S-integrable on $\overline{\mathcal{R}_\eta}$. Moreover ${}^\circ\hat{g}_\eta = st^*(\hat{g}_\infty)$, almost everywhere $L(\lambda_\eta)$, and;*

$${}^\circ\int_{\overline{\mathcal{R}_\eta}} \hat{g}_\eta d\lambda_\eta = \int_{\overline{\mathcal{R}_\eta}} st^*(\hat{g}_\infty)dL(\lambda_\eta) = \int_{\mathcal{R}} \hat{g}d\mu$$

Proof. By Lemma 0.23;

$$\mathcal{R} \models (\forall n_{(n>N(\epsilon))})(\forall L, N_{(LN\geq 0, N(\epsilon)<|L|,|N|<n)}) \int_L^N |\hat{g}_{n,st}|d\lambda_{n,st} < \epsilon$$

Hence, the corresponding statement is true in ${}^*\mathcal{R}$. In particular, if η is infinite, and $\{L, N\}$ are infinite, of the same sign, belonging to $\overline{\mathcal{R}_\eta}$, we have that;

$\int_L^N |\hat{g}_\eta| d\lambda_\eta < \epsilon$

As ϵ was arbitrary we conclude that;

$\int_L^N |\hat{g}_\eta| d\lambda_\eta \simeq 0$ $(*)$

for all infinite $\{L, N\}$, of the same sign, in $\overline{\mathcal{R}}_\eta$. Now, using Definition 0.15 and the fact that $|exp_\eta(-\pi ixt)| \leq 1$, by transfer, we have, for $t \in \overline{\mathcal{R}}_\eta$;

$|\hat{g}_\eta(t)| \leq \int_{\overline{\mathcal{R}_\eta}} |g_\eta(x) d\lambda_\eta = C$

where C is finite, as, by Theorem 0.17, g_η is S-integrable. It follows that for $n \in \mathcal{N}$, the functions $\hat{g}_\eta \chi_{[-n,n]}$ are finite, in the sense of Definition 1.7. Now, proceeding as in Theorem 0.17, we obtain that $\hat{g}_\eta$ is S-integrable. If $t \in \mathcal{R}_\eta$, the function $r_t(x) = g_\eta(x)exp_\eta(-\pi ixt)$ is S-integrable, by Theorem 1.8(i), as $|r_t| \leq |g_\eta|$, and g_η is S-integrable, by Theorem 0.17. Then, if t is finite, we have;

$^\circ\hat{g}_\eta(t) = {}^\circ \int_{\overline{\mathcal{R}_\eta}} g_\eta(x) exp_\eta(-\pi ixt) d\lambda_\eta(x)$

$= \int_{\overline{\mathcal{R}_\eta}} {}^\circ g_\eta(x) {}^\circ exp_\eta(-\pi ixt) dL(\lambda_\eta)(x)$

$= \int_{x finite} st^*(g_\infty)(x) exp_\eta(-\pi i {}^\circ x {}^\circ t) dL(\lambda_\eta)(x)$

$= \int_{x finite} st^*(g_\infty exp_{-\pi i {}^\circ t})(x) dL(\lambda_\eta)(x)$

$= \int_{\mathcal{R}} g(x) exp(-\pi i {}^\circ t x) d\mu(x) = \hat{g}({}^\circ t) = st^*(\hat{g}_\infty)(t)$ $(**)$

using Definition 0.15, Theorem 1.9, Theorem 0.17, continuity of exp, see Theorem 1.6, and Theorem 0.7. Now suppose there exists $B \in L(\mathfrak{C}_\eta)$, with $L(\lambda_\eta)(B) > 0$, such that $^\circ\hat{g}_\eta \neq st^*(\hat{g}_\infty)$ on B. Then, by $(**)$, we can assume that $B \subset st^{-1}(\{-\infty, +\infty\})$, and $|^\circ\hat{g}_\eta| > 0$ on B. We can, therefore, suppose that there exists a standard $n \in \mathcal{N}_{>0}$, with $|^\circ\hat{g}_\eta| > \frac{1}{n}$, on B. Then for all finite t', using [3](Theorem 1.32);

$^\circ \int_{|t|>t'} |\hat{g}_\eta|(t) d\lambda_\eta(t)$

$\geq \int_{|t|>t'} |^{\circ}\hat{g}_\eta|(t)dL(\lambda_\eta)(t) > \frac{1}{n}L(\lambda_\eta)(B)$

By the Overflow principle, see [3], we can find an infinite L such that;

$\int_{|t|>L} |\hat{g}_\eta|(t)d\lambda_\eta)(t) > \frac{1}{2n}L(\lambda_\eta)(B)$

This contradicts $(*)$. Hence $^{\circ}\hat{g}_\eta = st^*(\hat{g}_\infty)$ a.e $L(\lambda_\eta)$. The rest of the proof is the same as Theorem 0.17. □

Finally, we have;

Theorem 0.25. *For $g \in S(\mathcal{R})$, the Fourier Inversion Theorem holds and admits a non standard proof.*

Proof. By Lemma 0.16, we have that;

$g_\eta(x) = \frac{1}{2}\int_{\overline{\mathcal{R}_\eta}} \hat{g}_\eta(t)exp_\eta(\pi ixt)d\lambda_\eta(t)$ $(*)$

for $x \in \overline{\mathcal{R}_\eta}$. As in Theorem 0.24, the function $s_x(t) = \hat{g}_\eta(t)exp_\eta(\pi ixt)$ is S-integrable, because, by the same theorem, $\hat{g}_\eta$ is S-integrable. We now argue as before, and use the result that $^{\circ}g_\eta = st^*(\hat{g}_\infty)$, a.e $L(\lambda_\eta)$. We have, if x is standard, taking standard parts in $(*)$;

$g(x) = {}^{\circ}g_\eta(x) = \frac{1}{2}\int_{\overline{\mathcal{R}_\eta}} {}^{\circ}\hat{g}_\eta(t){}^{\circ}exp_\eta(\pi ixt)dL(\lambda_\eta)(t)$

$= \frac{1}{2}\int_{t finite} st^*(\hat{g}_\infty)(t)exp_\eta(\pi ix{}^{\circ}t)dL(\lambda_\eta)(t)$

$= \frac{1}{2}\int_{t finite} st^*(\hat{g}_\infty exp_{\pi ix})(t)dL(\lambda_\eta)(t)$

$= \frac{1}{2}\int_{\mathcal{R}} \hat{g}(t)exp(\pi ixt)d\mu(t)$

as required.

□

1. Appendix

We collect, here, some results in standard and nonstandard analysis which are required in the main proof.

Lemma 1.1. *If $(X, \mathfrak{M}, \mu) \subset (X, \mathfrak{M}', \mu')$ as standard measure spaces. Then, if g is $\mathfrak{M}$-measurable, and integrable with respect to $(X, \mathfrak{M}, \mu)$, then g is integrable with respect to $(X, \mathfrak{M}', \mu')$, and;*

$\int_B g d\mu = \int_B g d\mu'$

for any $B \in \mathfrak{M}$.

Proof. First check the result for $\mathfrak{M}$-measurable simple functions, $(*)$. Then, without loss of generality, assume $g \geq 0$. g can be written as in increasing limit of simple $\mathfrak{M}$-measurable functions, see [9](Theorem 1.17). Now apply the Monotone Convergence Theorem, see [9](Theorem 1.26), and $(*)$, to obtain the result. □

Lemma 1.2. *Change of Variables If* $\tau : (X_1, \mathfrak{C}_1, \mu_1) \rightarrow (X_2, \mathfrak{C}_2, \mu_2)$ *is measurable and measure preserving, so* $\mu_2 = \tau_* \mu_1$*, then a function* $\theta \in L^1(X_2, \mathfrak{C}_2, \mu_2)$ *iff* $\tau^* \theta \in L^1(X_1, \mathfrak{C}_1, \mu_1)$ *and then;*

$\int_C \theta d\tau_* \mu_1 = \int_{\tau^{-1}(C)} \tau^* \theta d\mu_1$

for $C \in \mathfrak{C}_2$.

Proof. This is a simple exercise, using the abstract definition of integration on measure spaces, see [9]. □

Definition 1.3. *If* X *is a hyperfinite interval, that is* $X = \{x \in {}^*\mathcal{R} : \frac{a}{\eta} \leq x < \frac{b}{\eta}\}$*, where* $a, b \in {}^*\mathcal{Z}$*,* $\mathfrak{A}$ *is the set of all internal unions of intervals of the form* $[\frac{j}{\eta}, \frac{j+1}{\eta})$*, where* $a \leq j < b$*,* $j \in {}^*\mathcal{Z}$ *and* ν *is the counting measure given by* $\nu([\frac{j}{\eta}, \frac{j+1}{\eta})) = \frac{1}{\eta}$*, then the internal integral takes the form;*

$\int_X f d\nu = \frac{1}{\eta} \sum_{j=a}^{b-1} f(\frac{j}{\eta})$

Some authors, see [3]*, prefer to use a discrete version of the hyperfinite interval, in which* $X = \{\frac{j}{\eta} : a \leq j < b, j \in {}^*\mathcal{Z}\}$*,* $\mathfrak{A}$ *is the set of internal subsets, and* ν *is the counting measure given by* $\nu(x) = \frac{1}{\eta}$*, for* $x \in X$*. Of course the two interpretations are equivalent and the internal integral takes the same form.*

Theorem 1.4. *Let* $(s_n)_{n \in \mathcal{N}}$ *be a standard infinite sequence, then the following are equivalent;*

(i). $lim_{n \rightarrow \infty} s_n = s$.

(ii). $s_n \simeq s$ *for all infinite* n.

Proof. See [8]. □

The following is a slight generalisation of Theorem 3.5.13 of [8];

Theorem 1.5. *Let $(s_n)_{n\in {}^*\mathcal{N}}$ be an internal sequence, enumerated by an internal g, not necessarily the transfer of a standard one. Suppose there exists an infinite $\omega \in {}^*\mathcal{N}$ with $s_{\omega'} \simeq 0$, for all infinite ω' with $\omega' < \omega$, then;*

$lim_{n\rightarrow\infty}({}^\circ s_n) = 0$ in the standard sense.

Proof. Let $\epsilon > 0$ be standard, then $|g(\omega')| < \epsilon$ for all infinite ω' with $\omega' < \omega$. Let;

$A = \{m \in {}^*\mathcal{N} : |g(n)| < \epsilon, if\ m \leq n < \omega\}$

Then A is internal and contains arbitrarily small positive infinite numbers. By the underflow principle, see [3], it contains a positive finite number $m_0 \in \mathcal{N}$. In particular, $|g(n)| < \epsilon$, for all $n \in \mathcal{N}$, with $n \geq m_0$. It follows that $|{}^\circ(g(n))| \leq \epsilon$, for all $n \in \mathcal{N}$, with $n \geq m_0$. Hence, as ϵ was arbitrary, the result follows. □

Theorem 1.6. *Let $f : \mathcal{R} \rightarrow \mathcal{R}$ be a standard function, and $b \in \mathcal{R}$, then the following are equivalent;*

(i). f is continuous at b.

(ii). ${}^*f(x) \simeq {}^*f(b)$ for all $x \in \mu(b)$.

where $\mu(b)$ is the monad of b. ([6]).

Definition 1.7. *Anderson*

Let $(X, \mathfrak{A}, \nu)$ be an internal measure space, in the sense of [5], *and let $f : X \rightarrow {}^*\mathcal{R}$ be $\mathfrak{A}$-measurable. Then we say f is finite if;*

(i). There exists an $n \in \mathcal{N}$, with $|f(x)| < n$, for all $x \in X$.

[6] This is often applied in the following form; if $x \simeq y$ belong to ${}^*\mathcal{R}_{fin}$, then ${}^*f(x) \simeq {}^*f(y)$. This follows from the facts that x and y have a standard part in $\mathcal{R}$, and $\simeq$ is an equivalence relation.

(ii). f is supported on a set A with $\nu(A)$ finite.

We have;

Theorem 1.8. *Anderson's Criteria*

Let $(X,\mathfrak{A},\nu)$ be as in Definition 1.7 and let $f : X \rightarrow^{*} \mathcal{R}$ be $\mathfrak{A}$-measurable.

(i). If F is S-integrable, with $|f| \leq F$, then f is S-integrable.

(ii). If $\mathfrak{A}$ is a $^{*}\sigma$-algebra, then f is S-integrable iff there exists a sequence of finite functions $(f_n)_{n\in\mathcal{N}}$ such that;

$^{\circ}(\int_X |f - f_n| d\nu) \rightarrow 0$ *as* $n \rightarrow \infty$.

Proof. See [1].

□

Theorem 1.9. *Let $(X,\mathfrak{A},\nu)$ be as in Definition 1.7, let $(X, L(\mathfrak{A}), L(\nu))$ be the corresponding Loeb space and let $f : X \rightarrow^{*} \mathcal{R}$ be $\mathfrak{A}$-measurable. Then the following are equivalent;*

(i). f is S-integrable.

(ii). $^{\circ}f$ is integrable with respect to $L(\nu)$, and;

$^{\circ}\int_A f d\nu = \int_A {^{\circ}f} dL(\nu)$ *for any* $A \in \mathfrak{A}$.

Proof. See [1] or Theorem 3.24 of [7]. □

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