A SIMPLE PROOF OF THE UNIFORM CONVERGENCE OF FOURIER SERIES USING NONSTANDARD ANALYSIS

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Abstract. We give a proof of the uniform convergence of Fourier series, using the methods of nonstandard analysis.

We use the following notation;

Definition 0.1. *We let $\mathcal{S}$ denote the circle of radius $\frac{1}{\pi}$, which we will model by the real interval $[-1,1]$, with the endpoints $\{-1,1\}$ identified. We denote by $C^\infty(\mathcal{S})$, the set of all functions $g:\mathcal{S}\rightarrow\mathcal{C}$, which are infinitely differentiable, in the sense of real manifolds. Equivalently, using L'Hospital's Rule, $C^\infty(\mathcal{S})$ consists of the functions $g:(-1,1)\rightarrow\mathcal{C}$, which are infinitely differentiable, and such that g and all its derivatives $\{g^{(n)}:n\in\mathcal{N}\}$ extend to continuous functions on $\mathcal{S}$.*

For such a function g, we define its m'th Fourier coefficient, where $m\in\mathcal{Z}$, by;

$$\hat{g}(m)=\int_{\mathcal{S}}g(x)e^{-\pi ixm}dx=\int_{-1}^{1}g(x)e^{-\pi ixm}dx$$

Theorem 0.2. *Uniform Convergence of Fourier Series*

Let $g\in C^\infty(\mathcal{S})$, then;

$$g(x)=\frac{1}{2}\sum_{m\in\mathcal{Z}}\hat{g}(m)e^{\pi ixm}\ \text{for all}\ x\in\mathcal{S}$$

where the infinite sum is considered as $lim_{n\rightarrow\infty}S_n(g)$, $n\in\mathcal{N}$, for the finite sum $S_n(g)=\sum_{m=-n}^{m=n}\hat{g}(m)e^{\pi ixm}$. Moreover, the convergence is uniform on $\mathcal{S}$.

Remarks 0.3. *The result of Theorem 0.2 generalises immediately, to obtain, if $g\in C^\infty([a,b])$, (with $L=b-a>0$ and the obvious extension of the above definition), that;*

$g(x) = \sum_{m \in \mathcal{Z}} \hat{g}(m) e^{2\pi i \frac{xm}{L}}$ *(uniform convergence)*

where, for $m \in \mathcal{Z}$;

$\hat{g}(m) = \frac{1}{L} \int_a^b g(x) e^{-2\pi i \frac{xm}{L}} dx$

This is achieved by a simple change of variables. Observe that this result also demonstrates that the functions $\{e_n(x) = e^{2\pi i \frac{xm}{L}} : m \in \mathcal{Z}\}$ *form an orthonormal "basis" of* $C^\infty([a,b]) \subset L^2([a,b])$*, with respect to the (normalised by* $\frac{1}{L}$*) inner product on* $L^2([a,b])$*. The more general result that they form an orthonormal basis of* $L^2([a,b]$ *follows from the fact that* $C^\infty([a,b])$ *is dense in* $L^2([a,b])$*. Proofs of these results (without using nonstandard analysis) can be found in* [7] *and* [3].

In order to prove Theorem 0.2, we follow the strategy of [5].

Definition 0.4. *Let* $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$*, then we define;*

$\overline{S}_\eta = \{\tau \in {}^*\mathcal{R} : -1 \leq \tau < 1\}$

As in Definition 0.5 of [5], $\mathfrak{C}_\eta$ *is the* $*$*-finite algebra consisting of internal unions of intervals of the form* $[\frac{i}{\eta}, \frac{i+1}{\eta})$*, for* $-\eta \leq i < \eta$, $i \in {}^*\mathcal{N}$. λ_η *is the counting measure on* $\mathfrak{C}_\eta$ *given by* $\lambda_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$*. Then* $(\overline{S}_\eta, \mathfrak{C}_\eta, \lambda_\eta)$ *is a hyperfinite measure space with* $\lambda_\eta(\overline{S}_\eta) = 2$*. We denote by* $(\overline{S}_\eta, L(\mathfrak{C}_\eta), L(\lambda_\eta))$*, the associated Loeb space. The existence of such a space and the extension of* ${}^\circ\lambda$ *to* $\sigma(\mathfrak{C}_\eta)$ *is shown in* [4]*. After producing the extension, we are then passing to the completion.*

We let $([0,1], \mathfrak{B}, \mu)$ *denote the completion of the restriction of the Borel field on* $\mathcal{R}$ *to* $[0,1]$*, with respect to Lebesgue measure* μ*. We let* $(S, \mathfrak{B}, \mu)$ *denote the obvious corresponding measure space on* S*, so the identification map* $G : [-1,1] \rightarrow S$ *is continuous, measurable and measure preserving.*

Lemma 0.5. *The standard part mapping;*

$st : (\overline{S}_\eta, L(\mathfrak{C}_\eta), L(\lambda_\eta)) \rightarrow ([-1,1], \mathfrak{B}, \mu)$

is measurable and measure preserving. In particular, if $g \in C^\infty(S)$*, then* $(G \circ st)^*(g)$ *is integrable with respect to* $L(\lambda_\eta)$ *and;*

$\int_{\overline{S}_\eta}(G\circ st)^*(g)dL(\lambda_\eta)=\int_{-1}^1 G^*(g)d\mu=\int_S gd\mu$

Proof. Using the proof of Theorem 0.7 in [5], or Theorem 14 of [1], and the remark in Definition 0.4. □

We recall Definition 0.8, Theorem 0.9, Definition 0.10, Theorem 0.11 and Lemma 0.12 from [5].

Definition 0.6. *Let $n\in\mathcal{N}_{>0}$, let $\mathfrak{G}$, $G_{n,n}$ be as in Definition 0.8 of* [5]*, and $g\in L^1(G_{n,n})$, with respect to the probability measure μ_G. Let λ_G be the rescaled measure, given by $\lambda_G=2\mu_G$. For $m\in\mathcal{N}$, with $-n\leq m\leq n-1$, we define the m'th discrete Fourier coefficient $\hat{g}(m)\in\mathcal{C}$ by;*

$\hat{g}(m)=\int_{G_{n,n}} g(x)exp(-\pi ixm)d\lambda_G\ (x\in G_{n,n})$

We then have;

Theorem 0.7. *Inversion Theorem for $G_{n,n}$*

Let $\{G_{n,n},\lambda_G,g,\hat{g}(m)\}$ be as in Definition 0.6, then;

$g(x)=\frac{1}{2}\sum_{-n\leq m\leq n-1}\hat{g}(m)exp(\pi ixm)$

Proof. By Lemma 0.12 of [5], the characters on $G_{n,n}$ are given by;

$\gamma_y(x)=exp(\frac{\pi in^2}{n}xy)=exp(\pi inxy)\ (*)$

for $x,y\in G_{n,n}$. Using Definition 0.10 of [5], and the fact that $\mu_G(x)=\frac{1}{2n}$, for $x\in G_{n,n}$, we have;

$\hat{g}(\gamma_y)=\frac{1}{2n}\sum_{w\in G_{n,n}} g(w)exp(-\pi inwy)\ (**)$

By Theorem 0.11 of [5], $(*)$, $(**)$ and the fact that $\lambda_G(x)=\frac{1}{n}$, for $x\in G_{n,n}$;

$g(x)=\sum_{y\in G_{n,n}}\hat{g}(\gamma_y)\gamma_y(x)$

$=\sum_{y\in G_{n,n}}\hat{g}(\gamma_y)exp(\pi inxy)$

$= \frac{1}{2}\sum_{y\in G_{n,n}}(\frac{1}{n}\sum_{w\in G_{n,n}} g(w)exp(-\pi inwy))exp(\pi inxy)$

$= \frac{1}{2}\sum_{-n\leq m\leq n-1}(\frac{1}{n}\sum_{w\in G_{n,n}} g(w)exp(-\pi iwm))exp(\pi ixm)$ $(y=\frac{m}{n})$

$= \frac{1}{2}\sum_{-n\leq m\leq n-1}\hat{g}(m)exp(\pi ixm)$

□

Definition 0.8. *We adopt similar notation to Definition 0.15 of* [5]. $(\overline{\mathcal{S}}_\eta, \mathfrak{C}_\eta, \lambda_\eta)$ *are as in Definition 0.4. We define;*

$\overline{\mathcal{Z}}_\eta = \{m \in {}^*\mathcal{N} : -\eta \leq m \leq \eta - 1\}$

$\mathfrak{D}_\eta$ *will denote the* **-finite algebra on* $\overline{\mathcal{Z}}_\eta$*, consisting of internal subsets, with counting measure* δ_η*, defined by* $\delta_\eta(m) = 1$*. We let* $\mathfrak{E}_\eta$ *denote the* **-finite algebra on* $\overline{\mathcal{S}_\eta}\times\overline{\mathcal{Z}_\eta}$*, consisting of internal unions of the form* $[\frac{i}{\eta}, \frac{i+1}{\eta}) \times m$*,* $-\eta \leq i, m \leq \eta - 1$*, and let* ϵ_η *be the counting measure* $\lambda_\eta \times \delta_\eta$*.*

We let $^*exp(\pi ixt), {}^*exp(-\pi ixt) : {}^*\mathcal{S} \times {}^*\mathcal{Z} \rightarrow {}^*\mathcal{C}$ *be the transfers of the functions* $exp(\pi ixt), exp(-\pi ixt) : \mathcal{S} \times \mathcal{Z} \rightarrow \mathcal{C}$*, and use the same notation to denote the restrictions of the transfers to* $\overline{\mathcal{S}_\eta} \times \overline{\mathcal{Z}}_\eta$*.*

We let $exp_\eta(\pi ixt), exp_\eta(-\pi ixt) : \overline{\mathcal{S}_\eta} \times \overline{\mathcal{Z}_\eta} \rightarrow {}^*\mathcal{C}$ *denote their* $\mathfrak{E}_\eta$*-measurable counterparts, defined by;*

$exp_\eta(\pi ixt) = {}^*exp(\pi i\frac{[\eta x]}{\eta}t)$, $(x, t) \in \overline{\mathcal{S}_\eta} \times \overline{\mathcal{Z}_\eta}$

and, similarly, for $exp_\eta(-\pi ixt)$*. Given* $f : \overline{\mathcal{S}_\eta} \rightarrow {}^*\mathcal{C}$*, which is* $\mathfrak{C}_\eta$*-measurable, we define the nonstandard m'th Fourier coefficient of* f*, for* $m \in \overline{\mathcal{Z}}_\eta$*, by;*

$\hat{f}_\eta(m) = \int_{\overline{\mathcal{S}_\eta}} f(x)exp_\eta(-\pi ixm)d\lambda_\eta$

so $\hat{f}_\eta : \overline{\mathcal{Z}_\eta} \rightarrow {}^*\mathcal{C}$ *is* $\mathfrak{D}_\eta$*-measurable.* $(*)$

Given $g : [-1, 1] \rightarrow \mathcal{C}$*, we let* $^*g : {}^*[-1, 1] \rightarrow {}^*\mathcal{C}$ *denote its transfer and its restriction to* $\overline{\mathcal{S}}_\eta$*. We let* g_η *denote its* $\mathfrak{C}_\eta$*-measurable counterpart, as above, and let* $\hat{g}_\eta$ *be as in* $(*)$*.*

For $n \in \mathcal{N}$*, we let* $\mathcal{S}_n = [-1, 1)$*. We let* $\mathfrak{C}_{n,st}$ *consist of all finite unions of intervals of the form* $[\frac{i}{n}, \frac{i+1}{n})$*, for* $-n \leq i \leq n - 1$*.* $\lambda_{n,st}$ *is* *defined on* $\mathfrak{C}_{n,st}$*, by setting* $\lambda_n([\frac{i}{n},\frac{i+1}{n})) = \frac{1}{n}$*. We let;*

$\mathcal{Z}_n = \{m \in \mathcal{N} : -n \leq m \leq n-1\}$

$\{\mathfrak{D}_{n,st}, \delta_{n,st}, \epsilon_{n,st}, exp_{n,st}(\pi ixt), exp_{n,st}(-\pi ixt)\}$ *are all defined as above, restricting to* $[-1,1)$ *and* $\mathcal{Z}$*. If* $g : [-1,1] \rightarrow \mathcal{C}$*, we similarly define,* $\{g_{n,st}, \hat{g}_{n,st}\}$*, (st is suggestive notation for standard). Observe that* $\lambda_{n,st}$ *is just the restriction of Lebesgue measure* μ *to* $\mathfrak{C}_{n,st}$*, and transfers to* λ_n*.*

$\{exp_{n,st}(\pi ixt), exp_{n,st}(-\pi ixt), g_{n,st}, \hat{g}_{n,st}\}$ *are all standard functions, which transfer to* $\{exp_n(\pi ixt), exp_n(-\pi ixt), g_n, \hat{g}_n\}$*.*

We now obtain;

Lemma 0.9. *Inversion Theorem for* $\overline{\mathcal{S}}_\eta$

Let $\{\overline{\mathcal{S}}_\eta, \overline{\mathcal{Z}}_\eta, f, \hat{f}_\eta\}$ *be as in Definition 0.8, then;*

$f(x) = \frac{1}{2}\sum_{m\in\overline{\mathcal{Z}}_\eta} \hat{f}_\eta(m) exp_\eta(\pi ixm)\ (x \in \overline{\mathcal{S}}_\eta)$

Proof. As $f(x)$ and $exp_\eta(\pi ixm)$ are both $\mathfrak{C}_\eta$-measurable, the two sides of the equation are unchanged if we replace x by $\frac{[\eta x]}{\eta}$. Now the result follows, by transfer, from Theorem 0.7, the definition of the internal integral $\int_{\overline{\mathcal{S}}_\eta}$ on $\overline{\mathcal{S}}_\eta$, see Definition 1.3 of [5], and the definition of a hyperfinite sum $\sum_{m\in\overline{\mathcal{Z}}_\eta}$ on $\overline{\mathcal{Z}}_\eta$, see Definition 2.19 of [6]. Again, the reader should consider footnote 5 of [5]. □

We now specialise the result of Lemma 0.9 to $(\overline{\mathcal{S}}_\eta, L(\mathfrak{C}_\eta), L(\lambda_\eta))$ and $(\overline{\mathcal{Z}}_\eta, L(\mathfrak{D}_\eta), L(\delta_\eta))$. As in [5], the problem is to obtain the S-integrability conditions.

Theorem 0.10. *Let* $g \in C^\infty(S)$*, and let* g *also denote the pullback* $\Gamma^*(g) \in C^\infty([0,1])$*, then* g_η*, as given in Definition 0.8, is* S*-integrable on* $\overline{\mathcal{S}}_\eta$*. Moreover,* ${}^\circ g_\eta = st^*(g)$*, everywhere* $L(\lambda_\eta)$*, and;*

${}^\circ\int_{\overline{\mathcal{S}}_\eta} g_\eta d\lambda_\eta = \int_{\overline{\mathcal{S}}_\eta} st^*(g) dL(\lambda_\eta) = \int_S g d\mu$

Proof. The proof is essentially contained in Theorem 0.17 of [5], but we give it, here, for the convenience of the reader. As g is continuous on the interval $[-1,1)$, by Darboux's Theorem, see [2], there exists $M \in \mathcal{N}$, such that for all $n \geq M$;

$|\int_{-1}^{1} g d\mu - \int_{-1}^{1} g_{n,st} d\lambda_{n,st}| < \epsilon$ $(*)$

Transferring the result $(*)$, see [5] (Theorem 0.17), and using Lemma 0.5, we obtain;

$\int_{\overline{\mathcal{S}_\eta}} g_\eta d\lambda_\eta \simeq \int_S g d\mu = \int_{\overline{\mathcal{S}_\eta}} st^*(g) dL(\lambda_\eta)$ $(**)$

As g is continuous on $[-1,1]$, by [5] (Theorem 1.6), we have $g_\eta(x) = {}^*g(\frac{[\eta x]}{\eta}) \simeq g({}^\circ x)$, for all $x \in \overline{\mathcal{S}}_\eta$. Hence, ${}^\circ g_\eta = st^*(g)$ on $\overline{\mathcal{S}}_\eta$, $(***)$. We have that g_η is $\mathfrak{C}_\eta$-measurable, and by $(**), (***)$, ${}^\circ g_\eta$ is integrable $L(\lambda_\eta)$ and;

${}^\circ \int_{\overline{\mathcal{S}_\eta}} g_\eta d\lambda_\eta = \int_{\overline{\mathcal{S}_\eta}} {}^\circ g_\eta dL(\lambda_\eta)$ $(\dagger)$

Using Remarks 3.21 of [6], it follows that g_η is S-integrable, and, clearly, the rest of the Theorem follows from $(**), (***), (\dagger)$.

□

We now show a corresponding result for $\hat{g}_\eta$. We require the following, observe that the definitions are slightly adjusted from Definition 0.18 of [5].

Definition 0.11. *If $n \in \mathcal{N}$, and $g_{n,st}$ is $\mathfrak{C}_{n,st}$-measurable, we define the discrete derivative $g'_{n,st}$ by;*

$g'_{n,st}(\frac{j}{n}) = n(g_{n,st}(\frac{j+1}{n}) - g_{n,st}(\frac{j}{n}))$ $(-n \leq j < n-1)$

$g'_{n,st}(\frac{n-1}{n}) = 0$

$g'_{n,st}(x) = g'_{n,st}(\frac{[nx]}{n})$ $(x \in \mathcal{S}_n)$

and the shift $g^{sh}_{n,st}$ by;

$g^{sh}_{n,st}(\frac{j}{n}) = g_{n,st}(\frac{j+1}{n})$ $(-n \leq j < n-1)$

$g^{sh}_{n,st}(\frac{n-1}{n}) = 0$

$g^{sh}_{n,st}(x) = g^{sh}_{n,st}(\frac{[nx]}{n})$ $(x \in \mathcal{S}_n)$

So both are $\mathfrak{C}_{n,st}$-measurable.

Lemma 0.12. *Discrete Calculus Lemmas*

Let $\{g_{n,st}, h_{n,st}\}$ be $\mathfrak{C}_{n,st}$-measurable and let $\{g'_{n,st}, h'_{n,st}, g^{sh}_{n,st}, h^{sh}_{n,st}\}$ be as in Definition 0.11. Then;

(i). $\int_{\mathcal{S}_n} g'_{n,st} d\lambda_{n,st} = g_{n,st}(\frac{n-1}{n}) - g_{n,st}(-1)$

(ii). $(g_{n,st}h_{n,st})' = g'_{n,st}h^{sh}_{n,st} + g_{n,st}h'_{n,st}$

(iii). $\int_{\mathcal{S}_n} g'_{n,st}h_{n,st} d\lambda_{n,st} = -\int_{\mathcal{S}_n} g^{sh}_{n,st}h'_{n,st} d\lambda_{n,st} + gh_{n,st}(\frac{n-1}{n}) - gh_{n,st}(-1)$

Proof. Just use Lemma 0.19 of [5], with n replacing n^2 in the proof. □

Definition 0.13. *For $n \in \mathcal{N}$, we let $\phi_n : \mathcal{N} \rightarrow \mathcal{C}$ be defined by $\phi_n(m) = n(exp(\frac{-\pi i m}{n}) - 1)$, and let $\psi_n : \mathcal{N} \rightarrow \mathcal{C}$ be defined by $\psi_n(m) = n(exp(\frac{\pi i m}{n}) - 1)$. Then, restricting to $\mathcal{Z}_n$, $\{\phi_n, \psi_n\}$ are $\mathfrak{D}_{n,st}$-measurable on $\mathcal{Z}_n$. If $g_{n,st}$ is $\mathfrak{C}_{n,st}$-measurable, for $-n \leq m \leq n-1$, we let;*

$C_n(m) = g_{n,st}(\frac{n-1}{n}) exp_{n,st}(-\pi i \frac{n-1}{n} m) - g_{n,st}(-1) exp_{n,st}(-\pi i(-1)m)$

$D_n(m) = -\frac{1}{n} g_{n,st}(-1) exp_{n,st}(\pi i \frac{m}{n}) exp_{n,st}(-\pi i(-1)m)$.

$C'_n(m) = -g'_{n,st}(-1) exp_{n,st}(-\pi i(-1)m)$

$D'_n(m) = -\frac{1}{n} g'_{n,st}(-1) exp_{n,st}(\pi i \frac{m}{n}) exp_{n,st}(-\pi i(-1)m)$.

$E_n(m) = \phi_n(m) D_n(m) - C_n(m)$

$E'_n(m) = \phi_n(m) D'_n(m) - C'_n(m)$

$F_n(m) = \psi_n(m)\phi_n(m)D_n(m) - \psi_n(m)C_n(m) + \phi_n(m)D'_n(m) - C'_n(m)$

considered as $\mathfrak{D}_{n,st}$-measurable functions.

Lemma 0.14. *Discrete Fourier transform*

Let $g_{n,st}$ be $\mathfrak{C}_{n,st}$-measurable. Then, for $m \neq 0$;

$\hat{g}_{n,st}(m) = \frac{\hat{g'}_{n,st}(m)+E_n(m)}{\psi_n(m)} = \frac{\hat{g''}_{n,st}(m)+F_n(m)}{\psi_n^2(m)}$

Proof. Again, use Lemma 0.21 of [5], with the simple adjustment of replacing n^2 by n in the proof. □

Lemma 0.15. *If $g \in C^\infty([-1,1])$, with $g(-1) = g(1) = 0$, then the functions $\hat{g''}_{n,st}(m)$ and $F_n(m)$ are uniformly bounded, independently of n, for $n \geq 1$.*

Proof. The proof is similar to Lemma 0.22 of [5], with some minor modifications. Observing that;

$D_n(m) = 0$, $(g(-1) = 0)$

$|D'_n(m)| \leq \frac{1}{n}|g'_{n,st}|(-1)$

$|\phi_n(m)| \leq 2n$, $|\psi_n(m)| \leq 2n$

$|C_n(m)| \leq |g_{n,st}|(\frac{n-1}{n})$, $(g(-1) = 0)$

$|C'_n(m)| \leq |g'_{n,st}|(-1)$

we obtain;

$|F_n(m)| \leq 2n|g_{n,st}|(\frac{n-1}{n}) + 3|g'_{n,st}|(-1)$

$= 2n|g_{n,st}|(\frac{n-1}{n}) + 3n|g_{n,st}|(\frac{1-n}{n})$, $(g(-1) = 0)$

Now assuming that g is real valued, otherwise take real and imaginary parts, we can apply the mean value theorem, and using the assumptions on g, we have that;

$-ng_{n,st}(\frac{n-1}{n}) = g'(c_n)$, $c_n \in (\frac{n-1}{n}, 1)$

$ng_{n,st}(\frac{1-n}{n}) = g'(d_n)$, $d_n \in (-1, \frac{1-n}{n})$

$|F_n(m)| \leq 2|g'(c_n)| + 3|g'(d_n)| \leq 5D$

where $D = ||g'||_{C(\mathcal{S})}$.

We now follow through the rest of the proof of Lemma 0.22 in [5], replacing n^2 by n, to obtain;

$|\hat{g''}_{n,st}(m)| \leq M + 2B$

where $M = ||g''||_{L^1(\mathcal{S})}$, and $B = ||g'||_{C(\mathcal{S})}$.

□

Lemma 0.16. *If $g \in C^\infty([-1,1])$, there exists a constant $H \in \mathcal{R}$ such that, for all $n \geq 1$, and $-n \leq m \leq n-1$, $m \neq 0$, with $n \in \mathcal{N}$, $m \in \mathcal{Z}$;*

$|\hat{g}_{n,st}(m)| \leq \frac{H}{m^2}$

Moreover, if $\epsilon > 0$ is standard, there exists a constant $N(\epsilon) \in \mathcal{N}_{>0}$, such that for all $n > N(\epsilon)$, for all $L, L' \in \mathcal{N}$ with $N(\epsilon) < |L| \leq |L'| \leq n$, $LL' > 0$;

$\int_L^{L'} |\hat{g}_{n,st}|(m) d\delta_n(m) < \epsilon$

Proof. As in Lemma 0.23 of [5], we have;

$|\psi_n(m)|^2 \geq 4m^2 \ (-n \leq m \leq n-1) \ (*)$

The function $h = g - c$, where $c = g(0) = g(1)$, satisfies the hypotheses of Lemma 0.15. Let W be the constant bound obtained there. Then, using Lemma 0.14, $(*)$, we obtain, for $m \neq 0$;

$|\hat{h}_{n,st}(m)| \leq \frac{W}{4m^2} \ (**)$

Now observe that $\hat{g}_{n,st} = \hat{h}_{n,st} + \hat{c}_{n,st}$ and, using, for example, Lemma 0.7, that $\hat{c}_{n,st}(m) = 0$, for $m \neq 0$, $(***)$. Then, combining $(**)$, $(***)$, we obtain the first result with $H = \frac{W}{4}$. Now;

$\int_L^{L'} |\hat{g}|_{n,st}(m) d\delta_n(m)$

$\leq \int_L^{n} |\hat{g}|_{n,st}(m) d\delta_n(m)$

$\leq \int_L^{n} \frac{H}{m^2} d\delta_n(m)$

$= \sum_{k=L}^{n-1} \frac{H}{m^2}$

$\leq \int_{L-1}^{n-1} \frac{H}{x^2} dx$

$= [\frac{-H}{x}]_{L-1}^{n-1} = \frac{H}{L-1} - \frac{H}{n-1} < \epsilon$

if $min(n,L) > N(\epsilon) = \frac{2H}{\epsilon} + 1$

□

We can now show the analogous result to Theorem 0.10. We require some further notation;

Definition 0.17. *If $g \in C^\infty([-1,1])$, with Fourier coefficients $\hat{g}(m)$, for $m \in \mathcal{Z}$, as given in 0.1, then we consider $\hat{g} : \mathcal{Z} \rightarrow \mathcal{C}$, as a measurable function on $(\mathcal{Z}, \mathfrak{D}, \delta)$, where $\mathfrak{D}$ is the σ-algebra of subsets of $\mathcal{Z}$, and δ is the counting measure, given by $\delta(m) = 1$, for $m \in \mathcal{Z}$. We let $\mathcal{Z}^{+-\infty}$ denote the extended integers $\mathcal{Z} \cup \{+\infty, -\infty\}$. We let $\hat{g}_\infty$ be the extension of $\hat{g}$ to $\mathcal{Z}^{+-\infty}$, obtained by setting $\hat{g}_\infty(+\infty) = \hat{g}_\infty(-\infty) = 0$, ([1]).*

Theorem 0.18. *Let $g \in C^\infty([-1,1])$, then $\hat{g}_\eta$, as given in Definition 0.8, is S-integrable on $\overline{\mathcal{Z}_\eta}$. Moreover ${}^\circ\hat{g}_\eta = st^*(\hat{g}_\infty)$, everywhere $L(\delta_\eta)$, and;*

$^\circ \int_{\overline{\mathcal{Z}_\eta}} \hat{g}_\eta d\delta_\eta = \int_{\overline{\mathcal{Z}_\eta}} st^*(\hat{g}_\infty) dL(\delta_\eta) = \int_{\mathcal{Z}} \hat{g} d\delta$

see Definition 0.17 and footnote 1 for relevant terminology.

Proof. By Lemma 0.16;

$\mathcal{R} \models (\forall n_{(n>N(\epsilon))})(\forall L, N_{(LN\geq 0, N(\epsilon)<|L|,|N|<n)}) \int_L^N |\hat{g}_{n,st}| d\delta_{n,st} < \epsilon$

[1] As in Lemma 0.6 of [5], it is easy to show there exists a unique σ-algebra $\mathfrak{D}'$ on $\mathcal{Z}^{+-\infty}$, which separates the points $+\infty$ and $-\infty$, and such that $\mathfrak{D}'|_{\mathcal{Z}} = \mathfrak{D}$. Moreover, there is a unique extension of δ to a complete measure δ' on $\mathfrak{D}'$, with $\delta'(+\infty) = \delta'(-\infty) = \infty$. As in Theorem 0.7 of [5], it is straightforward to show that;

$st : (\overline{\mathcal{Z}}_\eta, L(\mathfrak{D}_\eta), L(\delta_\eta)) \rightarrow (\mathcal{Z}^{+-\infty}, \mathfrak{D}', \delta')$

is measurable and measure preserving. In particular, if $\hat{g}$ is integrable δ (iff $st^*(\hat{g}_\infty)$ is integrable $L(\delta_\eta)$), we have;

$\int_{\overline{\mathcal{Z}}_\eta} st^*(\hat{g}_\infty) dL(\delta_\eta) = \int_{\mathcal{Z}^{+-\infty}} \hat{g}_\infty d\delta' = \int_{\mathcal{Z}} \hat{g} d\delta$

Hence, the corresponding statement is true in ${}^*\mathcal{R}$. In particular, if η is infinite, and $\{L, N\}$ are infinite, of the same sign, belonging to $\overline{\mathcal{Z}}_\eta$, we have that;

$\int_L^N |\hat{g}_\eta| d\delta_\eta < \epsilon$

As ϵ was arbitrary we conclude that;

$\int_L^N |\hat{g}_\eta| d\delta_\eta \simeq 0$ $(*)$

for all infinite $\{L, N\}$, of the same sign, in $\overline{\mathcal{R}}_\eta$. Now, using Definition 0.8 and the fact that $|exp_\eta(-\pi ixm)| \leq 1$, by transfer, we have, for $m \in \overline{\mathcal{Z}}_\eta$;

$|\hat{g}_\eta(m)| \leq \int_{\overline{\mathcal{S}_\eta}} |g_\eta(x) d\lambda_\eta = C$

where C is finite, as, by Theorem 0.10, g_η is S-integrable. It follows that for $n \in \mathcal{N}$, the functions $\hat{g}_\eta \chi_{[-n,n]}$ are finite, in the sense of Definition 1.7 of [5]. Now, proceeding as in Theorem 0.17 of [5], replacing $\overline{\mathcal{R}}_\eta$ by $\overline{\mathcal{Z}}_\eta$, we obtain that $\hat{g}_\eta$ is S-integrable.

If $m \in \mathcal{Z}_\eta$, the function $r_m(x) = g_\eta(x) exp_\eta(-\pi ixm)$ is S-integrable, by Corollary 5 of [1], as $|r_m| \leq |g_\eta|$, and g_η is S-integrable, by Theorem 0.10. Then, if $m \in \mathcal{Z}_\eta$ is finite, as in Theorem 0.24 of [5], just replacing $\overline{R}_\eta$ by $\overline{S}_\eta$, and using Definition 0.8, Theorem 1.9 of [5], Theorem 0.10, continuity of exp, see Theorem 1.6 of [5], and Lemma 0.5;

${}^\circ\hat{g}_\eta(m) = \hat{g}(m) = st^*(\hat{g}_\infty)(m)$ $(**)$

Now, using the first part of Lemma 0.16, we obtain, by transfer, that for infinite $m \in \overline{\mathcal{Z}}_\eta$, $\hat{g}_\eta(m) \simeq 0$. By Definition 0.17, we have $st^*(\hat{g}_\infty)(m) = 0$, $(***)$. Combining $(**), (***)$ gives ${}^\circ\hat{g}_\eta = st^*(\hat{g}_\infty)$, everywhere $L(\delta_\eta)$. The rest of the theorem follows from footnote 1. □

Finally, we have;

Theorem 0.19. *For $g \in C^\infty(\mathcal{S})$, there is a non standard proof of the uniform convergence of its Fourier series, Theorem 0.2.*

Proof. By Lemma 0.9, and using the definition of the internal integral $\int_{\mathcal{Z}_\eta}$ to replace the hyperfinite sum $\Sigma_{m \in \mathcal{Z}_\eta}$, we have that;

$g_\eta(x) = \frac{1}{2}\int_{\overline{\mathcal{Z}_\eta}} \hat{g}_\eta(m)exp_\eta(\pi ixm)d\delta_\eta(m)$ $(*)$

for $x \in \overline{\mathcal{S}_\eta}$. As in Theorem 0.18, the function $s_x(m) = \hat{g}_\eta(m)exp_\eta(\pi ixm)$ is S-integrable, because, by the same theorem, $\hat{g}_\eta$ is S-integrable. We now argue as before, and use the result that ${}^\circ g_\eta = st^*(\hat{g}_\infty)$, everywhere $L(\delta_\eta)$ $(**)$. Note that, as $\hat{g}_\eta$ is S-integrable, using Theorem 3.24 of [6] and $(**)$, $st^*(\hat{g}_\infty)$ is integrable $L(\delta_\eta)$ and, therefore, by footnote 1 $\hat{g}$ is integrable δ, $(***)$. We have, if $x \in [-1, 1)$, taking standard parts in $(*)$;

$g(x) = {}^\circ g_\eta(x) = \frac{1}{2}\int_{\overline{\mathcal{Z}_\eta}} {}^\circ\hat{g}_\eta(m){}^\circ exp_\eta(\pi ixm)dL(\delta_\eta)(m)$

$= \frac{1}{2}\int_{m finite} st^*(\hat{g}_\infty)(m)exp_\eta(\pi i{}^\circ xm)dL(\delta_\eta)(m)$

$= \frac{1}{2}\int_{m finite} st^*(\hat{g}_\infty exp_{\pi ix})(m)dL(\delta_\eta)(m)$

$= \frac{1}{2}\int_{\mathcal{Z}} \hat{g}(m)exp(\pi ixm)d\delta(m)$

$= \frac{1}{2}\sum_{m\in\mathcal{Z}} \hat{g}(m)exp(\pi ixm)$ (†), (2)

The sum in (†) can be taken as $lim_{n\rightarrow\infty}S_n(g)$, for $S_n(g) = \sum_{m=-n}^{n} \hat{g}(m)exp(\pi ixm)$, (††) because, for $n \in \mathcal{N}$, $g_{n,x}(m) = (\hat{g}exp_{\pi ix}\chi_{[-n,n]})(m)$, converges everywhere δ to $(\hat{g}exp_{\pi ix})(m)$, so (††) follows from $(***)$ and the DCT. In order to obtain uniform convergence in x, we use the fact, from Theorem 0.18, that, for $m \in \mathcal{Z}$, $\hat{g}(m) = {}^\circ\hat{g}_\eta(m)$ and, the first part of Lemma 0.16, which gives, by transfer, that $|\hat{g}_\eta(m)| \leq \frac{H}{m^2}$, for $m \in \mathcal{Z}_{\neq 0}$. Combining these results gives that $|\hat{g}(m)| \leq \frac{H}{m^2}$, for $m \neq 0$, so $|g_m(x)| = |\hat{g}(m)exp(\pi ixm)| \leq \frac{H}{m^2}$, $m \neq 0$, uniformly in $x \in [-1, 1]$, (†††). Applying Weierstrass M-test, see [2], and the estimate (†††), gives the required uniform convergence of the sums $S_n(g) = \sum_{m=-n}^{n} g_m$.

□

References

[1] A Non-Standard Representation for Brownian Motion and Ito Integration, Robert Anderson, Isreal Journal of Mathematics, (1976).

[2] Introduction to Real Analysis, Robert Bartle and Donald Schubert, John Wiley, (1982).

2 The case $x = 1$ obviously then follows from the fact that $g(1) = g(-1)$ and $exp(\pi im) = exp(-\pi im)$, for $m \in \mathcal{Z}$.

[3] Fourier Analysis, Tom Korner, CUP, (1998).

[4] Conversion from Nonstandard to Standard Measure Spaces and Applications in Probability Theory, Peter Loeb, Transactions of the American Mathematical Society, (1975).

[5] A Simple Proof of the Fourier Inversion Theorem using Nonstandard Analyis, Tristram de Piro, available on AP ArXiv(1311.1682), (2013).

[6] Applications of Nonstandard Analysis to Probability Theory, Tristram de Piro, MSc dissertation in "Financial Mathematics", Exeter University, (2013).

[7] Fourier Analysis, An Introduction, Elias Stein and Rami Shakarchi, Princeton Lectures in Analysis, (2003).

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