# A SIMPLE PROOF OF A MARTINGALE REPRESENTATION THEOREM USING NONSTANDARD ANALYSIS

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Abstract. We give a proof of a Martingale Representation Theorem using the methods of nonstandard analysis.

We introduce the following spaces;

**Definition 0.1.** *Let $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$, and set $\eta = 2^\nu$. Define;*

$\overline{\Omega_\eta} = \{x \in {}^*\mathcal{R} : 0 \leq x < 1\}$

$\overline{\mathcal{T}_\nu} = \{x \in {}^*\mathcal{R} : 0 \leq x \leq 1\}$

*We let $\mathcal{C}_\eta$ consist of internal unions of the intervals $[\frac{i}{\eta}, \frac{i+1}{\eta})$, for $0 \leq i \leq \eta - 1$, and let $\mathcal{D}_\nu$ consist of internal unions $[\frac{i}{\nu}, \frac{i+1}{\nu})$, for $0 \leq i \leq \nu - 1$, together with $\{1\}$*

*We define counting measures $\mu_\eta$ and $\lambda_\nu$ on $\mathcal{C}_\eta$ and $\mathcal{D}_\nu$ respectively, by setting $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$, $\lambda_\nu((\frac{i}{\nu}, \frac{i+1}{\nu}]) = \frac{1}{\nu}$ and $\lambda_\nu(\{1\}) = 0$*

*We let $(\overline{\Omega_\eta}, \mathcal{C}_\eta, \mu_\eta)$ and $(\overline{\mathcal{T}}_\nu, \mathcal{D}_\nu, \lambda_\nu)$ be the resulting $*$-finite measure spaces, in the sense of* [4], *and let $(\overline{\Omega_\eta}, L(\mathcal{C}_\eta), L(\mu_\eta))$, $(\overline{\mathcal{T}}_\nu, L(\mathcal{D}_\nu), L(\lambda_\nu))$ be the associated Loeb spaces.*

*We let $V(\mathcal{C}_\eta) = \{f : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}, f(x) = f(\frac{[\eta x]}{\eta})\}$ and $W(\mathcal{C}_\eta) \subset V(\mathcal{C}_\eta)$ be the set of measurable functions $f : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$, with respect to $\mathcal{C}_\eta$, in the sense of* [4]. *Then $W(\mathcal{C}_\eta)$ is a $*$-finite vector space over ${}^*\mathcal{C}$, of dimension $\eta$, ([1]). Similarly, we let $V(\mathcal{D}_\nu) = \{f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}, f(t) = f(\frac{[\nu t]}{\nu})\}$ and*

[1] By a $*$-vector space, one means an internal set $V$, for which the operations $+ : V \times V \rightarrow V$ of addition and scalar multiplication $. : {}^*\mathcal{C} \times V \rightarrow V$ are internal. Such spaces have the property that $*$-finite linear combinations ${}^*\Sigma_{i \in I} \lambda_i . v_i$, $(*)$, for a $*$-finite index set $I$, belong to $V$, by transfer of the corresponding standard result for vector spaces. We say that $V$ is a $*$-finite vector space, if there exists a $*$-finite

*$W(\mathcal{D}_\nu) \subset V(\mathcal{D}_\nu)$ be the set of measurable functions $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$, with respect to $\mathcal{D}_\nu$, in the sense of* [4]. *Then $W(\mathcal{D}_\nu)$ is a $*$-finite vector space over ${}^*\mathcal{C}$, of dimension $\nu + 1$.*

**Definition 0.2.** *Given $n \in \mathcal{N}_{>0}$, we let $\Omega_n = \{m \in \mathcal{N} : 0 \leq m < 2^n\}$, and let $C_n$ be the set of sequences of length $n$, consisting of 1's and $-1$'s. We let $\theta_n : \Omega_n \rightarrow \mathcal{N}^n$ be the map which associates $m \in \Omega_n$ with its binary representation, and let $\phi_n : \Omega_n \rightarrow C_n$ be the composition $\phi_n = (\gamma \circ \theta_n)$, where, for $\bar{m} \in \mathcal{N}^n$, $\gamma(\bar{m}) = 2.\bar{m} - \bar{1}$. For $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$, we let $\phi_\nu : \Omega_\nu \rightarrow C_\nu$ be the map, obtained by transfer of $\phi_n$, which associates $i \in {}^*\mathcal{N}$, $0 \leq i < 2^\nu$, with an internal sequence of length $\nu$, consisting of 1's and $-1$'s. Similarly, for $\eta = 2^\nu$, we let $\psi_\eta : \overline{\Omega_\eta} \rightarrow C_\nu$ be defined by $\psi_\eta(x) = \phi_\nu([\eta x])$. For $1 \leq j \leq \nu$, we let $\omega_j : C_\nu \rightarrow \{1, -1\}$ be the internal projection map onto the $j$'th coordinate, and let $\omega_j : \overline{\Omega_\eta} \rightarrow \{1, -1\}$ also denote the composition $(\omega_j \circ \psi_\eta)$, so that $\omega_j \in W(\overline{\Omega_\eta})$. By convention, we set $\omega_0 = 1$. For an internal sequence $\bar{t} \in C_\nu$, we let $\omega_{\bar{t}} : \overline{\Omega_\eta} \rightarrow \{1, -1\}$ be the internal function defined by;*

$$\omega_{\bar{t}} = \prod_{1 \leq j \leq \nu} \omega_j^{\frac{\bar{t}(j)+1}{2}}$$

*Again, it is clear that $\omega_{\bar{t}} \in W(\overline{\Omega_\eta})$.*

**Lemma 0.3.** *The functions $\{\omega_j : 1 \leq j \leq \nu\}$ are $*$-independent in the sense of* [2], *(Definition 19), in particular they are orthogonal with respect to the measure $\mu_\eta$. Moreover, the functions $\{\omega_{\bar{t}} : \bar{t} \in C_\nu\}$ form an orthogonal basis of $V(\overline{\Omega_\eta})$, and, if $\bar{t} \neq \overline{-1}$, $E_\eta(\omega_{\bar{t}}) = 0$, and $Var_\eta(\omega_{\bar{t}}) = 1$, where, $E_\eta$ and $Var_\eta$ are the expectation and variance corresponding to the measure $\mu_\eta$.*

*Proof.* According to the definition, we need to verify that for an internal index set $J = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, \nu\}$, and an internal tuple $(\alpha_1, \ldots, \alpha_s)$, where $s = |J|$;

$$\mu_\eta(x : \omega_{j_1}(x) < \alpha_1, \ldots, \omega_{j_k}(x) < \alpha_k, \ldots, \omega_{j_s}(x) < \alpha_s)$$

index set $I$ and elements $\{v_i : i \in I\}$ such that every $v \in V$ can be written as a combination $(*)$, and the elements $\{v_i : i \in I\}$ are independent, in the sense that if $(*) = 0$, then each $\lambda_i = 0$. It is clear, by transfer of the corresponding result for finite dimensional vector space over $\mathcal{C}$, that $V$ has a well defined dimension given by $Card(I)$, see [5], even though $V$ may be infinite dimensional, considered as a standard vector space.

$= \prod_{k=1}^{s} \mu_\eta(x : \omega_{j_k}(x) < \alpha_k)$ $(*)$

Without loss of generality, we can assume that each $\alpha_{j_k} > -1$, as if some $\alpha_{j_k} \leq -1$, both sides of $(*)$ are equal to zero. Let $J' = \{j' \in J : -1 < \alpha_{j'} \leq 1\}$ and $J'' = \{j'' \in J : 1 < \alpha_{j''}\}$, so $J = J' \cup J''$. Then;

$\mu_\eta(x : \omega_{j_1}(x) < \alpha_1, \ldots, \omega_{j_s}(x) < \alpha_s)$

$= \frac{1}{\eta} Card(z \in C_\nu : z(j') = -1 \ for \ j' \in J', z(j'') \in \{-1, 1\} \ for \ j'' \in J'')$

$= \frac{1}{2^\nu} Card(z \in C_\nu : z(j') = -1 \ for \ j' \in J') = \frac{2^{\nu - s'}}{2^\nu} = 2^{-s'}$

where $s' = Card(J')$. Moreover;

$\prod_{k=1}^{s} \mu_\eta(x : \omega_{j_k}(x) < \alpha_k) = \prod_{j' \in J'} \mu_\eta(x : \omega_{j'}(x) = -1) = 2^{-s'}$

as $\mu_\eta(x : \omega_j(x) = -1) = \frac{1}{2}$, for $1 \leq j \leq \nu$. Hence, $(*)$ is shown. That $*$-independence implies orthogonality follows easily by transfer, from the corresponding fact, for finite measure spaces, that $E(X_{j_1} X_{j_2}) = E(X_{j_1}) E(X_{j_2})$, for the standard expectation $E$ and independent random variables $\{X_{j_1}, X_{j_2}\}$, $(**)$. Hence, by $(**)$;

$E_\eta(\omega_{j_1} \omega_{j_2}) = E_\eta(\omega_{j_1}) E_\eta(\omega_{j_2}) = 0$, $(j_1 \neq j_2)$ $(***)$

as clearly $E_\eta(\omega_j) = 0$, for $1 \leq j \leq \nu$. If $\bar{t} \neq \overline{-1}$, let $J' = \{j' : 1 \leq j' \leq \nu, \bar{t}(j') = 1\}$, then;

$E_\eta(\omega_{\bar{t}}) = E_\eta(\prod_{1 \leq j \leq \nu} \omega_j^{\frac{\bar{t}(j)+1}{2}}) = E_\eta(\prod_{j' \in J'} \omega_{j'}) = \prod_{j' \in J'} E_\eta(\omega_{j'}) = 0$ $(\sharp)$

where, in $(\sharp)$, we have used the facts that $J' \neq \emptyset$ and internal, and a simple generalisation of $(***)$, by transfer from the corresponding fact for finite measure spaces. Hence, $1 = \omega_{\overline{-1}}$ is orthogonal to $\omega_{\bar{t}}$, for $\bar{t} \neq \overline{-1}$. If $\overline{t_1} \neq \overline{t_2}$ are both distinct from $\overline{-1}$, then, if $J_1 = \{j : 1 \leq j \leq \nu, \overline{t_1}(j) = 1\}$ and $J_2 = \{j : 1 \leq j \leq \nu, \overline{t_2}(j) = 1\}$, so $J_1 \neq J_2$ and $J_1, J_2 \neq \emptyset$, we have;

$E_\eta(\omega_{\overline{t_1}} \omega_{\overline{t_2}})$

$= E_\eta(\prod_{j \in J_1} \omega_j . \prod_{j \in J_2} \omega_j)$ $(\sharp\sharp)$

$= E_\eta(\prod_{j\in(J_1\setminus J_2)}\omega_j.\prod_{j\in(J_2\setminus J_1)}\omega_j)$ $(\sharp\sharp\sharp)$

$= E_\eta(\prod_{j\in(J_1\setminus J_2)}\omega_j)E_\eta(\prod_{j\in(J_2\setminus J_1)}\omega_j) = 0$ $(\sharp\sharp\sharp\sharp)$

In $(\sharp\sharp)$, we have used the definition of $J_1$ and $J_2$, and in $(\sharp\sharp\sharp)$, we have used the fact that $(J_1\cup J_2) = (J_1\cap J_2)\sqcup(J_1\setminus J_2)\sqcup(J_2\setminus J_1)$, and $\omega_j^2 = 1$, for $1\leq j\leq \nu$. Finally, in $(\sharp\sharp\sharp\sharp)$, we have used the facts that $(J_1\setminus J_2)$ and $(J_2\setminus J_1)$ are disjoint, and at least one of these sets is nonempty, the result of $(\sharp)$ and a similar generalisation of $(***)$. This shows that the functions $\{\omega_{\bar{t}} : \bar{t}\in C_\nu\}$ are orthogonal, $(****)$. That they form a basis for $V(\overline{\Omega_\eta})$ follows immediately, by transfer, from $(****)$ and the corresponding fact for finite dimensional vector spaces. The final calculation is left to the reader. □

We require the following;

**Definition 0.4.** *For $0\leq l\leq\nu$, we define $\sim_l'$, on $C_\nu$, to be the internal equivalence relation given by;*

$\bar{t}_1\sim_l'\bar{t}_2$ *iff* $\bar{t}_1(j) = \bar{t}_2(j)$ $(\forall j\leq l)$

*We extend this to an internal equivalence relation on $\overline{\Omega}_\eta$, which we denote by $\sim_l$;*

$x_1\sim_l x_2$ *iff* $\psi_\eta(x_1)\sim_l\psi_\eta(x_2)$ $(*)$

*We let $\mathcal{C}_\eta^l$ be the $*$-finite algebra generated by the partition of $\overline{\Omega}_\eta$ into the $2^l$ equivalence classes with respect to $\sim_l$, $(*)$. As is easily verifed, we have $\mathcal{C}_\eta^{l_1}\subseteq\mathcal{C}_\eta^{l_2}$, if $l_1\leq l_2$, $\mathcal{C}_\eta^0 = \{\emptyset,\overline{\Omega}_\eta\}$ and $\mathcal{C}_\eta = \mathcal{C}_\eta^\nu$. For $0\leq l\leq\nu$, we let $W(\mathcal{C}_\eta^l)\subseteq W(\mathcal{C}_\eta)$ be the set of measurable functions $f:\overline{\Omega}_\eta\rightarrow {}^*\mathcal{C}$, with respect to $\mathcal{C}_\eta^l$. We will refer to the collection $\{\mathcal{C}_\eta^l : 0\leq l\leq\nu\}$ of $*$-finite algebras, as the nonstandard filtration associated to $\overline{\Omega}_\eta$. We produce a standard filtration $\{\mathfrak{D}_t : t\in[0,1]\}$, $(**)$, by following the method of [2], see Definition 7.14 of [5], (replacing the equivalence relation $\sim$ there, by $\sim_l$, as given in $(*)$, and being careful to use the index $\nu$ instead of $\eta$. Note that Lemma 7.15 of [5] still applies in this case.) We also require a slight modification of the construction of Brownian motion in [2]. Namely, we take;*

$\chi(t,x) = \frac{1}{\sqrt{\nu}}({}^*\sum_{i=1}^{[\nu t]}\omega_i)$, ([2])

*and* $W(t,x) = {}^\circ\chi(t,x)$, $(t,x) \in [0,1] \times \overline{\Omega_\eta}$ $(**)$.

One of the advantages of the non-standard approach to stochastic calculus, is that it allows one to show easily that every stochastic integral is a martingale. We follow the notation from Chapter 7 of [5], again using the filtration $(**)$ of Definition 0.4 to replace the one from Definition 7.14, and its subsequent applications;

**Theorem 0.5.** *If $g \in \mathcal{G}_0$, and $f$ is a 2-lifting of $g$, then $I(t,x)$, as in Definition 7.20 of* [5], *is equivalent, as a stochastic process, to a martingale, with respect to the filtration $\mathfrak{D}_t$,* ([3]).

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[2] We adopt the convention that the sum is zero, when $t = 0$

[3] By which I mean a function $I : [0,1] \times \overline{\Omega_\eta} \rightarrow \mathcal{R}$, such that;

$(i)$. $I$ is $\mathfrak{B} \times \mathfrak{D}$ measurable (complete product).

$(ii)$. $I_t$ is measurable with respect to $\mathfrak{D}_t$, for $t \in [0,1]$.

$(iii)$. $E(|I_t|) < \infty$, for $t \in [0,1]$.

$(iv)$. $E(I_t|\mathfrak{D}_s) = I_s$, if $s < t$ belong to $[0,1]$.

$(v)$. For $C \subset \overline{\Omega_\eta}$, with $L(\mu_\eta)(C) = 1$, and $x \in C$, the paths $\gamma_x : [0,1] \rightarrow \mathcal{R}$, where $\gamma_x(t) = I(t,x)$, are continuous.

Most of this definition can be found in [7], see also [8] for a thorough discussion of discrete time martingales. We call a martingale tame if it satisfies the additional conditions that;

$(vi)$. $I_1 \in L^2(\overline{\Omega}_\eta, L(\mu_\eta))$ and, for $0 \leq s < t \leq 1$;

$\int_{\overline{\Omega}_\eta}(I_t^2 - I_s^2)dL(\mu_\eta) \leq C(t-s)$

where $C \in \mathcal{R}_{\geq 0}$

$(vii)$ $(UI)$ For $a.a.s$, $0 \leq s < 1$ and sufficiently small $h > 0$, $\frac{[I]_{s+h}-[I]_s}{h}$ is strongly uniformly integrable in the sense that that there exists $f : \mathcal{R} \rightarrow \mathcal{R}$, with $f \geq 0$ and $lim_{x\rightarrow\infty} f(x) = 0$ such that, for $K > 0$, $K \in \mathcal{R}$;
$\int_{\frac{[I]_{s+h}-[I]_s}{h} > K} \frac{[I]_{s+h}-[I]_s}{h} dL(\mu_\eta) < f(K)$.

where $[I]$ denotes the quadratic variation of the process $I$.

*Proof.* Let $I'$ be the modification of $I$, as given in the proof of Theorem 7.25 of [5]. Then $I'$ and agree $I$ on $[0,1]\times C$, where $P(C)=1$, and $P=L(\mu_\eta)$, so they are equivalent as stochastic processes. We show that $I'$ is a martingale.

$(i)$ follows from the fact that $I$ is $\mathfrak{B}\times\mathfrak{D}$ measurable, and $I=I'$ a.e $\mu\times L(\mu_\eta)$, $(*)$. Here, completeness of the product is required.

$(ii)$. By the construction in the proof of Theorem 7.25 of [5], $I'_t$ is measurable with respect to $\mathfrak{D}'_t\subset\mathfrak{D}_t$.

$(iii)$. We have, for $t\in[0,1]$;

$$\int_{\overline{\Omega_\eta}} I'^2(t,x)dL(\mu_\eta)=\int_{\overline{\Omega_\eta}} I^2(t,x)dL(\mu_\eta)$$

$$=\int_{\overline{\Omega_\eta}} {}^\circ F^2(t,x)d\mu_\eta$$

$$\leq {}^\circ\int_{\overline{\Omega_\eta}} F^2(t,x)d\mu_\eta$$

$$={}^\circ\int_{\overline{\Omega_\eta}}\int_0^t f^2(t,x)d\lambda_\nu d\mu_\eta=||g||^2_{L^2([0,t]\times\overline{\Omega_\eta})}\ (\dagger)$$

using $(*)$, Definition 7.20, (see notation in Theorem 7.24), Theorem 3.16 and the proof of Theorem 7.22 in [5]. Hence $I'_t\in L^2(\overline{\Omega_\eta},\mathcal{C}_\eta,P)$, so $I'_t\in L^1(\overline{\Omega_\eta},\mathcal{C}_\eta,P)$, by Holder's inequality, see [6].

$(iv)$. Suppose $s<t$. We first show that $E(I'_t|\mathfrak{D}'_s)=I'_s$, $(\dagger\dagger)$. Suppose $i\in{}^*\mathcal{N}$, with $\frac{i}{\nu}\simeq s$, then we claim that $E(I'_t|\sigma(\mathcal{C}^i_\eta)^{comp})=I'_s$, $(**)$. As $I_t=I'_t$ a.e $P$, we have $E(I'_t|\sigma(\mathcal{C}^i_\eta)^{comp}))=E(I_t|\sigma(\mathcal{C}^i_\eta)^{comp}))$. We can also see that $F_t\in SL^2(\overline{\Omega_\eta},\mathcal{C}_\eta,\mu_\eta)$. This follows from the calculation $(\dagger)$, Theorem 3.34(i) of [5], and the fact that;

$$\int_{\overline{\Omega_\eta}} I^2(t,x)dL(\mu_\eta)=||g||^2_{L^2([0,t]\times\overline{\Omega_\eta})}$$

by Ito's isometry, as $g\in\mathcal{G}_0$. Hence, by Theorem 3.34(iv) of [5], $F_t\in SL^1(\overline{\Omega_\eta},\mathcal{C}_\eta,\mu_\eta)$, $(***)$. Applying Theorem 7.3(ii) of [5] and $(***)$;

$$E(I_t|\sigma(\mathcal{C}^i_\eta)^{comp})=E({}^\circ F_t|\sigma(\mathcal{C}^i_\eta)^{comp})={}^\circ E(F_t|\mathcal{C}^i_\eta)$$

We have;

$E(F_t|\mathcal{C}_\eta^i) = \sum_{j=0}^{i-1} f(\frac{j}{\nu}, x)\frac{\omega_{j+1}}{\sqrt{\nu}}$

by $^*$-independence of the sequence $\{\omega_j\}_{0\leq j\leq[\nu t]+1}$. Letting $s' = \frac{i-1}{\nu}$, so $s' \simeq s$, $E(F_t|\mathcal{C}_\eta^i) = F_{s'}$. We have, using Theorem 7.24 of [5], that $I_s = I_{s'}$ a.e $P$, so $I'_s = I_s = I_{s'}$ a.e $P$. As $I'_s$ is $\sigma(\mathcal{C}_\eta^i)^{comp}$-measurable, we have $E(I'_t|(\mathcal{C}_\eta^i)^{comp}) = I'_s$, showing $(**)$. As $\mathfrak{D}'_s \subset \sigma(\mathcal{C}_\eta^i)^{comp}$, and $I'_s$ is $\mathfrak{D}'_s$-measurable, we have $E(I'_t|\mathfrak{D}'_s) = I'_s$, showing $(\dagger\dagger)$.

If $A \in \mathfrak{D}_s$, then, by Lemma 7.15(i) of [5], $A \in \mathfrak{D}'_{s_1}$, for $s < s_1 < t$. As $E(I'_t|\mathfrak{D}'_{s_1}) = I'_{s_1}$, to show $(iv)$, it is sufficient to prove that;

$\int_A I'_s dL(\mu_\eta) = lim_{s_1\to s}\int_A I'_{s_1} dL(\mu_\eta)$ $(\dagger\dagger\dagger)$

To show $(\dagger\dagger\dagger)$, observe that $||I'_{s_1} - I'_s||_2^2 \leq ||g_{[0,s_1]} - g_{[0,s]}||_2^2$ by $(\dagger)$, where $g_{[0,s_1]}$ is obtained by truncating the function $g$ to the interval $[0, s_1]$, ([4]). Using Holder's inequality and the DCT, we have $lim_{s_1\to s}||I'_{s_1} - I'_s||_1 \leq lim_{s_1\to s}||g_{[0,s_1]} - g_{[0,s]}||_1 = 0$. Therefore, $(\dagger\dagger\dagger)$ is shown. This proves $(iv)$.

$(v)$. This is Theorem 25 of [2].

□

We proceed to show the converse, that every martingale can be represented as a stochastic integral, using the nonstandard approach.

**Lemma 0.6.** *For $0 \leq l \leq \nu$, a basis of the $*$-finite vector space $W(\mathcal{C}_\eta^l)$ is given by $D_l = \bigcup_{0\leq m\leq l} B_m$, where, for $1 \leq m \leq \nu$, $B_m = \{\omega_{\bar{t}} : \bar{t}(m) = 1, \bar{t}(m') = -1, m < m' \leq \nu\}$, and $B_0 = \{\omega_{\overline{-1}}\}$.*

*Proof.* The case when $l = 0$ is clear as $\omega_{\overline{-1}} = 1$, and using the description of $\mathcal{C}_\eta^0$ in Definition 0.4. Using the observation $(*)$ there, we have, for $1 \leq l \leq \nu$, that $W(\mathcal{C}_\eta^l)$ is a $*$-finite vector space of dimension $2^l$. Using Lemma 0.3, and the fact that $Card(D_l) = 2^l$, it is sufficient to show each $\omega_{\bar{t}} \in D_l$ is measurable with respect to $\mathcal{C}_\eta^l$. We have that, for $1 \leq j \leq l$, $\omega_j$ is measurable with respect to $\mathcal{C}_\eta^j \subseteq \mathcal{C}_\eta^l$. Hence, the result follows easily, by transfer of the result for finite measure spaces, that the product $X_{j_1}X_{j_2}$, of two measurable random variables $X_{j_1}$ and $X_{j_2}$ is measurable.

[4]Technically, you need to show that $I_{s_1}$ is the non standard stochastic integral of $g_{[0,s_1]}$, and then apply Theorem 7.22 of [5], however, this is clear by truncating the corresponding lift of $g$.

□

**Definition 0.7.** *We define a nonstandard martingale to be a $\mathcal{D}_\nu \times \mathcal{C}_\eta$-measurable function $Y : \overline{\mathcal{T}_\nu} \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$, such that;*

*(i). For $t \in \overline{\mathcal{T}_\nu}$, $Y_{\frac{[\nu t]}{\nu}}$ is measurable with respect to $\mathcal{C}_\eta^{[\nu t]}$.*

*(ii). $E_\eta(Y_{\frac{[\nu t]}{\nu}}|\mathcal{C}_\eta^{[\nu s]}) = Y_{\frac{[\nu s]}{\nu}}$, for $(0 \leq s \leq t \leq 1)$.*

*(iii). $E_\eta(|Y_{\frac{[\nu t]}{\nu}}|)$ is finite.*

*We say that $Y$ is $S$-continuous, if there exists $C \subset \overline{\Omega}_\eta$ with $L(\mu_\eta)(C) = 1$, such that for $x \in C$, $Y(t,x) \simeq Y(s,x)$, when $s \simeq t$, and each $Y(t,x)$ is near standard. We say that $Y$ has infinitesimal increments if, for all $x \in \overline{\Omega}_\eta$, and $t \in \overline{\mathcal{T}_\nu}$, $t \neq 1$, $Y(\frac{[t\nu]+1}{\nu}, x) \simeq Y(\frac{[t\nu]}{\nu}, x)$.*

**Lemma 0.8.** *Let $Y : \overline{\mathcal{T}_\nu} \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{R}$ be a $\mathcal{D}_\nu \times \mathcal{C}_\eta$-measurable function, satisfying (i) and (ii) of Definition 0.7, then;*

$Y_t(x) = \sum_{j=0}^{[\nu t]} c_j(t,x)\omega_j(x)$ $(*)$

*where $c_0 : [0,1] \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$ is $\mathcal{D}_\nu \times \mathcal{C}_\eta^0$-measurable, $c_j : [\frac{j}{\nu}, 1] \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$ is $\mathcal{D}_\nu \times \mathcal{C}_\eta^{j-1}$-measurable, for $1 \leq j \leq \nu$, and $c_0(s,x) = c_0(t,x)$, for $0 \leq s \leq t \leq 1$, $c_j(s,x) = c_j(t,x)$, for $\frac{j}{\nu} \leq s \leq t \leq 1$. Conversely, if $\{c_j : 0 \leq j \leq \nu\}$ is a collection of functions satisfying the above conditions, then the definition $(*)$ produces a $\mathcal{D}_\nu \times \mathcal{C}_\eta$-measurable function, satisfying (i) and (ii) of Definition 0.7.*

*Proof.* Using $(ii)$, we have that $E_\eta(Y_t) = E_\eta(Y_t|\mathcal{C}_\eta^0) = Y_0$. Replacing $Y_t$ by $Y_t - Y_0$, we can, without loss of generality, assume that $E_\eta(Y_t) = 0$, for $t \in {}^*[0,1]$. By $(i)$ and Lemma 0.6;

$Y_t = \sum_{j=1}^{[\nu t]} c_j(t,x)\omega_j(x)$

where;

$c_j(t,x) = \sum_{a=0}^{j-1} \sum_{i_0<\ldots<i_a;0}^{j-1} p_j^{(i_0,\ldots,i_a)}(t)\omega_{i_0}\ldots\omega_{i_a}(x)$

Clearly, $c_j$ is $\mathcal{D}_\nu \times \mathcal{C}_\eta^{j-1}$-measurable. Again, using $(ii)$, and the fact that $c_k(t,x)\omega_k$ is orthogonal to the basis $D_{[\nu s]}$ of $W(\mathcal{C}_\eta^{[\nu s]}$, for $[\nu s] < k \leq [\nu t]$, $(\dagger)$, we have;

$\sum_{j=1}^{[\nu s]} c_j(t,x)\omega_j(x) = \sum_{j=1}^{[\nu s]} c_j(s,x)\omega_j(x)$

Equating coefficients, and using the fact that $D_j$ is a basis for $W(\mathcal{C}_\eta^j)$, for $1 \leq j \leq [\nu s]$, we obtain $c_j(s,x) = c_j(t,x)$, for all $\frac{j}{\nu} \leq s \leq t \leq 1$.

The converse is easy to check. $(i)$ is obtained, observing that for $t \in \mathcal{T}_\nu$, all the functions $c_{j,t}$ and $\omega_j$ are measurable with respect to $\mathcal{C}_\eta^{[\nu t]}$, for $0 \leq j \leq [\nu t]$. To obtain $(ii)$, just take the conditional expectation of $(*)$ and make the observation † again.

□

**Lemma 0.9.** *Let $X$ be a martingale, see footnote 3 for the definition, with the extra condition that $X_1 \in L^2(\overline{\Omega_\eta})$, then there exists a nonstandard martingale $\overline{X}$, see Definition 0.7, with ${}^\circ(\overline{X}_t) = X_{{}^\circ t}$, for $t \in \overline{\mathcal{T}_\nu}$, a.e $L(\mu_\eta)$, and such that the sequence $\{\overline{X}_{\frac{i}{\nu}} : 0 \leq i \leq \nu\} \subset SL^2(\overline{\Omega_\eta}, \mu_\eta)$. Moreover, $\overline{X}$ is $S$-continuous, and we can take $\overline{X}$ to have infinitesimal increments.*

*Proof.* By $(i)$ of footnote 3, we have $X$ is $\mathfrak{B} \times \mathfrak{D}$-measurable. We claim that $X \in L^1([0,1] \times \overline{\Omega_\eta})$, $(*)$. Without loss of generality, we can assume that $X \geq 0$, ([5]) Then $(*)$ follows from the fact that, for $0 \leq t \leq 1$, $E(X_t) = E(X_t|\mathfrak{D}_0) = X_0$, by $(iv)$ of footnote 3, and so;

$\int_{[0,1]\times\overline{\Omega_\eta}} X(t,x)d(L(\lambda_\nu) \times L(\mu_\eta)) = X_0 < \infty$

by $(iii)$ of footnote 3 and Fubini's theorem, see [6]. By the hypothesis that $X_1 \in L^2(\overline{\Omega_\eta})$, and using Theorem 7 of [2], see also Theorems 3.31 and 3.34 of [5], we can find $\overline{V} \in SL^2(\overline{\Omega_\eta}, \mu_\eta)$, with $({}^\circ\overline{V}) = X_1$, a.e $L(\mu_\eta)$, (†). We now define $\overline{X} : \overline{T}_\nu \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ by taking $\overline{X}(t,x) = (E_\eta(\overline{V}|\mathcal{C}_\eta^{[\nu t]}))(x)$. We may assume that $\overline{X}$ is $\mathcal{D}_\nu \times \mathcal{C}_\eta$ measurable, by the definition of $E_\eta(|)$, see footnote 25 of Chapter 7, [5], and

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[5] In order to see this, it is sufficient to show that $X^+$ is a martingale, $(*)$. We have $X = X^+ - X^-$, and, by $(iv)$, for $0 \leq t \leq 1$;

$X_t = X_t^+ - X_t^- = E(X_1|\mathfrak{D}_t) = E(X_1^+ - X_1^-|\mathfrak{D}_t) = Y_t - Y_t'$ $(**)$

where $Y_t = E(X_1^+|\mathfrak{D}_t)$ and $Y_t' = E(X_1^-|\mathfrak{D}_t)$. It follows easily, modifying $Y$ to $Y^1$, and $Y'$ to $Y^{',1}$, a.e $L(\lambda_\nu) \times L(\mu_\eta)$, if necessary, and, using the tower law and definition of conditional expectations, see [8], that $Y, Y'$ are martingales and $Y, Y' \geq 0$. We then have, by $(**)$, that $X_t^+ = Y_t$ and $X_t^- = Y_t'$ a.e $L(\mu_\eta)$. Hence, $(*)$ is shown.

transfer of the corresponding result for finite measure spaces. Then, by Theorem 7.3 of [5];

$(^\circ\overline{X})(t,x) = {}^\circ(E_\eta(\overline{V}|\mathcal{C}_\eta^{[\nu t]}))(x) = E((^\circ\overline{V})|\sigma(\mathcal{C}_\eta^{[\nu t]})^{comp})$ $(**)$

Moreover, if $A \in \sigma(\mathcal{C}_\eta^{[\nu t]})^{comp}$, we have;

$\int_A X_{^\circ t} dL(\mu_\eta) = lim_{t'\rightarrow {}^\circ t} \int_A X_{t'} dL(\mu_\eta) = \int_A X_1 dL(\mu_\eta)$ $(***)$

using $(iv)$,$(v)$ of footnote 3 and the result of $(*)$ to apply the $DCT$. Hence, as $\mathfrak{D}_{^\circ t} \subset \sigma(\mathcal{C}_\eta^{[\nu t]})^{comp} \subset \mathfrak{D}_{t'}$, for $0 \leq {}^\circ t < t'$, using $(**)$ in Definition 0.4, we have;

$E((^\circ\overline{V})|\sigma(\mathcal{C}_\eta^{[\nu t]})^{comp}) = E((^\circ\overline{V})|\mathfrak{D}_{^\circ t}) = E(X_1|\mathfrak{D}_{^\circ t}) = X_{^\circ t}$

by $(***)$, $(\dagger)$ and $(iv)$ of footnote 3. By $(**)$, we then have $(^\circ\overline{X}_t) = X_{^\circ t}$, a.e $L(\mu_\eta)$. We now verify conditions $(i),(ii),(iii)$ of Definition 0.7. $(i)$ is clear by Definition of $\overline{X}$ and footnote 25 of Chapter 7, [5]. $(ii)$ follows by transfer of the tower law for the conditional expectation $E_\eta(|)$, see again footnote 25 of Chapter 7. $(iii)$ follows immediately from the fact that $\overline{V} \in SL^2(\overline{\Omega_\eta},\mu_\eta)$, and;

$|E_\eta(\overline{X}_t)| = |E_\eta(\overline{V})| \leq E_\eta(|\overline{V}|) \leq ||\overline{V}||_{SL^2} \simeq ||X_1||_{L^2} < \infty$ (for $t \in \mathcal{T}_\nu$)

by transfer of Holders inequality, the definition of $E_\eta(|)$, and property $(ii)$ in Definition 0.7. Finally, using Theorem 7.3 of [5], we have that the sequence $\{\overline{X}_{\frac{i}{\nu}} : 0 \leq i \leq \nu\} \subset SL^2(\overline{\Omega_\eta},\mu_\eta)$. The $S$-continuity claim follows from the proof of Theorem 8.1 in [3]. We omit the details. For the final claim, we modify $\overline{X}$ to obtain the final condition, while preserving the other properties. For $n \in {}^*\mathcal{N}$, we let;

$V_n = \{x : \exists t(|\Delta\overline{X}(t,x)| \geq \frac{1}{n})\}$, ([6])

By $S$-continuity of $\overline{X}$, we have that the internal set $A = \{n \in {}^*\mathcal{N} : \mu_\eta(V_n) \leq \frac{1}{n}\}$ contains $\mathcal{N}$, hence, it contains an infinite element $\kappa$. For $x \in V_\kappa$, we let $\tau(x)$ be the first $t$ such that $|\Delta\overline{X}(t,x)| \geq \frac{1}{\kappa}$ and let

[6]We use the notation $\Delta\overline{X}(t,x)$ to denote the increment $\overline{X}(t+\frac{1}{\nu},x) - \overline{X}(t,x)$, for $0 \leq t \leq 1-\frac{1}{\nu}$

$\tau(x) = 1$ otherwise. We let $\overline{W}$ be the internal process defined by;

$\overline{W}_0 = \overline{X}_0$

$\Delta\overline{W}(x,t) = \Delta\overline{X}(x,t)$, if $t < \tau(x)$.

$\Delta\overline{W}(x,t) = 0$, if $t \geq \tau(x)$.

We claim that $\overline{W}$ is a nonstandard martingale in the sense of Definition 0.7. For $(i)$, by hyperfinite induction, and the fact that $\overline{W}_0 = \overline{X}_0$, it is sufficient to show that if $\overline{W}_{\frac{i-1}{\nu}}$ is measurable with respect to $\mathcal{C}_\eta^{i-1}$, then $\overline{W}_{\frac{i}{\nu}}$ is measurable with respect to $\mathcal{C}_\eta^i$, for $1 \leq i \leq \nu$, $(\dagger\dagger)$. If $x \sim_i x'$, we have that $\frac{i-1}{\nu} < \tau(x)$ iff $\frac{i-1}{\nu} < \tau(x')$, as this is an internal definition depending only on information up to time $\frac{i}{\nu}$, hence must contain the equivalence class $[x]_{\sim_i}$. In this case, we have that $\overline{W}(x, \frac{i}{\nu}) = \overline{W}(x, \frac{i-1}{\nu}) + \Delta\overline{X}(x, \frac{i-1}{\nu})$, which is constant on $[x]_{\sim_i}$, using the inductive hypothesis and measurability of $\overline{X}$. The case when $\frac{i-1}{\nu} \geq \tau(x)$ is similar. Hence, $(\dagger\dagger)$ and $(i)$ are shown. For $(ii)$, it is sufficient to show that if $x \in \overline{\Omega}_\eta$, then;

$\int_{[x]_{\sim_{i-1}}} \overline{W}_{\frac{i-1}{\nu}} d\mu_\eta = \int_{[x]_{\sim_{i-1}}} \overline{W}_{\frac{i}{\nu}} d\mu_\eta$, for $1 \leq i \leq \nu$ $(\dagger\dagger\dagger)$

Clearly, if $\frac{i-1}{\nu} \geq \tau(x')$, for all $x' \in [x]_{\sim_{i-1}}$, then $\Delta\overline{W}(x, \frac{i-1}{\nu})|_{[x]_{\sim_{i-1}}} = 0$, and the result $(\dagger\dagger\dagger)$ follows trivially. Similarly, if $\frac{i-1}{\nu} < \tau(x')$, for all $x' \in [x]_{\sim_{i-1}}$, then $\overline{W}|_{[x]_{\sim_{i-1}} \times [\frac{i-1}{\nu}, \frac{i+1}{\nu})} = \overline{X}|_{[x]_{\sim_{i-1}} \times [\frac{i-1}{\nu}, \frac{i+1}{\nu})}$, and the result $(\dagger\dagger\dagger)$ follows from the martingale property of $\overline{X}$. We can, therefore, write $[x]_{\sim_{i-1}} = [x_1]_{\sim_i} \cup [x_2]_{\sim_i}$, and assume that $\frac{i-1}{\nu} < \tau(x')$, for all $x' \in [x_1]_{\sim_i}$, and $\frac{i-1}{\nu} \geq \tau(x')$, for all $x' \in [x_2]_{\sim_i}$. If $\frac{i-2}{\nu} \geq \tau(x')$, for all $x' \in [x_2]_{\sim_i}$, then the same must hold for all $x' \in [x_1]_{\sim_i}$, contradicting the assumption. Hence, we can also assume that $\frac{i-2}{\nu} < \tau(x')$, for all $x' \in [x_2]_{\sim_i}$. It follows that $|\Delta\overline{X}(x, \frac{i-1}{\nu})|_{[x_1]_{\sim_i}}| \leq \frac{1}{\kappa}$ and $|\Delta\overline{X}(x, \frac{i-1}{\nu})|_{[x_2]_{\sim_i}}| > \frac{1}{\kappa}$, but this contradicts the martingale property $(\dagger\dagger\dagger)$ for $\overline{X}$. Hence, this case can't happen, so $(\dagger\dagger\dagger)$ and $(ii)$ is shown. Property $(iii)$ follows from the fact that $\overline{W}_1 \in SL^2(\overline{\Omega}_\eta)$, $(\dagger\dagger\dagger\dagger)$, which we show below, and the inequality;

$E_\eta(|\overline{W}_{\frac{\nu t}{\nu}}|) \leq E_\eta(\overline{W}^2_{\frac{\nu t}{\nu}})^{\frac{1}{2}} \leq E_\eta(\overline{W}^2_1)^{\frac{1}{2}}$.

which uses Cauchy-Schwartz, and the martingale property $(ii)$. By construction $\overline{W}$ has infinitesimal increments. As we are only modifying $\overline{X}$ inside $V_\kappa \times \mathcal{T}_\nu$, where $L(\mu_\eta)(V_\kappa) = 0$ it is clear that $S$-continuity is preserved. Similarly, we must have that ${}^\circ(\overline{W}_t) = X_{{}^\circ t}$, for $t \in \overline{\mathcal{T}_\nu}$, a.e $L(\mu_\eta)$. It remains to show $(\dagger\dagger\dagger\dagger)$. By the above remark on modification, it is sufficient to show that $\int_{V_\kappa} \overline{W}_1^2 d\mu_\eta \simeq 0$. We can define a relation on $\overline{\Omega}_\eta$ by $x \sim x'$ if $x' \in [x]_{\tau(x)-1}$. If $x \sim x'$, then, by the above discussion, $\tau(x) = \tau(x')$, and so $\sim$ defines an equivalence relation. We clearly have that $V_\kappa = \bigcup_{1 \leq j \leq r} [x_j]_\sim$ is an internal union of such equivalence classes. A simple calculation gives that;

$$\int_{V_\kappa} \overline{W}_1^2 d\mu_\eta = {}^*\sum_{1 \leq j \leq r} \int_{[x_j]_\sim} \overline{W}_1^2 d\mu_\eta$$

$$= {}^*\sum_{1 \leq j \leq r} \int_{[x_j]_{\tau(x_j)-1}} \overline{X}_{\tau(x_j)-1}^2 d\mu_\eta$$

$$\leq {}^*\sum_{1 \leq j \leq r} \int_{[x_j]_{\tau(x_j)-1}} \overline{X}_1^2 d\mu_\eta$$

$$= \int_{V_\kappa} \overline{X}_1^2 d\mu_\eta \simeq 0$$

where we have used the definition of $\overline{W}$, and the calculation of Theorem 12(ii) in [2]. This gives the result.

□

**Lemma 0.10.** *Let $X$ be a tame martingale, and let $\overline{X}$ be as in Lemma 0.9. Then we can find $\kappa \in {}^*\mathcal{N} \setminus \mathcal{N}$ such that $\kappa | \nu$, and for all $t \in \mathcal{T}_\nu$;*

$$\int_{\overline{\Omega}_\eta} (\overline{X}_{t+\frac{1}{\kappa}}^2 - \overline{X}_t^2) d\mu_\eta \leq \frac{C+1}{\kappa}$$

*where $C \in \mathcal{R}_{\geq 0}$ is as given in footnote 3. Moreover, we can find $D \subset \overline{\Omega}_\eta$, with $\mu_\eta(D) \simeq 1$, $E \subset \mathcal{T}_\nu$ with $\mu_\eta(E) \simeq 0$, such that for all $t \in \mathcal{T}_\nu \setminus E$;*

$$1_D \kappa([\overline{X}]_{t+\frac{1}{\kappa}} - [\overline{X}]_t) \in SL^1(\overline{\Omega}_\eta, \mu_\eta)$$

*Proof.* Without loss of generality we can assume that $n|\nu$, for all $n \in \mathcal{N}$. If $t \in \mathcal{T}_\nu$ and $n \in \mathcal{N}$, we have that $\{\overline{X}_t, \overline{X}_{t+\frac{1}{n}}\} \subset SL^2(\overline{\Omega}_\eta, \mu_\eta)$, hence $(\overline{X}_{t+\frac{1}{n}}^2 - \overline{X}_t^2) \in SL^1(\overline{\Omega}_\eta, \mu_\eta)$. We, therefore, have that;

$${}^\circ(\int_{\overline{\Omega}_\eta} (\overline{X}_{t+\frac{1}{n}}^2 - \overline{X}_t^2) d\mu_\eta)$$

$= \int_{\overline{\Omega}_\eta}({}^\circ(\overline{X}_{t+\frac{1}{n}})^2 - {}^\circ(\overline{X}_t)^2)dL(\mu_\eta)$

$= \int_{\overline{\Omega}_\eta}(X^2_{{}^\circ t+\frac{1}{n}} - X^2_{{}^\circ t})dL(\mu_\eta) \leq \frac{C}{n}$

It follows that;

$\int_{\overline{\Omega}_\eta}(\overline{X}^2_{t+\frac{1}{n}} - \overline{X}^2_t)d\mu_\eta \leq \frac{C+1}{n}$

As this holds for all $n \in \mathcal{N}$, and the property is internal, we can find an infinite $\kappa|\nu$, such that;

$\int_{\overline{\Omega}_\eta}(\overline{X}^2_{t+\frac{1}{\kappa}} - \overline{X}^2_t)d\mu_\eta \leq \frac{C+1}{\kappa}$

for all $t \in \mathcal{T}_\nu$, as required, for the first part.

For the second condition, using Proposition 4.4.12 in [1], we can assume that there exists $C \subset \overline{\Omega}_\eta$, with $L(\mu_\eta)(C) = 1$, such that $[\overline{X}]$ lifts the standard process $[X]$ on $C \times \mathcal{T}_\nu$. For ease of notation, for $m \in \mathcal{N}$ and $t \in \mathcal{T}_\nu$, let $[\overline{X}]_{t,m}$ denote the increment $m([\overline{X}]_{t+\frac{1}{m}} - [\overline{X}]_t)$ and $[X]_{t,m}$ the corresponding standard increment, for $t \in [0,1]$. We clearly have that ${}^\circ[\overline{X}]_{t,m} = [X]_{{}^\circ t,m}$ on $C \times \mathcal{T}_\nu$. Choose a sequence of $\{C_m : m \in \mathcal{N}\}$, with $C_m \subset C$, such that each $C_m \in \mathcal{C}_\eta$ and $\mu_\eta(C_m) = 1 - \frac{1}{m}$. As $C_m$ is internal and $[\overline{X}]$ lifts $X$ on $C_m \times \mathcal{T}_\nu$, by compactness, we must have that $[\overline{X}]$ is bounded on $C_m \times \mathcal{T}_\nu$, $|[\overline{X}]| \leq D(m)$, where $D(m) \in \mathcal{R}$. Let $V \subset [0,1]$ be the set on which the incremental condition $(vii)$ in Definition 3 does not hold. Then $L(\lambda_\nu)(st^{-1}(V)) = 0$, and we can choose $E_m \in \mathcal{C}_\nu$, with $\lambda_\nu(E_m) = \frac{1}{m}$, such that $E_m \supset st^{-1}(V)$. Then, we have that, for all $t \in \mathcal{T}_\nu \setminus E_m$, for all $K \leq 2D(m)m$, that;

${}^\circ\int_{[\overline{X}]_{t,m}>K} 1_{C_m}[\overline{X}]_{t,m}d\mu_\eta$

$= {}^\circ\int_{\overline{\Omega}_\eta} 1_{([\overline{X}]_{t,m}>K)\cap C_m}[\overline{X}]_{t,m}d\mu_\eta$

$= \int_{\overline{\Omega}_\eta} 1_{([\overline{X}]_{t,m}>K)\cap C_m}[X]_{{}^\circ t,m}dL\mu_\eta$

It follows that;

$\int_{[\overline{X}]_{t,m}>K} 1_{C_m}[\overline{X}]_{t,m}d\mu_\eta$

$< \int_{[X]_{{}^\circ t,m}>K-1} 1_{C_m}[X]_{{}^\circ t,m}dL(\mu_\eta) + \frac{1}{m}$

$< \int_{[X]_{\circ t,m}|>K-1} [X]_{\circ t,m} dL(\mu_\eta) + \frac{1}{m}$

$= f^*(K-1) + \frac{1}{m}$

where we have used condition $(vii)$ in the definition from footnote 3. The condition $(*)$ holds trivially when $K > 2D(m)m$, as then;

$\int_{[\overline{X}]_{t,m}>K} 1_{C_m}[\overline{X}]_{t,m} d\mu_\eta = 0$

It follows that;

${}^*\mathcal{R} \models (\forall t \in \mathcal{T}_\nu \setminus E_m)(\forall K,)$

$\int_{[\overline{X}]_{t,m}>K} 1_{C_m}[\overline{X}]_{t,m} d\mu_\eta < f^*(K-1) + \frac{1}{m}$

for all sufficiently large $m \in \mathcal{N}$. By overflow, we can satisfy the condition for the same infinite $\kappa \in {}^*\mathcal{N}$ as above. In particular, we obtain, for infinite $K$, $t \in \mathcal{T}_\nu \setminus E_\kappa$ that;

$\int_{[\overline{X}]_{t,\kappa}>K} 1_{C_\kappa}[\overline{X}]_{t,\kappa} d\mu_\eta < f^*(K-1) + \frac{1}{\kappa} \simeq 0$

It follows, using the criterion in Lemma 3.19 of [5], that $1_{C_\kappa}[\overline{X}]_{t,\kappa} \in SL^1(\overline{\Omega}_\eta, \mu_\eta)$, for all $t \in \mathcal{T}_\nu \setminus E_\kappa$ as required. Letting $D = C_\kappa$ $E = E_\kappa$ and noting that $\mu_\eta(D) = 1 - \frac{1}{\kappa} \simeq 1$, $\mu_\eta(E) = \frac{1}{\kappa} \simeq 0$ we obtain the result.

□

**Definition 0.11.** *Let $\overline{X}$ be as in Definition 0.7, with $E_\eta(\overline{X}_0) = 0$, and let $\{c_j(t,x) : 1 \leq j \leq \nu\}$ be given as in Lemma 0.8. Then we define;*

$\overline{H} : \overline{\mathcal{T}_\nu} \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$, $\overline{Z} : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$, $\overline{Y} : \overline{\mathcal{T}_\kappa} \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$, $\overline{W} : \overline{\mathcal{T}_\kappa} \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$, $\{d_j(t,x) : 1 \leq j \leq \nu\}$, $\overline{S} : \overline{\mathcal{T}_\nu} \times \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$, $\overline{Q} : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$ *by;*

$\overline{H}(t,x) = \sqrt{\nu} c_{[\nu t]+1}(s,x)$

*where* $s \geq \frac{[\nu t]+1}{\nu}$, *for* $0 \leq t < 1$ *and;*

$\overline{H}(t,x) = 0$, *for* $t = 1$

$\overline{Z}(x) = {}^* \sum_{0 \leq j \leq \nu-1} (\overline{X}_{\frac{j+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2$

$\overline{Y}(t,x)=0$, *for* $0\leq[\nu t]<\frac{\nu}{\kappa}-1$

$\overline{Y}(t,x)=\frac{k}{\nu}(\overline{H}^2_{\frac{[\nu t]}{\nu}}+\overline{H}^2_{\frac{[\nu t]-1}{\nu}}+\ldots+\overline{H}^2_{\frac{[\nu t]-\frac{\nu}{\kappa}+1}{\nu}})$, *for* $\frac{\nu}{\kappa}-1\leq[\nu t]\leq 1$

$\overline{W}=\sqrt{\overline{Y}}$

$d_j(s,x)=\frac{1}{\sqrt{\nu}}\overline{W}_{\frac{j-1}{\nu}}(x)$, *for* $1\leq j\leq\nu$, *and* $\frac{j}{\nu}\leq s\leq 1$.

$\overline{S}(t,x)={}^*\sum_{j=1}^{[\nu t]}d_j(1,x)\omega_j$

$\overline{Q}(x)={}^*\sum_{0\leq j\leq\nu-1}(\overline{S}_{\frac{j+1}{\nu}}(x)-\overline{S}_{\frac{j}{\nu}}(x))^2$

**Lemma 0.12.** *If* $\overline{X}$ *is as in Lemma 0.9, and* $X$ *is tame, then* $\overline{Y}\in SL^1(\mathcal{T}_\nu\times\overline{\Omega}_\eta,\lambda_\nu\times\mu_\eta)$, $\overline{Z}\in SL^1(\overline{\Omega}_\eta,\mu_\eta)$ *and* $\overline{S}$ *is a nonstandard martingale, with* $\overline{S}_1\in SL^2(\overline{\Omega}_\eta,\mu_\eta)$.

*Proof.* The fact that $\overline{Z}\in SL^1(\overline{\Omega}_\eta,\mu_\eta)$, (†), follows from Proposition 4.4.3 of [1] and the properties of $\overline{X}$. This does not require that $\overline{X}$ is $S$-continuous or has infinitesimal increments.

For the last claim, it is easily seen that the functions $d_j:[\frac{j}{\nu},1]\times\overline{\Omega_\eta}\rightarrow{}^*\mathcal{C}$ are $\mathcal{D}_\nu\times\mathcal{C}_\eta^{j-1}$-measurable, for $1\leq j\leq\nu$. Hence, using Lemma 0.8, we have that $\overline{S}$ satisfies conditions $(i)$ and $(ii)$ of Definition 0.7. By Proposition 4.4.3 of [1], it is sufficient to show that $\overline{Q}\in SL^1(\overline{\Omega}_\eta,\mu_\eta)$, as $\overline{S}_0=0$. (explain why we can assume this?) We compute;

$\overline{Q}(x)={}^*\sum_{0\leq j\leq\nu-1}(\overline{S}_{\frac{j+1}{\nu}}(x)-\overline{S}_{\frac{j}{\nu}}(x))^2$

$={}^*\sum_{1\leq j\leq\nu-1}d_j^2(1,x)$

$=\frac{1}{\nu}{}^*\sum_{1\leq j\leq\nu-1}\overline{W}^2_{\frac{j-1}{\nu}}(x)$

$=\frac{1}{\nu}{}^*\sum_{1\leq j\leq\nu-1}\overline{Y}_{\frac{j-1}{\nu}}(x)$

$=\frac{1}{\nu}{}^*\sum_{\frac{\nu}{\kappa}-1\leq j\leq\nu-2}\frac{k}{\nu}(\overline{H}^2_{\frac{j}{\nu}}+\overline{H}^2_{\frac{j-1}{\nu}}+\ldots+\overline{H}^2_{\frac{j-\frac{\nu}{\kappa}+1}{\nu}})$

$=\frac{1}{\nu}{}^*\sum_{0\leq j\leq\nu-1}\overline{H}^2_{\frac{j}{\nu}}d\mu_\eta+r(x)$

$={}^*\sum_{1\leq j\leq\nu}c_j^2(1,x)d\mu_\eta+r(x)$

$= {}^{*}\sum_{0\leq j\leq\nu-1}(\overline{X}_{\frac{j+1}{\nu}}(x)-\overline{X}_{\frac{j}{\nu}}(x))^{2}+r(x)$

$=\overline{Z}(x)+r(x)$

where $r(x)\geq 0$ is a remainder term. We have that $E_{\eta}(r(x))\simeq 0$, and $r(x)\leq\overline{Z}(x)$. It follows, easily, that $r(x)\simeq 0$, a.e $L(\mu_{\eta})$, $\overline{Q}(x)\simeq\overline{Z}(x)$ a.e $L(\mu_{\eta})$, and $\overline{Q}(x)\in SL^{2}(\overline{\Omega}_{\eta},\mu_{\eta})$ as required.

For the first part, observe first that $\overline{H}$ is progressively measurable, that is $\overline{H}_{t}$ is measurable with respect to $\mathcal{C}_{\eta}^{[\nu t]}$, hence, so is $\overline{Y}$.

By Lemma 3.19 of [5], it is sufficient to prove that;

$\int_{\overline{Y}>K}\overline{Y}d(\lambda_{\nu}\times\mu_{\eta})\simeq 0$, for $K$ infinite

As $\overline{Y}$ is progressively measurable, the set $\overline{Y}>K$ is progressively measurable. Moreover, it has infinitesimal measure. This clearly follows from showing that;

$\int_{\mathcal{T}_{\nu}\times\overline{\Omega}_{\eta}}\overline{Y}d\lambda_{\nu}d\mu_{\eta}$ is finite, $(*)$

To see $(*)$, we compute;

$\int_{\mathcal{T}_{\nu}\times\overline{\Omega}_{\eta}}\overline{Y}(t,x)d\lambda_{\nu}d\mu_{\eta}$

$=\frac{1}{\nu}{}^{*}\sum_{0\leq j\leq\nu-1}\int_{\overline{\Omega}_{\eta}}\overline{Y}(\frac{j}{\nu},x)d\mu_{\eta}$

$=\frac{1}{\nu}{}^{*}\sum_{\frac{\nu}{k}-1\leq j\leq\nu-1}\int_{\overline{\Omega}_{\eta}}\overline{Y}(\frac{j}{\nu},x)d\mu_{\eta}$

$=\frac{1}{\nu}{}^{*}\sum_{\frac{\nu}{k}-1\leq j\leq\nu-1}\int_{\overline{\Omega}_{\eta}}(\frac{k}{\nu}(\overline{H}^{2}_{\frac{j}{\nu}}+\overline{H}^{2}_{\frac{j-1}{\nu}}+\ldots+\overline{H}^{2}_{\frac{j-\frac{\nu}{\kappa}+1}{\nu}}))d\mu_{\eta}$

$\leq\frac{1}{\nu}{}^{*}\sum_{0\leq j\leq\nu}\int_{\overline{\Omega}_{\eta}}\overline{H}^{2}_{\frac{j}{\nu}}d\mu_{\eta}$

$=\frac{1}{\nu}{}^{*}\sum_{0\leq j\leq\nu-1}\int_{\overline{\Omega}_{\eta}}\nu|c_{j}(1,x)|^{2}d\mu_{\eta}$ $(\dagger\dagger)$

$={}^{*}\sum_{0\leq j\leq\nu-1}\int_{\overline{\Omega}_{\eta}}|c_{j}(1,x)|^{2}d\mu_{\eta}$

$=\int_{\overline{\Omega}_{\eta}}|\overline{X}_{1}|^{2}d\mu_{\eta}$ $(\dagger\dagger\dagger)$

where, in $(\dagger\dagger)$, we have used Definition 0.11, and, in $(\dagger\dagger\dagger)$, we have used the fact that $\overline{X}_{1}={}^{*}\sum_{0\leq j\leq\nu-1}c_{j}(1,x)\omega_{j}$, by Lemma 0.8, and the orthogonality observation $(*)$ there. Hence, $(*)$ is shown, by the assumption that $\overline{X}_1 \in SL^2(\overline{\Omega}_\eta, \mu_\eta)$. Therefore, it is sufficient to prove that;

$\int_A \overline{Y} d(\lambda_\nu \times \mu_\eta) \simeq 0$, for a progressively measurable set $A$ with $\lambda_\nu \times \mu_\eta(A) \simeq 0$. $(**)$

We now verify $(**)$;

Case 1. Let $A \subset \overline{\Omega_\eta}$, with $\mu_\eta(A) \simeq 0$, then;

$\int_{A \times \overline{T}_\eta} \overline{Y} d\mu_\eta d\lambda_\nu$

$= \frac{1}{\nu}{}^* \sum_{0 \leq j \leq \nu-1} \int_A \overline{Y} d\mu_\eta$

$\leq \frac{1}{\nu}{}^* \sum_{0 \leq j \leq \nu-1} \int_A \overline{H}^2_{\frac{j}{\nu}} d\mu_\eta$ (as above)

$= \int_A {}^* \sum_{0 \leq j \leq \nu-1} c_j(1,x)^2 d\mu_\eta$

$= \int_A {}^* \sum_{0 \leq j \leq \nu-1} (\overline{X}_{\frac{j+1}{\nu}} - \overline{X}_{\frac{j}{\nu}})^2 d\mu_\eta = \int_A \overline{Z} \simeq 0$

by $(\dagger)$.

Case 2. Let $B \subset \overline{T}_\nu$, with $B \in \mathcal{D}_\nu$ and $\lambda_\nu(B) \simeq 0$. We can write $B = \bigcup_{1 \leq j \leq s} I_j$, where $I_j$ is an interval of the form $[\frac{i_j}{\nu}, \frac{i_j+1}{\nu})$, for some $0 \leq i_j \leq \nu - 1$, and $\frac{s}{\nu} \simeq 0$. We compute, for $i_j \geq \frac{\nu}{\kappa} - 1$;

$\int_{\overline{\Omega}_\eta \times I_j} \overline{Y}(t,x) d\lambda_\nu d\mu_\eta$

$= \frac{1}{\nu} \int_{\overline{\Omega}_\eta} (\frac{k}{\nu}(\overline{H}^2_{\frac{i_j}{\nu}} + \overline{H}^2_{\frac{i_j-1}{\nu}} + \ldots + \overline{H}^2_{\frac{i_j - \frac{\nu}{\kappa}+1}{\nu}})) d\mu_\eta$

$= \frac{k}{\nu} \int_{\overline{\Omega}_\eta} (c^2_{i_j+1} + \ldots c^2_{i_j - \frac{\nu}{\kappa}+2}) d\mu_\eta$

We have that;

$\overline{X}(t,x) = \sum_{0 \leq j \leq t\nu} c_j(1,x)\omega_j(x)$

$\overline{X}(t,x)^2 = \sum_{0 \leq j,k \leq t\nu} c_j(1,x) c_k(1,x) \omega_j(x) \omega_k(x)$ $(\sharp)$

Then;

$\int_{\overline{\Omega}_\eta}(\overline{X}_t)^2(x)d\mu_\eta$

$= \sum_{0\leq j,k\leq[t\nu]}\int_{\overline{\Omega}_\eta} c_j(1,x)c_k(1,x)\omega_j\omega_k d\mu_\eta$ (using ($\sharp$))

$= \sum_{0\leq j\leq[t\nu]}\int_{\overline{\Omega}_\eta} c_j^2(1,x)d\mu_\eta$ (using Lemma 0.8) ($\sharp\sharp$)

It follows that;

$\int_{\overline{\Omega}_\eta}(c^2_{i_j+1}+\ldots c^2_{i_j-\frac{\nu}{\kappa}+2})d\mu_\eta$

$= \int_{\overline{\Omega}_\eta}(\overline{X}^2_{\frac{i_j+1}{\nu}} - \overline{X}^2_{\frac{i_j+1-\frac{\nu}{\kappa}}{\nu}})d\mu_\eta$

and, therefore, that;

$\int_{\overline{\Omega}_\eta\times I_j}\overline{Y}(t,x)d\lambda_\nu d\mu_\eta$

$\frac{\kappa}{\nu}\int_{\overline{\Omega}_\eta}(\overline{X}^2_{\frac{i_j+1}{\nu}} - \overline{X}^2_{\frac{i_j+1}{\nu}-\frac{1}{\kappa}})d\mu_\eta \leq \frac{C+1}{\nu}$

using Lemma 0.10. We then have that;

$\int_{\overline{\Omega}_\eta\times B}\overline{Y}(t,x)d\lambda_\nu d\mu_\eta$

$= {}^*\sum_{1\leq j\leq s}\int_{\overline{\Omega}_\eta\times I_j}\overline{Y}(t,x)d\lambda_\nu d\mu_\eta \leq \frac{s(C+1)}{\nu}\simeq 0$

as required.

Case 3. Let $B\in\mathcal{D}_\nu\times\mathcal{C}_\eta$, with $(\lambda_\nu\times\mu_\eta)(B)=\delta\simeq 0$. Let;

$I=\{i: 0\leq i\leq\nu, \mu_\eta(pr_\eta(B\cap pr_\nu^{-1}(\frac{i}{\nu})))>\delta^{\frac{1}{2}}\}$

Let $C=\bigcup_{i\in I}[\frac{i}{\nu},\frac{i+1}{\nu})$, so $C\in\mathcal{D}_\nu$, and let $B_1=B\cap pr_\nu^{-1}(C)$. As $B_1\subset B$, and by construction of $C$, we have that;

$\delta\geq(\lambda_\nu\times\mu_\eta)(B_1)>\delta^{\frac{1}{2}}\lambda_\nu(C)$

It follows that $\lambda_\nu(C)<\delta^{\frac{1}{2}}\simeq 0$. By Case 2, we have that;

$\int_{B_1}\overline{Y}d(\lambda_\nu\times\mu_\eta)$

$\leq\int_{\overline{\Omega}_\eta\times C}\overline{Y}d(\lambda_\nu\times\mu_\eta)\simeq 0$

Let $B_2 = B \cap B_1^c$, then $B_2 \in \mathcal{D}_\nu \times \mathcal{C}_\eta$ and $(\lambda_\nu \times \mu_\eta)(B_2) \simeq 0$, and to show Case 3, it is sufficient to prove that;

$\int_{B_2} \overline{Y} d(\lambda_\nu \times \mu_\eta) \simeq 0$

We say that $B \in \mathcal{D}_\nu \times \mathcal{C}_\eta$ is wide, if there exists $\epsilon \simeq 0$, with $\mu_\eta(pr_\eta(B \cap pr_\nu^{-1}(t))) \leq \epsilon$, for $t \in \mathcal{T}_\nu$, and note that $B_2$ is wide. We are thus reduced to;

Case 4. Suppose $B$ is progressively measurable and wide, and let;

$I_j = \{i \in {}^*\mathcal{N} : 0 \leq i \leq \nu - 1, rem(2, i) = j, B \cap pr_\nu^{-1}(\frac{i}{\nu}) \neq \emptyset\}$, for $0 \leq j \leq 1$

$S_j = \bigcup_{i \in I_j} [\frac{i}{\nu}, \frac{i+1}{\nu})$, $0 \leq j \leq 1$

$B_j = B \cap pr_\nu^{-1}(S_j)$, $0 \leq j \leq 1$

Then $B = \bigcup_{0 \leq j \leq 1} B_j$, and each $B_j$ is progressively measurable and wide. Let;

$V_j = \{(i, s) \in {}^*\mathcal{N}^2 : 1 \leq i \leq \nu - 1, 0 \leq s < 2^i, rem(2, s) = j, B \cap pr_\nu^{-1}(\frac{i}{\nu}) \neq \emptyset, B \cap pr_\eta^{-1}(\frac{s}{\eta}) \neq \emptyset\}$, for $0 \leq j \leq 1$

$W_j = \bigcup_{(i,s) \in V_j} [\frac{i}{\nu}, \frac{i+1}{\nu}) \times [\frac{s}{2^i}, \frac{s+1}{2^i})$, $0 \leq j \leq 1$

By the progressive measurability of $B$, $B = \bigcup_{0 \leq j \leq 1} W_j$ and each $W_j$ is progressively measurable and wide. Let $B_{ij} = B_i \cap W_j$, $0 \leq i \leq 1$, $0 \leq j \leq 1$. Then $B = \bigcup_{0 \leq i,j \leq 1} B_{ij}$ and each $B_{ij}$ is progressively measurable and wide. We say that $B \in \mathcal{D}_\nu \times \mathcal{C}_\eta$ is separated if, for all $(t, x) \in B$, $(t + \frac{1}{\nu}) \notin pr_\nu(B)$, and $(t, \frac{[x2^{[t\nu]}]+1}{2^{[t\nu]}}) \notin B$, for $[\nu t] \geq 1$ and $0 \leq [[x2^{[t\nu]}] \leq 2^{[t\nu]} - 2$. . By construction, each $B_{ij}$ is separated, for $0 \leq i, j \leq 1$. We are thus reduced to;

Case 5. Suppose $B$ is progressively measurable, wide and separated.

Observe that;

$\kappa([\overline{X}]_t - [\overline{X}]_{t-\frac{1}{\kappa}})$

$= \kappa(\sum_{j=0}^{[\nu t]-1}(\overline{X}_{\frac{j+1}{\nu}} - \overline{X}_{\frac{j}{\nu}})^2 - \sum_{j=0}^{[\nu(t-\frac{1}{\kappa})]-1}(\overline{X}_{\frac{j+1}{\nu}} - \overline{X}_{\frac{j}{\nu}})^2)$

$= \kappa(\sum_{j=[\nu t]-\frac{\nu}{\kappa}}^{[\nu t]-1}(c_{j+1})^2)$

$= \frac{\kappa}{\nu}(\sum_{j=[\nu t]-\frac{\nu}{\kappa}}^{[\nu t]-1}(\overline{H}_{\frac{j}{\nu}})^2)$

$= \overline{Y}_{t-\frac{1}{\nu}}$

It follows from Lemma 0.10, that there exists $E'$ with $\mu_\eta(E') = 0$, such that $1_D\overline{Y}_t \in SL^1(\overline{\Omega}_\eta, \mu_\eta)$, (††) for all $t \in \mathcal{T}_\nu \setminus E'$. We now compute;

$\int_B \overline{Y} d(\lambda_\nu \times \mu_\eta)$

$\leq \int_{B\cap(D^c\times\mathcal{T}_\nu)} \overline{Y} d(\lambda_\nu \times \mu_\eta) + \int_{B\cap(\overline{\Omega}_\eta\times E')} \overline{Y} d(\lambda_\nu \times \mu_\eta)$

$+ \int_{B\cap(D\times\mathcal{T}_\nu\setminus E')} \overline{Y} d(\lambda_\nu \times \mu_\eta)$

$\simeq \int_{B\cap(D\times\mathcal{T}_\nu\setminus E')} \overline{Y} d(\lambda_\nu \times \mu_\eta)$ (by Cases 1,2)

$= \int_{\mathcal{T}_\nu\setminus E'} \int_{\overline{\Omega}_\eta} 1_D\overline{Y}_t d\mu_\eta d\lambda_\nu$

$= \int_{\mathcal{T}_\nu\setminus E'} g(t) d\lambda_\nu$ (where $g \simeq 0$ on $\mathcal{T}_\nu \setminus E'$)

$\simeq 0$

where we have used the assumption (††) and the fact that $B$ is wide in the penultimate line. It follows that $\overline{Y} \in SL^1(\overline{\Omega}_\eta \times \mathcal{T}_\nu)$ as required.

□

**Theorem 0.13.** *Any tame martingale $X$ is representable as a stochastic integral;*

$X(t,x) = \int_0^t F(s,x) d\beta_s$

*where $F : [0,1] \times \overline{\Omega}_\eta \rightarrow \mathcal{R} \in L^2([0,1] \times \overline{\Omega}_\eta, L(\mu_\eta))$, and $\beta_s$ is a Brownian motion.*

*Proof.* By Lemma 0.9, there exists a nonstandard martingale $\overline{X}$, with ${}^\circ(\overline{X}_t) = X_{{}^\circ t}$, for $t \in \overline{\mathcal{T}}_\nu$, a.e $L(\mu_\eta)$. Let notation be as in Definition 0.11. Then by Lemma 0.12, we have shown that $\overline{Y} \in SL^1(\overline{\mathcal{T}_\nu} \times \overline{\Omega}_\eta)$.

We have that $\overline{S} = \int \overline{W} d\chi$, , where $\chi$ is Anderson's random walk, and, therefore, the quadratic variation;

$[\overline{S}] = \overline{Q} = \int \overline{W}^2 dt.$

We claim that;

${}^\circ[\overline{S}](x,t) = \int_0^t f(x,s)ds$ a.e $dL(\mu_\eta)$ $(*)$

where $f \in L^1(\overline{\Omega}_\eta \times [0,1])$. To see this, we first claim that $\overline{W}_x^2 \in SL^1(\mathcal{T}_\nu)$ $a.e$ $dL(\mu_\eta)$, $(**)$. Suppose not, then, using Theorem 9 of [2], there exists $A$ with $L(\mu_\eta)(A) > 0$, such that;

${}^\circ \int_0^1 \overline{W}_x^2 d\lambda_\nu > \int_0^1 {}^\circ\overline{W}_x^2 dL(\lambda_\nu).$

But then;

${}^\circ \int_A \int_0^1 \overline{W}^2 d\lambda_\nu d\mu_\eta$

$\geq \int_A {}^\circ \int_0^1 \overline{W}^2 d\lambda_\nu dL(\mu_\eta)$

$> \int_A \int_0^1 {}^\circ\overline{W}^2 dL(\lambda_\nu) dL(\mu_\eta)$

contradicting the fact that $\overline{W} \in SL^2(\overline{\mathcal{T}_\nu} \times \overline{\Omega_\eta}, \lambda_\nu \times \mu_\eta)$. Hence, $(**)$ is shown. Let $V_x(t) = \int_0^t \overline{W}_x^2 d\lambda_\nu$, for $t \in [0,1]$. By $(**)$, we have that;

${}^\circ V_x(t) = \int_0^t {}^\circ\overline{W}_x^2 dL(\lambda_\nu)$

We claim that ${}^\circ V_x$ is absolutely continuous, $(***)$. Suppose not, then there exist internal $B_n \subset \mathcal{T}_\nu$, with each $B_n$ a finite union of intervals with real endpoints, such that $\lambda(B_n \cap [0,1]) < \frac{1}{n}$, where $\lambda$ is Lebesgue measure, and $\epsilon \in \mathcal{R}_{>0}$, such that;

$\int_{B_n} {}^\circ\overline{W}_x^2 dL(\lambda_\nu) > \epsilon$

Then ${}^\circ \int_{B_n} \overline{W}_x^2 d\lambda_\nu \geq \int_{B_n} {}^\circ\overline{W}_x^2 dL(\lambda_\nu > \epsilon$

and $\lambda_\nu(B_n) \simeq \lambda(B_n \cap [0,1]) < \frac{1}{n}$

as each $B_n$ is a finite union of intervals. We can extend the sequence $(B_n)_{n \in \mathcal{N}}$ to an internal sequence indexed by ${}^*\mathcal{N}$. By overflow, we can find an infinite $\rho \in {}^*\mathcal{N} \setminus \mathcal{N}$, with $B_\rho \in \mathcal{D}_\nu$, such that $\lambda_\nu(B_\rho) < \frac{1}{\rho} \simeq 0$ and;

$\int_{B_\rho} \overline{W}_x^2 d\lambda_\nu > \epsilon$

This contradicts $(**)$. Hence, $(***)$ is shown. By real analysis, see [6] Theorem 7.18, the derivative $f_x = ({}^\circ V_x)'$ exists a.e $d\lambda$, $f_x \in L^1([0,1])$ and;

${}^\circ[\overline{S}](x,t) = \int_0^t f(x,s)ds$ a.e $dL(\mu_\eta)$

We compute;

$\int_{\overline{\Omega}_\eta} \int_0^1 f(x,s)ds$

$= \int_{\overline{\Omega}_\eta} \int_{\overline{\mathcal{T}}_\nu} {}^\circ\overline{W}^2 dL(\lambda_\nu)dL(\mu_\eta)$

$= {}^\circ \int_{\overline{\Omega}_\eta} \int_{\overline{\mathcal{T}}_\nu} \overline{W}^2 d\lambda_\nu d\mu_\eta$

which is finite, as $\overline{W} \in SL^2(\overline{\mathcal{T}_\nu} \times \overline{\Omega_\eta}, \lambda_\nu \times \mu_\eta)$, hence $f \in L^1(\overline{\Omega_\eta} \times [0,1])$, thus $(*)$ is shown.

We have that;

$[\overline{S}]_t \simeq [\overline{X}]_t = \overline{Z}_t \ a.edL(\mu_\eta$

This follows by computing the remainder term $r(x)$ in the proof of Lemma 0.12 and using the fact that $\overline{Z}$ is $S$-continuous. This last is a consequence of the fact that $\overline{X}$ is $S$-continuous and $\overline{X}_1 \in SL^2(\overline{\Omega}_\eta)$, using Theorem 4.2.16 of [1]. Hence, we have;

${}^\circ[\overline{X}](x,t) = \int_0^t f(x,s)ds$ a.e $dL(\mu_\eta)$ $(****)$

Define a new adapted process $g$ by;

$g(x,t) = f^{\frac{-1}{2}}(x,t)$ if $f(x,t) \neq 0$, and $g(x,t) = 0$ otherwise.

Let $1_g$ be the characteristic function of the set $\{(x,t) : g(x,t) = 0\}$. We have that;

$E(\int_0^1 g(x,s)^2 d^\circ[\overline{X}]) = E(\int_0^1 g(x,s)^2 f(x,s)ds) \leq 1$

hence, $g \in L^2(\nu_{\circ[\overline{X}]})$. Let $G \in SL^2(\overline{X})$ be a 2-lifting of $g$, and $1_G$ a 2-lifting of $1_g$. We can assume that $G.1_G = 0$. Define;

$\beta(x,t) = {}^\circ(\int_0^t G(x,s)d\overline{X}(x,s) + \int_0^t 1_G(x,s)d\chi(x,s))$

Since, $G$ and $1_G$ have disjoint supports;

$[\beta](x,t) = {}^\circ[\int Gd\overline{X}](x,t) + {}^\circ[\int 1_G d\chi](x,t)$

$= {}^\circ(\int G^2 d\overline{X})(x,t) + {}^\circ[\int 1_G dt](x,t)$

$= \int_0^t g^2 f ds + \int_0^t 1_g^2 ds = \int_0^t 1 ds = t$

It follows, using Proposition 4.4.13 and 4.4.18 of [1], this requires that $\overline{X}$ has infinitesimal increments, that $\beta$ is a Brownian motion, adapted to the filtration $(\overline{\Omega}_\eta, \mathcal{D}_t, L(\mu_\eta))$. We have that $f^{\frac{1}{2}} \in L^2(\nu_\beta)$ and;

$\int f^{\frac{1}{2}} d\beta = \int f^{\frac{1}{2}} g d^\circ\overline{X} + \int f^{\frac{1}{2}} 1_g d^\circ\chi = \int f^{\frac{1}{2}} g d^\circ\overline{X}$

since $f^{\frac{1}{2}} 1_g = 0$. It remains to show that ${}^\circ\overline{X} = \int f^{\frac{1}{2}} g d^\circ\overline{X}$, since, we then get the result by setting $F = f^{\frac{1}{2}}$. Using Doob's inequality;

$E(sup_{q\leq 1, q\in\mathcal{Q}}({}^\circ\overline{X}(q) - \int_0^q f^{\frac{1}{2}} g d^\circ\overline{X})^2)$

$\leq 4E(({}^\circ\overline{X}(1) - \int_0^1 f^{\frac{1}{2}} g d^\circ\overline{X})^2)$

$= 4E(\int_0^1 (1 - f^{\frac{1}{2}} g)^2 d^\circ\overline{X})$

$= 4E(\int_0^1 (1 - f^{\frac{1}{2}} g)^2 dt) = 0$

as $f^{\frac{1}{2}} g = 1$, whenever $f \neq 0$.

□

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