SOLVING THE HEAT EQUATION USING NONSTANDARD ANALYSIS

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Abstract. We use the nonstandard Fourier transform method, see [6], along with an established nonstandard approach to ODE's, see [2] and [7], to find a solution to the heat equation, on $(0,\infty)\times\mathcal{R}$, with a given boundary condition g at $t=0$. We use this result to find an algorithm, converging to a solution of this equation, with applications to derivatives pricing in finance.

We adopt the following notation;

Definition 0.1. *For $\eta\in{}^{*}\mathcal{N}\setminus\mathcal{N}$, we let $(\overline{\mathcal{R}_\eta},\mathfrak{C}_\eta,\lambda_\eta)$ be as in Definition 0.15 of* [6].

We let $(\overline{\mathcal{R}_\eta},L(\mathfrak{C}_\eta),L(\lambda_\eta))$ denote the associated Loeb space, see Definition 0.5 of [6].

$(\mathcal{R},\mathfrak{B},\mu),(\mathcal{R}^{+-\infty},\mathfrak{B}',\mu')$ *are as in Lemma 0.6 of* [6].

$\overline{\mathcal{T}_\eta}=\{\tau\in{}^{*}\mathcal{R}_{\geq 0}:0\leq\tau<\eta\}$ *and we again denote by $\mathfrak{C}_\eta$, the restriction of $\mathfrak{C}_\eta$ to $\overline{\mathcal{T}_\eta}$, and λ_η the restriction of the counting measure.*

$(\overline{\mathcal{T}_\eta},L(\mathfrak{C}_\eta),L(\lambda_\eta))$ *is the corresponding Loeb space.*

$\mathcal{T}=\mathcal{R}_{\geq 0}$ *and $(\mathcal{T},\mathfrak{B},\mu),(\mathcal{T}^{+\infty},\mathfrak{B}',\mu')$ are defined analogously to Lemma 0.6 of* [6].

$(\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta},\mathfrak{C}_\eta^2,\lambda_\eta^2)$ *is as in Definition 0.15 of* [6].

$(\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta},L(\mathfrak{C}_\eta^2),L(\lambda_\eta^2))$ *is the corresponding Loeb space.*

$(\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta},L(\mathfrak{C}_\eta)\times L(\mathfrak{C}_\eta),L(\lambda_\eta)\times L(\lambda_\eta))$ *is the complete product of the Loeb spaces $(\overline{\mathcal{T}_\eta},L(\mathfrak{C}_\eta),L(\lambda_\eta))$ and $(\overline{\mathcal{R}_\eta},L(\mathfrak{C}_\eta),L(\lambda_\eta))$.*

Similarly, $(\mathcal{T}^{+\infty}\times\mathcal{R}^{+-\infty},\mathfrak{B}'\times\mathfrak{B}',\mu'\times\mu')$ *and* $(\mathcal{T}\times\mathcal{R},\mathfrak{B}\times\mathfrak{B},\mu\times\mu)$ *are the complete products of* $(\mathcal{T}^{+\infty},\mathfrak{B}',\mu')$, $(\mathcal{R}^{+-\infty},\mathfrak{B}',\mu')$ *and* $(\mathcal{T},\mathfrak{B},\mu)$, $(\mathcal{R},\mathfrak{B},\mu)$ *respectively.*

We let $({}^*\mathcal{R},{}^*\mathfrak{D})$ *denote the hyperreals, with the transfer of the Borel field* $\mathfrak{D}$ *on* $\mathcal{R}$. *A function* $f:(\overline{\mathcal{R}_\eta},\mathfrak{C}_\eta)\rightarrow({}^*\mathcal{R},{}^*\mathfrak{D})$ *is measurable, if* $f^{-1}:{}^*\mathfrak{D}\rightarrow\mathfrak{C}_\eta$. *Similarly,* $f:(\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta},\mathfrak{C}_\eta^2)\rightarrow({}^*\mathcal{R},{}^*\mathfrak{D})$ *is measurable, if* $f^{-1}:{}^*\mathfrak{D}\rightarrow\mathfrak{C}_\eta^2$. *Observe that this is equivalent to the definition given in* [4]. *We will abbreviate this notation to* $f:\overline{\mathcal{R}_\eta}\rightarrow{}^*\mathcal{R}$ *or* $f:\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}\rightarrow{}^*\mathcal{R}$ *is measurable,* $(*)$. *The same applies to* $({}^*\mathcal{C},{}^*\mathfrak{D})$, *the hyper complex numbers, with the transfer of the Borel field* $\mathfrak{D}$, *generated by the complex topology. Observe that* $f:\overline{\mathcal{R}_\eta}\rightarrow{}^*\mathcal{C}$ *or* $f:\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}\rightarrow{}^*\mathcal{C}$ *is measurable, in this sense, iff* $Re(f)$ *and* $Im(f)$ *are measurable in the sense of* $(*)$.

We have the following lemma, generalising Theorem 0.7 of [6] and Theorem 22 of [1];

Lemma 0.2. *The identity;*

$$i:(\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta},L(\mathfrak{C}_\eta^2),L(\lambda_\eta^2))\rightarrow(\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta},L(\mathfrak{C}_\eta)\times L(\mathfrak{C}_\eta),L(\lambda_\eta)\times L(\lambda_\eta))$$

and the standard part mapping;

$$st:(\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta},L(\mathfrak{C}_\eta)\times L(\mathfrak{C}_\eta),L(\lambda_\eta)\times L(\lambda_\eta))$$
$$\rightarrow(\mathcal{T}^{+\infty}\times\mathcal{R}^{+-\infty},\mathfrak{B}'\times\mathfrak{B}',\mu'\times\mu')$$

are measurable and measure preserving.

Proof. To show that i is measurable and measure preserving, it is sufficient to prove that;

(i). $L(\mathfrak{C}_\eta)\times L(\mathfrak{C}_\eta)\subset L(\mathfrak{C}_\eta^2)$.

(ii). $L(\lambda_\eta^2)|_{L(\mathfrak{C}_\eta)\times L(\mathfrak{C}_\eta)}=L(\lambda_\eta)\times L(\lambda_\eta)$.

As in [1], if $A\in\mathfrak{C}_\eta$;

$$\{M\in\sigma(\mathfrak{C}_\eta):M\times A\in\sigma(\mathfrak{C}_\eta\times\mathfrak{C}_\eta)\}$$

is a σ-algebra, containing $\mathfrak{C}_\eta$, hence, it equals $\sigma(\mathfrak{C}_\eta)$. Similarly, if $B \in \sigma(\mathfrak{C}_\eta)$;

$$\{M \in \sigma(\mathfrak{C}_\eta) : B \times M \in \sigma(\mathfrak{C}_\eta \times \mathfrak{C}_\eta)\}$$

is a σ-algebra, and equals $\sigma(\mathfrak{C}_\eta)$. Therefore;

$$\mathfrak{C}_\eta \times \mathfrak{C}_\eta \subset \sigma(\mathfrak{C}_\eta) \times \sigma(\mathfrak{C}_\eta) = \sigma(\mathfrak{C}_\eta \times \mathfrak{C}_\eta)$$

Now, using Ward Henson's result, see footnote 1 of [6], it follows that $L(\lambda_\eta^2) = L(\lambda_\eta) \times L(\lambda_\eta)$ on $\sigma(\mathfrak{C}_\eta) \times \sigma(\mathfrak{C}_\eta)$, $(*)$. Now, suppose that $\{C, D\} \subset L(\mathfrak{C}_\eta)$ then, there exists $\{C_1, C_2, D_1, D_2\} \subset \sigma(\mathfrak{C}_\eta)$, with $C_1 \subset C \subset C_2$, $D_1 \subset D \subset D_2$, $L(\lambda_\eta)(C_2 \setminus C_1) = 0$, $L(\lambda_\eta)(D_2 \setminus D_1) = 0$, $(**)$, and $C_1 \times D_1 \subset C \times D \subset C_2 \times D_2$. Moreover, $(C_2 \times D_2 \setminus C_1 \times D_1) \subset ((C_2 \setminus C_1) \times D_2) \cup (C_2 \times (D_2 \setminus D_1))$, $(***)$. By $(*), (**), (***)$, $L(\lambda_\eta^2)(C_2 \times D_2 \setminus C_1 \times D_1) = 0$. Therefore, $C \times D \in L(\mathfrak{C}_\eta^2)$, and the product σ-algebra $L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta) \subset L(\mathfrak{C}_\eta^2)$, $(\dagger)$. Using $(*), (\dagger)$, $L(\lambda_\eta^2)$ agrees with $L(\lambda_\eta) \times L(\lambda_\eta)$ on this algebra, hence, the complete product $L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta) \subset L(\mathfrak{C}_\eta^2)$, showing (i), and $L(\lambda_\eta^2)|_{L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta)} = L(\lambda_\eta) \times L(\lambda_\eta)$, by the definition of a completion, showing (ii).

We recall the result, Theorem 0.7, of [6], that;

$$st : (\overline{\mathcal{R}_\eta}, L(\mathfrak{C}_\eta), L(\lambda_\eta)) \rightarrow (\mathcal{R}^{+-\infty}, \mathfrak{B}', \mu')$$

is measurable and measure preserving, $(\sharp)$. Similarly, one can show that;

$$st : (\overline{\mathcal{T}_\eta}, L(\mathfrak{C}_\eta), L(\lambda_\eta)) \rightarrow (\mathcal{T}^{+\infty}, \mathfrak{B}', \mu')$$

is measurable and measure preserving, $(\sharp\sharp)$. The rest of the argument is fairly straightforward, if $\{B_1, B_2\} \subset \mathfrak{B}'$, then, using $(\sharp), (\sharp\sharp)$, $st^{-1}(B_1 \times B_2) \in L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta)$, and $L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta)(st^{-1}(B_1 \times B_2)) = \mu' \times \mu'(B_1 \times B_2)$. It follows, using the usual argument, as in the first part of the proof, that the push forward measure $st_*(L(\mathfrak{C}_\eta) \times L(\mathfrak{C}_\eta))$ agrees with $\mu' \times \mu'$ on $\mathfrak{B}' \times \mathfrak{B}'$, considered as a product σ-algebra. Then, the result follows easily from the definition of a complete product.

$\square$

The following definition is based on Definition 0.18 of [6];

Definition 0.3. *Discrete Partial Derivatives*

Let $f:\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$ be measurable. Then we define $\frac{\partial f}{\partial t}$ to be the unique measurable function satisfying;

$\frac{\partial f}{\partial t}(\frac{j}{\eta},x)=\eta(f(\frac{j+1}{\eta},x)-f(\frac{j}{\eta},x))$ *for* $j\in {}^*\mathcal{N}_{0\leq j\leq \eta^2-2}, x\in\overline{\mathcal{R}_\eta}$

$\frac{\partial f}{\partial t}(\frac{\eta^2-1}{\eta},x)=0$

$\frac{\partial f}{\partial x}(t,\frac{j}{\eta})=\eta(f(t,\frac{j+1}{\eta})-f(t,\frac{j}{\eta}))$ *for* $j\in {}^*\mathcal{N}_{-\eta^2\leq j\leq \eta^2-2}, t\in\overline{\mathcal{T}_\eta}$

$\frac{\partial f}{\partial x}(t,\frac{\eta^2-1}{\eta})=0$

Remarks 0.4. *If f is measurable, then so are $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$. This follows immediately, by transfer, from the corresponding result for the discrete derivatives of discrete functions $f:\mathcal{T}_n\times\mathcal{R}_n\rightarrow\mathcal{C}$, where $n\in\mathcal{N}$, see Definition 0.15 and Definition 0.18 of* [6].

Lemma 0.5. *Given a measurable boundary condition $g:\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$, there exists a unique measurable $f:\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$, satisfying the nonstandard heat equation;*

$\frac{\partial f}{\partial t}-\frac{\partial^2 f}{\partial x^2}=0$ *on* $(\overline{\mathcal{T}_\eta}\setminus[\frac{\eta^2-1}{\eta},\eta))\times\overline{\mathcal{R}_\eta}$

with $f(0,x)=g(x)$, for $x\in\overline{\mathcal{R}_\eta}$, $(*)$.

Proof. Observe that, by Definition 0.3, if $f:\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$ is measurable, then;

$\frac{\partial^2 f}{\partial x^2}(t,\frac{j}{\eta})=\eta^2(f(t,\frac{j+2}{\eta})-2f(t,\frac{j+1}{\eta})+f(t,\frac{j}{\eta}))$, $(-\eta^2\leq j\leq\eta^2-3)$.

$\frac{\partial^2 f}{\partial x^2}(t,\frac{\eta^2-2}{\eta})=-\eta^2(f(t,\frac{\eta^2-1}{\eta})-f(t,\frac{\eta^2-2}{\eta}))$

$\frac{\partial^2 f}{\partial x^2}(t,\frac{\eta^2-1}{\eta})=0$

Therefore, if f satisfies $(*)$, we must have;

$$f(\frac{i+1}{\eta},\frac{j}{\eta})=f(\frac{i}{\eta},\frac{j}{\eta})+\eta(f(\frac{i}{\eta},\frac{j+2}{\eta})-2f(\frac{i}{\eta},\frac{j+1}{\eta})+f(\frac{i}{\eta},\frac{j}{\eta})),$$

$$(0\leq i\leq\eta^2-2,-\eta^2\leq j\leq\eta^2-3).$$

$f(\frac{i+1}{\eta},\frac{\eta^2-2}{\eta})=f(\frac{i}{\eta},\frac{\eta^2-2}{\eta})-\eta(f(\frac{i}{\eta},\frac{\eta^2-1}{\eta})-f(\frac{i}{\eta},\frac{\eta^2-2}{\eta})),$

$$(0\leq i\leq \eta^2-2).$$

$f(\frac{i+1}{\eta},\frac{\eta^2-1}{\eta})=f(\frac{i}{\eta},\frac{\eta^2-1}{\eta})$, $(0\leq i\leq \eta^2-2)$.

$f(0,\frac{j}{\eta})=g(\frac{j}{\eta})$, $(-\eta^2\leq j\leq \eta^2-1)$. $(**)$

If $\eta=n\in\mathcal{N}$, then given any measurable $g:\mathcal{R}_n\rightarrow\mathcal{C}$, the condition $(**)$, clearly determines a unique measurable, see Definition 0.15 of [6], $f:\mathcal{T}_n\times\mathcal{R}_n\rightarrow\mathcal{C}$, satisfying $(*)$. As the condition $(*)$ can be written down uniformly, in Robinson's higher order logic, we obtain the result, immediately, by transfer.

□

Definition 0.6. *We recall the definition from* [6], *Definition 0.15. Given a measurable* $f:\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$, *we define* $exp_\eta(-\pi ixy)$ *and* $exp(\pi ixy)$ *to be the* $\mathfrak{C}^2_\eta$ *measurable counterparts of the transfers of* $exp(\pi ixy)$ *and* $exp(-\pi ixy)$ *to* $\overline{\mathcal{R}_\eta}^2$. *We define the nonstandard Fourier transform in space;*

$\hat{f}(t,y)=\int_{\overline{\mathcal{R}_\eta}}f(t,x)exp_\eta(-\pi ixy)d\lambda_\eta(x)$

and the nonstandard inverse Fourier transform in space;

$\check{f}(t,y)=\int_{\overline{\mathcal{R}_\eta}}f(t,x)exp_\eta(\pi ixy)d\lambda_\eta(x)$

As in Definition 0.20 of [6], *we let* $\phi_\eta,\psi_\eta:\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$ *be defined by;*

$\phi_\eta(x)=\eta(exp_\eta(-\pi i\frac{x}{\eta})-1)$

$\psi_\eta(x)=\eta(exp_\eta(\pi i\frac{x}{\eta})-1)$

If f *is measurable, we let;*

$C_\eta(t,x)=f(t,\frac{\eta^2-1}{\eta})exp_\eta(-\pi i\frac{\eta^2-1}{\eta}x)-f(t,-\eta)exp_\eta(-\pi i(-\eta)x)$

$D_\eta(t,x)=-\frac{1}{\eta}f(t,-\eta)exp_\eta(\pi i\frac{x}{\eta})exp_\eta(-\pi i(-\eta)x)$.

$C'_\eta(t,x)=-\frac{\partial f}{\partial x}(t,-\eta)exp_\eta(-\pi i(-\eta)x)$

$D'_\eta(t,x)=-\frac{1}{\eta}\frac{\partial f}{\partial x}(t,-\eta)exp_\eta(\pi i\frac{x}{\eta})exp_\eta(-\pi i(-\eta)x)$.

$E_\eta(t,x)=\phi_\eta(x)D_\eta(t,x)-C_\eta(t,x)$

$E'_\eta(t,x)=\phi_\eta(x)D'_\eta(t,x)-C'_\eta(t,x)$

$F_\eta(t,x)=\psi_\eta(x)\phi_\eta(x)D_\eta(t,x)-\psi_\eta(x)C_\eta(t,x)+\phi_\eta(x)D'_\eta(t,x)-C'_\eta(t,x)$

Remarks 0.7. *If f is measurable, then so are $\hat{f}$ and $\check{f}$. Again this follows, by transfer, from the finite case, as in Remark 0.4. By Lemma 0.16 of* [6]*, if f is measurable, then, we have the nonstandard inversion theorems;*

$\check{\hat{f}}=2f$

$\hat{\check{f}}=2f$

Lemma 0.8. *If $f:\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$ is measurable, then;*

(i). $\hat{\frac{\partial f}{\partial t}}=\frac{\partial\hat{f}}{\partial t}$.

(ii). $\hat{\frac{\partial^2 f}{\partial x^2}}=\psi_\eta^2\hat{f}-F_\eta$.

Proof. (i). Using Definition 0.3 and Definition 0.6, we have:

$$\hat{\frac{\partial f}{\partial t}}(t',y)=\int_{\overline{\mathcal{R}_\eta}}\frac{\partial f}{\partial t}(t',x)exp_\eta(-\pi ixy)d\lambda_\eta(x)$$

$$=\eta(\int_{\overline{\mathcal{R}_\eta}}f(t'+\tfrac{1}{\eta},x)exp_\eta(-\pi ixy)d\lambda_\eta(x)-\int_{\overline{\mathcal{R}_\eta}}f(t',x)exp_\eta(-\pi ixy)d\lambda_\eta(x)),\quad (0\leq t'<\tfrac{\eta^2-1}{\eta})$$

$$=\eta(\hat{f}(t'+\tfrac{1}{\eta},y)-\hat{f}(t',y))=\frac{\partial\hat{f}}{\partial t}(t',y),\qquad (0\leq t'<\tfrac{\eta^2-1}{\eta})$$

$$\hat{\frac{\partial f}{\partial t}}(t',y)=\frac{\partial\hat{f}}{\partial t}(t',y)=0,\qquad (\tfrac{\eta^2-1}{\eta}\leq t'<\eta)$$

(ii). Using the definition of ψ_η and F_η in Definition 0.6, and the transfer of the result in Lemma 0.21 of [6].

□

Theorem 0.9. *Let $f:\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}\rightarrow {}^*\mathcal{C}$ satisfy the conditions of Lemma 0.5. Then $\hat{f}$ is determined by;*

$$\hat{f}(\tfrac{i}{\eta},x)=\hat{g}(x)(1+\tfrac{\psi_\eta^2(x)}{\eta})^i-\tfrac{1}{\eta}{}^*\textstyle\sum_{0\leq j\leq i-1}F_\eta(\tfrac{j}{\eta},x)(1+\tfrac{\psi_\eta^2(x)}{\eta})^{i-j-1},$$

$$(0\leq i\leq \eta^2-1),\ (*)$$

In particular, if the boundary condition g satisfies;

$$g(\tfrac{\eta^2-1}{\eta})=0$$

$g(x)=0$, *for* $-\eta\leq x<-\eta+\omega$, *where* $\omega\in{}^*\mathcal{N}\setminus\mathcal{N}$

then, $\hat{f}$ is determined by;

$$\hat{f}(\tfrac{i}{\eta},x)=\hat{g}(x)(1+\tfrac{\psi_\eta^2(x)}{\eta})^i,\ (0\leq i\leq n\eta, n\in\mathcal{N})$$

Proof. We have that;

$$\tfrac{\partial f}{\partial t}-\tfrac{\partial^2 f}{\partial x^2}=0 \text{ on } (\overline{\mathcal{T}_\eta}\setminus[\tfrac{\eta^2-1}{\eta},\eta))\times\overline{\mathcal{R}_\eta}$$

Applying the nonstandard Fourier transform, and using Lemma 0.8, we have;

$$\tfrac{\partial\hat{f}}{\partial t}-(\psi_\eta^2\hat{f}-F_\eta)=0 \text{ on } (\overline{\mathcal{T}_\eta}\setminus[\tfrac{\eta^2-1}{\eta},\eta))\times\overline{\mathcal{R}_\eta}$$

Using Definition 0.3, we have;

$$\eta(\hat{f}(\tfrac{k+1}{\eta},x)-\hat{f}(\tfrac{k}{\eta},x))=\psi_\eta^2(x)\hat{f}(\tfrac{k}{\eta},x)-F_\eta(\tfrac{k}{\eta},x)$$

$$\hat{f}(\tfrac{k+1}{\eta},x)=\hat{f}(\tfrac{k}{\eta},x)(1+\tfrac{\psi_\eta^2(x)}{\eta})-\tfrac{1}{\eta}F_\eta(\tfrac{k}{\eta},x),\ (0\leq k\leq\eta^2-2).\ (**)$$

Let $A=\{i\in{}^*\mathcal{N}:0\leq i\leq\eta^2-1,\ for\ which\ (*)\ holds\}$. Then A is internal, $A(0)$ holds, as $\hat{f}(0,x)=\hat{g}(x)$, by the boundary condition in Lemma 0.5, and if $A(i)$ holds, for $0\leq i\leq\eta^2-2$, then, using $(**)$;

$$\hat{f}(\tfrac{i+1}{\eta},x)=[\hat{g}(x)(1+\tfrac{\psi_\eta^2(x)}{\eta})^i$$

$$-\tfrac{1}{\eta}{}^*\textstyle\sum_{0\leq j\leq i-1}F_\eta(\tfrac{j}{\eta},x)(1+\tfrac{\psi_\eta^2(x)}{\eta})^{i-j-1}](1+\tfrac{\psi_\eta^2(x)}{\eta})-\tfrac{1}{\eta}F_\eta(\tfrac{i}{\eta},x)$$

$= \hat{g}(x)(1+\frac{\psi_\eta^2(x)}{\eta})^{i+1} - \frac{1}{\eta}{}^*\sum_{0\leq j\leq i-1} F_\eta(\frac{j}{\eta},x)(1+\frac{\psi_\eta^2(x)}{\eta})^{i-j} - \frac{1}{\eta}F_\eta(\frac{i}{\eta},x)$

$= \hat{g}(x)(1+\frac{\psi_\eta^2(x)}{\eta})^{i+1} - \frac{1}{\eta}{}^*\sum_{0\leq j\leq i} F_\eta(\frac{j}{\eta},x)(1+\frac{\psi_\eta^2(x)}{\eta})^{i-j}$

so $A(i+1)$ holds. It follows, by hyperfinite induction, see [7], that $A = \{i \in {}^*\mathcal{N} : 0 \leq i \leq \eta^2 - 1\}$, and $\hat{f}$ is determined by the condition $(*)$.

Now suppose that the boundary condition g satisfies the requirements in the second part of the Theorem, then, using Lemma 0.5, we have;

$f(\frac{i}{\eta}, \frac{\eta^2-1}{\eta}) = f(0, \frac{\eta^2-1}{\eta}) = g(\frac{\eta^2-1}{\eta}) = 0$, $(0 \leq i \leq \eta^2 - 1)$

Moreover, again by Lemma 0.5, $f(\frac{i}{\eta}, \frac{-\eta^2+1}{\eta})$ and $f(\frac{i}{\eta}, -\eta)$ are hyperfinite linear combinations of the values $g(\frac{j}{\eta})$, for $-\eta^2 \leq j \leq -\eta^2+1+2i$. For such j, and $0 \leq i \leq n\eta$, $\frac{j}{\eta} \leq -\eta + \frac{1+2i}{\eta} \leq -\eta + 1 + 2n < -\eta + \omega$, so $g(\frac{j}{\eta}) = 0$ by hypothesis, and, then, $f(\frac{i}{\eta}, \frac{-\eta^2+1}{\eta}) = f(\frac{i}{\eta}, -\eta) = 0$, for $0 \leq i \leq n\eta$. Checking the Definition 0.6, it follows that $F_\eta(\frac{i}{\eta}, x) = 0$, for $0 \leq i \leq n\eta$, $n \in \mathcal{N}$. Then, using the first part of the Theorem, we obtain the final result.

□

Definition 0.10. *Convolution*
Suppose that $f, g : \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ *are measurable. Then we define the nonstandard convolution by;*

$(f * g)(t,x) = \int_{\overline{\mathcal{R}_\eta}} f(t, \frac{[\eta x]}{\eta} - y) g(t,y) d\nu_\eta(y)$

Theorem 0.11. *Nonstandard Convolution Theorem*
Let hypotheses be as in 0.10, then;

$f \hat{*} g = \hat{f}\hat{g} \qquad f \check{*} g = \check{f}\check{g}$

Proof. This is a straightforward computation. We have, for $x \in \overline{\mathcal{R}_\eta}$, using Definition 0.15 of [6], that;

$\hat{f}\hat{g}(t,x) = \frac{1}{\eta^2}[{}^*\sum_{j=-\eta^2}^{\eta^2-1} f(t,\frac{j}{\eta}) exp_\eta(-\pi i(\frac{j}{\eta})x)][{}^*\sum_{k=-\eta^2}^{\eta^2-1} g(t,\frac{k}{\eta}) exp_\eta(-\pi i(\frac{k}{\eta})x)]$

$= \frac{1}{\eta^2} {}^*\sum_{j,k=-\eta^2}^{\eta^2-1} f(t, \frac{j}{\eta}) g(t, \frac{k}{\eta}) exp_\eta(-\pi i(\frac{j+k}{\eta})x)$

$= \frac{1}{\eta^2} {}^*\sum_{l,k=-\eta^2}^{\eta^2-1} f(t, \frac{l-k}{\eta}) g(t, \frac{k}{\eta}) exp_\eta(-\pi i(\frac{l}{\eta})x) \ (l = j + k)$

$= \frac{1}{\eta} {}^*\sum_{l=-\eta^2}^{\eta^2-1} (\int_{\overline{\mathcal{R}_\eta}} f(t, \frac{l}{\eta} - w) g(w) d\nu_\eta(w)) exp_\eta(-\pi i(\frac{l}{\eta})x)$

$= \frac{1}{\eta} {}^*\sum_{l=-\eta^2}^{\eta^2-1} (f * g)(t, \frac{l}{\eta}) exp_\eta(-\pi i(\frac{l}{\eta})x)$

$= f \hat{*} g(t, x)$

A similar calculation shows that $f \check{*} g = \check{f}\check{g}$

□

Definition 0.12. *For $\omega' \in {}^*\mathcal{N} \setminus \mathcal{N}$, with $\omega' < \eta$, we let $F_{\omega'} : \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{R}$ be the measurable function defined by;*

$F_{\omega'}(t, \frac{j}{\eta}) = \frac{1}{2}$, *if* $-\omega'\eta \leq j \leq \omega'\eta$

$F_{\omega'}(t, \frac{j}{\eta}) = 0$, *otherwise*

and let $F_\eta = \frac{1}{2} Id_{\overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}}$

Lemma 0.13. *Let f satisfy the hypotheses of Theorem 0.9, with the extra requirement on the boundary condition g, then, for finite t;*

$\check{F_{\omega'}} * f = (hF_{\omega'})\check{} * g$

where h is given by;

$h(t, x) = (1 + \frac{1}{\eta}\psi_\eta(x)^2)^{[\eta t]}$

Proof. By Theorem 0.9, for finite t, $\hat{f} = h\hat{g}$, and so, $\hat{f} F_{\omega'} = hF_{\omega'}\hat{g}$. Let $a = (hF_{\omega'})\check{}$ and $b = \check{F_{\omega'}}$, then, by Theorem 0.11 and Remark 0.7, $a \hat{*} g = \hat{a}\hat{g} = 2hF_{\omega'}\hat{g} = 2F_{\omega'}\hat{f}$, and, similarly, $b \hat{*} f = \hat{b}\hat{f} = 2F_{\omega'}\hat{f}$. Therefore, $a \hat{*} g = b \hat{*} f$, and, again using Remark 0.7, we obtain $b * f = a * g$, as required. □

Definition 0.14. *We call $\Psi_{\omega'}(t, x, y) = (hF_{\omega'})\check{}(t, x - y)$ a nonstandard heat kernel on $\overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}^2$.*

Lemma 0.15. *Let $\Psi(t,x,y)$ be as in Definition 0.14. Then, for finite $(t,x,y)\in\overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}_\eta}^2$, with ${}^\circ t\neq 0$, and $\omega'\leq log(\eta)^{\frac{1}{2}}$;*

$${}^\circ\Psi(t,x,y)=\frac{1}{\sqrt{4\pi^\circ t}}exp(\frac{-({}^\circ x-{}^\circ y)^2}{4^\circ t})$$

Proof. We first claim that, for finite $x\in\overline{\mathcal{R}_\eta}$, ${}^\circ\gamma_\eta(x)=exp(-\pi^{2\circ}x^2)$, where $\gamma_\eta(x)=(1+\frac{1}{\eta}\psi_\eta(x)^2)^\eta$, $(*)$. For $y\in\mathcal{R}$, let $(s_n)_{n\in\mathcal{N}}$ be the standard sequence, defined by;

$$s_n(y)=\frac{exp(\pi i\frac{y}{n})-1}{\frac{1}{n}}=y(\frac{exp(\pi i\frac{y}{n})-1}{\frac{y}{n}}),\ (y\neq 0)$$

Then, for $y\neq 0$;

$$lim_{n\rightarrow\infty}(s_n(y))=lim_{h\rightarrow 0}y(\frac{exp(\pi ih)-1}{h})=y\frac{d}{ds}|_{s=0}exp(\pi is)=\pi iy$$

and the sequence converges uniformly in y, on bounded intervals, ([1]). Now, it is standard result,([2]), that the sequence of functions $(r_n(w))_{n\in\mathcal{N}}$, defined by;

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[1] We estimate the rate of convergence of the sequence $p_n=n(exp(\pi i\frac{1}{n})-1)-i\pi$. We have;

$$p_n=\sum_{m\geq 2}\frac{(\pi i)^m(\frac{1}{n})^{m-1}}{m!}=\frac{-\pi^2}{n}\sum_{m\geq 0}\frac{(\frac{\pi i}{n})^m}{(m+2)!}$$

$$|p_n|\leq\frac{\pi^2}{n}\sum_{m\geq 0}\frac{(\frac{\pi}{n})^m}{m!}=\frac{\pi^2}{n}exp(\frac{\pi}{n})\leq\frac{\pi^2 exp(\pi)}{n}$$

In particular, $|p_n|<\epsilon$, and, therefore, $|s_n(y)-i\pi y|<\epsilon|y|$, if $n\geq\frac{\pi^2exp(\pi)}{\epsilon}$. Hence, $|s_n(y)-i\pi y|<\epsilon$, if $n\geq\frac{\pi^2exp(\pi)|y|}{\epsilon}$

[2] We estimate the rate of convergence of the sequence $q_n(w)=r_n(w)-exp(w)$, for $w\in\mathcal{C}$. We have, taking a branch of the logarithm with $log(1)=0$, and cutting the complex plane from $-\infty$ to -1, for $n>|w|$;

$$log(r_n(w))-w=nlog(1+\frac{w}{n})-w$$

$$|log(r_n(w))-w|\leq\sum_{m=1}^\infty\frac{|w|^{m+1}}{(m+1)n^m}\leq\frac{|w|^2}{n}\sum_{m=0}^\infty\frac{|w|^m}{n^m}=\frac{|w|^2}{n}\frac{1}{1-\frac{|w|}{n}}=\frac{|w|^2}{n-|w|}$$

Moreover, observe that, for $w\in\mathcal{C}$;

$$|exp(w)-1|\leq\sum_{m=1}^\infty\frac{|w|^m}{m!}=|w|\sum_{m=0}^\infty\frac{|w|^m}{(m+1)!}\leq|w|exp(|w|)$$

Therefore, for $\epsilon>0$, $|exp(w)-1|<\epsilon$, if $|w|<min(\frac{\epsilon}{e},1)$, and, for $w',w\in\mathcal{C}$, with $Re(w)\leq 0$, $|exp(w')-exp(w)|<\epsilon$ if $|exp(w'-w)-1|<\epsilon\leq\epsilon|exp(-w)|$. Hence, $|exp(w')-exp(w)|<\epsilon$, if $|w'-w|<min(\frac{\epsilon}{e},1)$, for $Re(w)\leq 0$.

$r_n(w) = (1 + \frac{w}{n})^n$

converges uniformly to $exp(w)$ on bounded subsets of $\mathcal{C}$. Therefore, if $(t_n)_{n \in \mathcal{N}}$ is the sequence defined by $t_n = r_n(s_n^2)$, then;

$lim_{n \to \infty} t_n = exp(-\pi^2 y^2)$

It follows that the sequence of functions $t_n(y)$ converges uniformly to $exp(-\pi^2 y^2)$ on bounded intervals of $\mathcal{R}$, ([3]) In particular, given $N, \epsilon > 0$ standard the statement;

$\forall y \leq N \exists M \forall n \geq M(|t_n(y) - exp(-\pi^2 y^2)| < \epsilon)$

is true in $\mathcal{R}$, therefore, by transfer, is true in ${}^*\mathcal{R}$. As ϵ and N were arbitrary, it follows that, for all finite $x \in \overline{\mathcal{R}_\eta}$;

$\gamma_\eta(x) \simeq t_\eta(x) \simeq {}^*exp(-\pi^2 x^2) \simeq exp(-\pi^{2\circ} x^2)$

using continuity of exp, and Theorem 2.25 of [7] or [5]. Therefore, $(*)$ holds. Now, by continuity of the function $q(w) = w^s$, for $s \in \mathcal{R}$, and the fact that $\frac{\eta^\circ t - [\eta t]}{\eta} \simeq 0$, for finite $t \in \overline{\mathcal{R}_\eta}$, it follows, again using [5] or Theorem 2.25 of [7], that, for finite $(t, x) \in \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}$, ${}^\circ h(t, x) = exp(-\pi^{2\circ} t^\circ x^2)$, $(**)$.

So, for $Re(w) \leq 0$, if $\frac{|w|^2}{n - |w|} < min(\frac{\epsilon}{e}, 1)$, that is $n > |w| + |w|^2 max(1, \frac{e}{\epsilon})$, then $|q_n(w)| = |r_n(w) - exp(w)| < \epsilon$.

[3] We estimate the rate of convergence of the sequence $b_n(y) = t_n(y) - exp(-\pi^2 y^2)$, for $y \in \mathcal{R}$, $y \neq 0$. It is a straightforward calculation, to show that, if $|s_n(y) - i\pi y| < min(2|y|, \frac{\epsilon}{3|y|})$, then $|s_n^2(y) - (-\pi^2 y^2)| < \epsilon$. Combining this with the result of footnote 1, we obtain that if $n > max(\frac{\pi^2 exp(\pi)}{2}, \frac{3\pi^2 exp(\pi) y^2}{\epsilon})$, $(*)$, then $|s_n^2(y) - (-\pi^2 y^2)| < \epsilon$. Using footnote 2, we also have that if $\epsilon < min(\frac{\delta}{e}, 1)$, then $|exp(s_n^2(y)) - exp(-\pi^2 y^2)| < \delta$, $(**)$. Now, assuming $(*)$ is satisfied, we have $|s_n(y)| < (\epsilon + \pi^2 y^2)^{\frac{1}{2}}$. Then, using footnote 2, if $\epsilon < min(\frac{\delta}{e}, 1, \frac{\pi^2 y^2}{2})$, $(\dagger)$, $n > max((\epsilon + \pi^2 y^2)^{\frac{1}{2}} + (\epsilon + \pi^2 y^2) max(1, \frac{e}{\delta}), max(\frac{\pi^2 exp(\pi)}{2}, \frac{3\pi^2 exp(\pi) y^2}{\epsilon}))$, $(\dagger\dagger)$, then $|exp(s_n^2(y)) - exp(-\pi^2 y^2)| < \delta$, $|r_n(s_n^2(y)) - exp(s_n^2(y))| < \delta$, so $|b_n(y)| = |t_n(y) - exp(-\pi^2 y^2) = |r_n(s_n^2(y)) - exp(-\pi^2 y^2)| < 2\delta$, $(***)$. Now, if $\delta < min(e, \pi^2 y^2 e)$, we can satisfy $(\dagger)$ by taking $\epsilon = \frac{\delta}{2e}$. Substituting into $(\dagger\dagger)$, we obtain, if $n > max((1 + \pi^2 y^2)^{\frac{1}{2}} + (1 + \pi^2 y^2)\frac{e}{\delta}, \frac{\pi^2 exp(\pi)}{2}, \frac{6\pi^2 exp(\pi) y^2 e}{\delta})$, then $(***)$ holds. Taking $\delta = \frac{1}{2|y|^r}$, for $r \in \mathcal{N}$, there exist constants $C_2, C_3 > 0$, such that, for all $y \in \mathcal{R}$, if $n > max(C_2, C_3 |y|^{r+2})$, then $|b_n(y)| < \frac{1}{|y|^r}$.

Now if $\omega'' \in {}^*\mathcal{R}\setminus\mathcal{R}$, with $|\omega''| \leq \eta^{\frac{1}{4}}$, then, in particular, $\eta > max(C_2, C_3|\omega''|^3)$, see footnote 3. Hence, we have, by transfer, that;

$|t_\eta(\omega'') - {}^*exp(-\pi^2\omega''^2)| < \frac{1}{|\omega''|} \simeq 0$

for infinite ω''. As $lim_{x\rightarrow\infty}exp(-\pi^2x^2) = 0$, it is a standard result, see [5], that ${}^*exp(-\pi^2\omega''^2) \simeq 0$, hence $|t_\eta(\omega'')| \simeq 0$, and ${}^\circ t_\eta(\omega'') = 0$. Now, by a similar argument to the above, for finite t, with ${}^\circ t \neq 0$, we have $h(t, \omega'') \simeq 0$. Combining these results, we have that;

${}^\circ hF_{\omega'}|_{st^{-1}(\mathcal{T}_{>0})\times\overline{\mathcal{R}}_\eta} = st^*(exp(-\pi^2tx^2)_\infty)$ ($\sharp$)

for $\omega' \in {}^*\mathcal{N}\setminus\mathcal{N}$, with $\omega' \leq \eta^{\frac{1}{4}}$. Here, we adopt the notation in Definition 0.5 of [6], letting $exp(-\pi^2tx^2)_\infty$ denote the extension of $exp(-\pi^2tx^2)$ on $\mathcal{T}_{>0}\times\mathcal{R}$ to $\mathcal{T}_{>0}^{+\infty}\times\mathcal{R}^{+-\infty}$, by setting $exp(-\pi^2tx^2)_\infty = 0$, at infinite values.

Now, for finite $(t, x) \in \overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}}_\eta$, we have, by $(**)$, that;

$|h(t, x)| \leq 2^*exp(-\pi^2tx^2)$ ($\dagger$)

For $(t, x) \in \overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}}_\eta$, with $t > 1$ finite, and x infinite, with $x \leq \eta^{\frac{1}{5}}$, we have, using footnote 3, that;

$|t_\eta(x)|^t < |t_\eta(x)| \leq |t_\eta(x)| \leq {}^*exp(-\pi^2x^2) + \frac{1}{x^2} \leq \frac{C}{x^2}$ ($\dagger\dagger$)

where $C \in \mathcal{R}_{>0}$.

For $(t, x) \in \overline{\mathcal{T}_\eta}\times\overline{\mathcal{R}}_\eta$, with $\frac{1}{r} < {}^\circ t \leq 1$, $r \in \mathcal{N}$ and x infinite, with $x \leq \eta^{\frac{1}{2r+3}}$, we have, by footnote 3, that;

$|t_\eta(x)| \leq {}^*exp(-\pi^2x^2) + \frac{1}{x^{2r}} \leq \frac{C'}{x^{2r}}$

$|t_\eta(x)|^t < |t_\eta(x)|^{\frac{1}{r}} \leq \frac{C''}{x^2}$ ($\dagger\dagger\dagger$)

where $C', C'' \in \mathcal{R}_{>0}$. Combining the estimates, ($\dagger$), ($\dagger\dagger$), ($\dagger\dagger\dagger$), and, using the fact that $h(x, t)$ is the measurable counterpart of $t_\eta(x)^t$, we have, for $\omega' \leq (log(\eta))^{\frac{1}{2}}$, and t finite, $0 < {}^\circ t$, that;

$|(hF_{\omega'})_t| \leq f_{t,\eta}$

Here, $f_{t,\eta} : \overline{\mathcal{R}_\eta} \to \overline{\mathcal{R}_\eta}$ is the measurable counterpart of the *-continuous function $f_t : {}^*\mathcal{R} \to {}^*\mathcal{R}$ given by;

$f_t(x) = C_t$ if $|x| \leq 1$

$f_t(x) = \frac{C_t}{x^2}$ if $|x| > 1$

and $C_t \in \mathcal{R}_{>2}$, depends on t. Now, using the proof of Theorem 0.17 in [6], it follows that $f_{t,\eta}$ is S-integrable. Then, using [1], Corollary 5, it follows that $(hF_{\omega'})_t(w)$ and $(hF_{\omega'})_t(w)exp_\eta(\pi iwz)$ are S-integrable, $d\lambda_\eta(w)$, for finite $z \in \overline{\mathcal{R}_\eta}$. Moreover, using [7], Theorem 3.24, and ($\sharp$), we have, for finite $(t,z) \in \overline{\mathcal{T}_\eta} \times \overline{\mathcal{R}_\eta}$, with ${}^\circ t \neq 0$, that;

${}^\circ(hF_{\omega'})\check{}(t,z) = {}^\circ \int_{\overline{\mathcal{R}_\eta}} hF_{\omega'}(t,w)exp_\eta(\pi iwz)d\lambda_\eta(w)$

$= \frac{1}{2}\int_{w finite} {}^\circ h exp_\eta(\pi iw({}^\circ z))dL(\lambda_\eta)(w)$

$= \frac{1}{2}\int_{\mathcal{R}} exp(-\pi^{2\circ}t^\circ w^2)exp(\pi iw({}^\circ z))d\mu(w)$ ($\sharp\sharp$)

$= \frac{1}{\sqrt{4\pi^\circ t}}exp(\frac{-({}^\circ z)^2}{4^\circ t})$, ([4]). Now substituting $x-y$ for z, we obtain the result.

□

Definition 0.16. *Let $g : \mathcal{R} \to \mathcal{C}$ be a continuous function, satisfying the growth condition;*

---

[4] Taking standard parts, the fact that;

$\int_{\mathcal{R}} e^{i\pi wz-\pi^2tw^2}dw = \frac{1}{\sqrt{\pi t}}e^{\frac{-z^2}{4t}}$

is a standard result, which we include for want of a convenient reference. We have $i\pi wz - \pi^2tw^2 = -\pi^2t(w-\frac{iz}{2\pi t})^2 - \frac{z^2}{4t}$. Hence;

$\int_{\mathcal{R}} e^{i\pi wz-\pi^2tw^2}dw$

$= e^{\frac{-z^2}{4t}}\int_{\mathcal{R}} e^{-\pi^2t(w-\frac{iz}{2\pi t})^2}dw$

$= e^{\frac{-z^2}{4t}}\int_{Im(w')=\frac{-z}{2\pi t}} e^{-\pi^2tw'^2}dw'$ ($w' = w - \frac{iw}{2\pi t}$)

$= \frac{e^{\frac{-z^2}{4t}}}{\pi\sqrt{t}}\int_{Im(w'')=\frac{-\pi\sqrt{t}z}{2\pi t}} e^{-w''^2}dw''$ ($w'' = \pi\sqrt{t}w'$)

$= \frac{1}{\sqrt{\pi t}}e^{\frac{-z^2}{4t}}$

$|g(x)| \leq Aexp(B|x|^{\rho})$, $(x \in \mathcal{R})$

for some constants A, B and $\rho < 2$. Then the function $H : \mathcal{T} \times \mathcal{R} \rightarrow \mathcal{C}$, defined by;

$H(0,x) = g(x)$

$H(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathcal{R}} exp(\frac{-(x-y)^2}{4t}) g(y) d\mu(y)$ $(t > 0)$

which is continuous, and satisfies the standard heat equation;

$\frac{\partial H}{\partial t} - \frac{\partial^2 H}{\partial x^2} = 0$

on $\mathcal{T}_{>0} \times \mathcal{R}$, ([5]), is known as the classical solution to the heat equation with boundary condition g.

Theorem 0.17. *Let g be as in Definition 0.16, let g_η denote its measurable extension to $\overline{\mathcal{R}_\eta}$, and, let $g_{\eta,\omega}$ be the truncation of g_η, given by;*

$g_{\eta,\omega} = g_\eta \chi_{[-\omega,\omega)}$

for a nonstandard step function $\chi_{[-\omega,\omega)}$, with $\omega \in {}^*\mathcal{N} \setminus \mathcal{N}$, $\eta - \omega$ infinite and $\omega < \omega^{'\frac{1}{2}}$. Then, with f determined by Lemma 0.5, for $g_{\eta,\omega}$ as the boundary condition, we have;

${}^\circ(\check{F_{\omega'}} * f)|_{st^{-1}(\mathcal{T}_{>0} \times \mathcal{R})} = st^*(H_\infty)$

if $\omega' < log(\eta)^{\frac{1}{2}}$, and H_∞ is obtained from the classical solution H of the heat equation, with boundary condition g, given in Definition 0.16.

Proof. Using the following footnote 6, we obtain, by transfer and the measurability observation at the end of Lemma 0.15, that, for any given $\delta', t \in {}^*\mathcal{R}_{>0}, x \in {}^*\mathcal{R}$;

$|h(t,x) - exp_\eta(-\pi^2 t x^2)| < \delta'$, $(*)$

if $\eta > C_4(x, \delta', t)$. In particular, observing that the function $C_4 : {}^*\mathcal{R} \times {}^*\mathcal{R}^2_{>0}$ is increasing in x and t, we have, for a given infinite $\omega' \in {}^*\mathcal{N}$,

[5] A good proof of this fact can be found in [8], (Theorem 2.1), if $g \in S(\mathcal{R})$. For the more general case, see [3]

that $(*)$ holds for all $|x| \leq \omega'$, and finite t, if $\eta > C_4(\omega', \delta', \omega')$,([6]).

In particular, we obtain that;

$|(hF_{\omega'})\check{}(t,z) - \check{\theta}(t,z)| \leq \int_{\overline{\mathcal{R}}_\eta} |hF_{\omega'}(t,w) - \theta(t,w)| d\lambda(w)$

$\leq 2 \cdot \frac{1}{2}\delta'\omega' = \delta'\omega'$ $(**)$

for all $z \in \overline{\mathcal{R}}_\eta$ and finite $t \in \overline{\mathcal{T}}_\eta$, $t \neq 0$, where $\theta(t,w) = exp_\eta(-\pi^2 t w^2) F_{\omega'}(t,w)$. Now, substituting $x-y$ for z in $(**)$, and multiplying through by

---

[6] Taking a principal branch of the logarithm, we have, for $w, w' \in \mathcal{C}$, with $w \neq 0$ and $|w'-w| < \frac{|w|}{2}$, that the function $\theta(t) = log(w + t(w'-w))$ is continuously differentiable on the interval $[0,1]$, with;

$\theta'(t) = \frac{w'-w}{w+t(w'-w)}$

Applying the mean value theorem to the real and imaginary parts of f, we obtain;

$|log(w) - log(w')| = |\theta(1) - \theta(0)| \leq 2\frac{|w-w'|}{min_{t\in[0,1]}|w+t(w'-w)|} \leq 4\frac{|w-w'|}{|w|}$ $(*)$

Using footnote 2, we have, for $w, w' \in \mathcal{C}$, with $|w-w'| < 1$ and $Re(w) \leq 0$, that;

$|exp(w) - exp(w')| = |exp(w)||exp(w'-w) - 1| \leq |w'-w| exp(|w'-w|)|exp(w)| \leq e|w-w'|$ $(**)$

Now, for $t \in \mathcal{R}$, $t > 0$, we can satisfy the condition $|tlog(w') - tlog(w)| < 1$, using $(*)$ and assuming that $|w'-w| < \frac{|w|}{2}$, by taking $|w'-w| < \frac{|w|}{4t}$. Then, assuming, that $|w| \leq 1$, so that $Re(tlog(w)) \leq 0$, and $w \neq 0$, we have, combining $(*), (**)$, that;

$|w'^t - w^t| = |exp(tlog(w')) - exp(tlog(w))| < 4et\frac{|w'-w|}{|w|}$

if $|w'-w| < min(\frac{|w|}{2}, \frac{|w|}{4t})$, $(\dagger)$. We now estimate the rate of convergence of the sequence $v_n(y) = t_n(y)^t - exp(-\pi^2 t y^2)$, for $t \in \mathcal{R}_{>0}$. Let $C(\delta, y)$ be the constant obtained in footnote 3, so that there $(***)$ holds. Then, it is easy to see, using $(\dagger)$ and the fact that $0 < exp(-\pi^2 y^2) \leq 1$, that, if, $n > max(C(\frac{exp(-\pi^2 y^2)}{4}, y), C(\frac{exp(-\pi^2 y^2)}{8t}, y), C(\delta'\frac{exp(-\pi^2 y^2)}{8et}, y))$, then;

$|v_n(y)| = |t_n(y)^t - exp(-\pi^2 t y^2)| < \delta'$ $(\dagger\dagger)$

In particular, substituting into the expression for $C(\delta, y)$, we can find constants $C_2, C_3 \in \mathcal{R}$, such that $(\dagger\dagger)$ holds, for;

$n > max(C_2, \frac{C_3 y^2 exp(\pi^2 y^2) t}{\delta'}) = C_4(y, \delta', t)$

$g_{\eta,\omega}(y)$, we have, from $(**)$, that;

$$|(hF_{\omega'})\check{}(t,x-y)g_{\eta,\omega}(y)| \leq |\check{\theta}(t,x-y)g_{\eta,\omega}(y)| + \delta'\omega'|g_{\eta,\omega}(y)| \ (***)$$

for all $x, y \in \overline{\mathcal{R}}_\eta$ and t as above. Now using the growth condition in Definition 0.16, we have that $|\delta'\omega' g_{\eta,\omega}| \leq \frac{\chi_{[-\omega,\omega)}}{\omega^2}$, if $\delta' \leq \frac{{}^*exp(-B|\omega|^\rho)}{A\omega^2\omega'}$. Using [7](Theorem 3.24), the fact that $\int_{\overline{\mathcal{R}}_\eta} \frac{\chi_{[-\omega,\omega)}}{\omega^2} d\lambda = \frac{2}{\omega} \simeq 0$, and [1], we have $\delta'\omega'|g_{\eta,\omega}|$ is S-integrable, and $\int_{\overline{\mathcal{R}_\eta}} \delta'\omega'|g_{\eta,\omega}|d\lambda \simeq 0$. In particular, using Definition 0.10 and Lemma 0.13, this implies that;

$$(\check{F_{\omega'}} * f)(t,x) = (hF_{\omega'})\check{} * g_{\eta,\omega}(t,x) \simeq \check{\theta} * g_{\eta,\omega}(t,x) \ (****)$$

for all finite $t \in \overline{\mathcal{R}}_{\eta,>0}$ and $x \in \mathcal{R}_\eta$. Let $\tau(t,w) = exp_\eta(-\pi^2 t w^2)$, then, using the following footnote 7, we obtain by transfer;

$$|\check{\tau}(t,z) - \frac{1}{\sqrt{\pi t}} exp_\eta(\frac{-z^2}{4t})| \leq \frac{K(t)}{\eta} + G(t)\frac{|z|}{\eta} + H\frac{z^2}{\eta}$$

for $t \in \overline{\mathcal{R}}_{\eta,>0}$ and $z \in \overline{\mathcal{R}}_{\eta}$,([7]). In particular, if $\omega'' \in {}^*\mathcal{N}$ is infinite, $\delta'' \in {}^*\mathcal{R}_{>0}$, then we obtain;

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[7] We require the following estimate, see [6] for relevant terminology. Let $f : \mathcal{R} \rightarrow \mathcal{R}$ be differentiable on $\mathcal{R}$ and increasing (decreasing) $(*)$ on the interval $[\frac{i}{n}, \frac{j}{n}]$, where $i, j \in \mathcal{Z}$, $-n^2 \leq i < j \leq n^2$, and $n \in \mathcal{N}$, then;

$$|\int_{[\frac{i}{n},\frac{j}{n}]} f_n d\lambda_n - \int_{[\frac{i}{n},\frac{j}{n}]} f d\mu|$$

$$\leq \frac{1}{n}\sum_{k=0}^{j-i-1} |f(\tfrac{i+k+1}{n}) - f(\tfrac{i+k}{n})| \ (**)$$

$$= \frac{1}{n}\sum_{k=0}^{j-i-1} |\int_{\frac{i+k}{n}}^{\frac{i+k+1}{n}} f' d\mu| \ (***)$$

$$\leq \frac{1}{n}\sum_{k=0}^{j-i-1} \int_{\frac{i+k}{n}}^{\frac{i+k+1}{n}} |f'| d\mu$$

$$= \frac{1}{n}\int_{\frac{i}{n}}^{\frac{j}{n}} |f'| d\mu \ (****)$$

where, in $(**)$, we have used the assumption $(*)$ and the definition of the relevant integrals, and, in $(***)$, we have used the Fundamental Theorem of Calculus. Now let $Y(x) = exp(-\pi^2 t x^2) cos(\pi x z)$ and let $Y_n(x)$ be its $\lambda_n(x)$ measurable counterpart on $\mathcal{R}_n$, where $t \in \mathcal{R}_{>0}$ and $z = \frac{j}{n}$, $0 < j \leq n^2 - 1$. Observe that the zeros of Y on $[0, n]$ are located at the points $p_k = \frac{(2k-1)n}{2j}$, for $k \in \mathcal{N} \cap [0, j]$, and, the local maxima (minima) of Y on $[0, n]$, are located at points q_k, where $p_k < q_k < p_{k+1}$, for $0 \leq k \leq j-1$, and $p_j < q_j < n$, for $n \geq D$, some $D \in \mathcal{R}$. Let p'_k denote the points $\frac{[np_k]}{n}$, p''_k the points $p'_k + \frac{1}{n}$, and, similarly, define q'_k, q''_k, then, it is easy to see (check this) that we can choose a constant $D(t)$, such that $0 < p'_k < p''_k < q'_k < q''_k < p'_{k+1} < n$, for $0 \leq k \leq j-1$, and $p'_j < q'_j < q''_j < n$, for $n \geq max(D(t), \sqrt{2j})$. Now, using $(****)$, and the fact that Y is monotone on the intervals $[p''_k, q'_k]$, $[q''_k, p'_{k+1}]$, for $0 \leq k \leq j-1$, and on $[0, p'_0]$, $[p''_j, q'_j]$, $[q''_j, n]$, we obtain;

$$|\int_{[p''_k, q'_k)} Y_n d\lambda_n - \int_{[p''_k, q'_k)} Y d\mu| \leq \frac{1}{n}\int_{[p''_k, q'_k)} |Y'| d\mu \ (\dagger)$$

and, similarly, for the other intervals. Choose a constant $A(t) \in \mathcal{R}$, such that $|Y(x)| \leq \frac{1}{x^2}$, $(\sharp)$, for $|x| > A(t)$. Let k_{max} be the largest k such that $p''_k \leq A(t)$, then $\frac{(2k_{max}-1)n}{2j} + \frac{1}{n} \leq A(t)$ and $k_{max} \leq \frac{jC(t)}{n} + 1$. Let $U = \bigcup_{0 \leq k \leq k_{max}} [p'_k, p''_k) \cup [q'_k, q''_k)$, then, using the bound $|Y| \leq 1$;

$$|\int_U Y_n d\lambda_n| \leq \frac{1}{n} 2k_{max} \leq \frac{2jC(t)}{n^2} + \frac{2}{n} \ (\dagger\dagger)$$

and, similarly, for $|\int_U Y d\mu|$. Let $V = \bigcup_{k_{max} < k \leq j} [p'_k, p''_k) \cup [q'_k, q''_k)$, then, using the bound $(\sharp)$;

$$|\int_V Y_n d\lambda_n| \leq \frac{2}{n}\sum_{k_{max} < k \leq j} \frac{1}{(\frac{(2k-1)n}{2j})^2} \leq 16\frac{j^2}{n^3} \ (\dagger\dagger\dagger)$$

and, similarly, for $|\int_V Y d\mu|$. Let $W = (\bigcup_{0 \leq k \leq j-1} [p''_k, q'_k) \cup [q''_k, p'_{k+1})) \cup [0, p'_0] \cup [p''_j, q'_j] \cup [q''_j, n]$. Then, using $(\dagger)$, we have;

$|\frac{1}{2}\check{\tau}(t,z)-\frac{1}{\sqrt{4\pi t}}exp_\eta(\frac{-z^2}{4t})|<\delta''$ (†)

for all finite $t\in\overline{\mathcal{R}}_{\eta>0}$, and $z\in\overline{\mathcal{R}}_\eta$, with $|z|\leq\omega''$, if $\eta>\frac{3\omega''^3}{\delta''}$, (††). Using Definition 0.6, and transfer of the following footnote 8, we have;

$|\frac{1}{2}\check{\tau}(t,z)-\check{\theta}(t,z)|\leq\frac{1}{2}\int_{|x|\geq\omega'}exp_\eta(-\pi^2tx^2)d\lambda_\eta(x)$

$\leq\frac{1}{2\pi\sqrt{t}}{}^*exp(-\pi^2t(\omega'-\frac{1}{\eta})^2)$

$|\int_W Y_nd\lambda_n-\int_W Yd\mu|\leq\frac{1}{n}\int_W|Y'|d\mu\leq\frac{1}{n}\int_{[0,n)}|Y'|d\mu$ (††††)

Using (††), (†††), (††††), and the fact that U,V,W is a partition of $[0,n)$, we obtain;

$|\int_{[0,n)}Y_nd\lambda_n-\int_{[0,n)}Yd\mu|\leq\frac{1}{n}\int_{[0,n)}|Y'|d\mu+\frac{4jC(t)}{n^2}+\frac{4}{n}+32\frac{j^2}{n^3}$ (♯♯)

Then, as Y is even, $|Y|\leq 1$, $|Y'|\leq exp(-\pi^2tx^2)(2\pi^2t|x|+\frac{\pi j}{n})$, we obtain, using (♯♯);

$|\int_{\mathcal{R}_n}Y_nd\lambda_n-\int_{[-n,n]}Yd\mu|$

$\leq 2|\int_{[0,n)}Y_nd\lambda_n-\int_{[0,n)}Yd\mu|+\frac{1}{n}(|Y(-n)|+|Y(0)|)$

$\leq\frac{1}{n}\int_{\mathcal{R}}|Y'|d\mu+\frac{8jC(t)}{n^2}+\frac{8}{n}+64\frac{j^2}{n^3}+\frac{2}{n}$

$\leq\frac{D(t)\pi j}{n^2}+\frac{2E(t)\pi^2t}{n}+\frac{8jC(t)}{n^2}+\frac{10}{n}+64\frac{j^2}{n^3}=\frac{F(t)}{n}+G(t)\frac{j}{n^2}+H\frac{j^2}{n^3}$ (♯♯♯)

where $F(t),G(t),H\in\mathcal{R}$. Choosing a constant $I(t)\in\mathcal{R}$ such that $exp(-\pi^2tx^2)\leq\frac{I(t)}{2x^2}$, for $|x|>1$, we obtain;

$|\int_{\mathcal{R}_n}Y_nd\lambda_n-\int_{\mathcal{R}}Yd\mu|$

$\leq\frac{F(t)}{n}+G(t)\frac{j}{n^2}+H\frac{j^2}{n^3}+\frac{I(t)}{n}=\frac{J(t)}{n}+G(t)\frac{j}{n^2}+H\frac{j^2}{n^3}$ (♯♯♯♯)

Let $Z(x)=exp(-\pi^2tx^2)sin(\pi xz)$, with hypotheses and $Z_n(x)$ as above. Then, as Z is odd, $|Z|\leq 1$, we have $\int_{\mathcal{R}}Zd\mu=0$, $\int_{\mathcal{R}_n}Z_nd\lambda_n=\frac{Z(-n)}{n}$, and;

$|\int_{\mathcal{R}_n}Z_nd\lambda_n-\int_{\mathcal{R}}Zd\mu|\leq\frac{1}{n}$ (♯♯♯♯♯)

Let $X(x)=exp(-\pi^2tx^2)exp(i\pi xz)$, with hypotheses and $X_n(x)$ as above. Then, using the estimates (♯♯♯♯), (♯♯♯♯♯) and footnote 4, we obtain;

$|\int_{\mathcal{R}_n}X_nd\lambda_n-\frac{1}{\sqrt{\pi t}}e^{\frac{-z^2}{4t}}|\leq\frac{K(t)}{n}+G(t)\frac{j}{n^2}+H\frac{j^2}{n^3}$ (♯♯♯♯♯♯)

where $K(t)=J(t)+1$ and $z=\frac{j}{n}$.

$\leq \omega'^{*}exp(-\pi^2 t(\omega'-1)^2)$ (†††)

for all $z \in \overline{\mathcal{R}_\eta}$, finite $t \in \overline{\mathcal{R}_{\eta>0}}$, $\omega' \in {}^*\mathcal{N}$ infinite,([8]).

Combining (†) and (†††), gives;

$|\breve{\theta}(t,z) - \Gamma(t,z)| \leq \delta'' + \omega'^{*}exp(-\pi^2 t(\omega'-1)^2)$ (††††)

for all finite $t \in \overline{\mathcal{R}}_{\eta>0}$, and $|z| \leq \omega''$, $z \in \overline{\mathcal{R}}_\eta$, if the condition (††) holds, where $\Gamma(t,z) = \frac{1}{\sqrt{4\pi t}} exp_\eta(\frac{-z^2}{4t})$. We have, using Definition 0.10 and (††††);

$|(\breve{\theta} * g_{\eta,\omega})(t,x) - (\Gamma * g_{\eta,\omega})(t,x)|$

$= |\int_{\overline{\mathcal{R}}_\eta} (\breve{\theta} - \Gamma)(x-y) g_{\eta,\omega}(y) d\lambda_\eta(y)|$

$\leq \int_{\overline{\mathcal{R}}_\eta} (\delta'' + \omega'^{*}exp(-\pi^2 t(\omega'-1)^2)) |g_{\eta,\omega}(y)| d\lambda_\eta(y)$ (♯)

for finite $t \in \overline{\mathcal{R}}_{\eta>0}$, finite $x \in \overline{\mathcal{R}}_\eta$, if $\omega'' = 2\omega$, that is, from (††), $\eta > \frac{24\omega^3}{\delta''}$, (♯♯). Following the same argument as above, we have $\delta'' g_{\eta,\omega}$ is S-integrable and $|\delta'' g_{\eta,\omega}| \leq \frac{\chi_{[-\omega,\omega)}}{\omega^2}$, if $\delta'' \leq \frac{{}^*exp(-B\omega^\rho)}{\omega^2}$, so we require, from (♯♯), that $\eta > 24\omega^{5*}exp(B\omega^\rho)$, (♯♯♯). Similarly, we have $\omega'^{*}exp(-\pi^2 t(\omega'-1)^2)) g_{\eta,\omega}$ is S-integrable and $|\omega'^{*}exp(-\pi^2 t(\omega'-1)^2)) g_{\eta,\omega}| \leq \frac{\chi_{[-\omega,\omega)}}{\omega^2}$, if ${}^*exp(-\pi^2 t(\omega'-1)^2+1) \leq \frac{{}^*exp(-B\omega^\rho)}{\omega^2}$. By a simple calculation, this can be achieved if $\omega' \geq Cmax(log(\omega)^{\frac{1}{2}}, \omega^{\frac{\rho+1}{2}})$,

---

[8] We make the following estimate, with $t \in \mathcal{R}_{>0}$;

$\int_{|x|\geq \frac{j}{n}, x\in\mathcal{R}_n} exp_n(-\pi^2 t x^2) d\lambda_n(x)$

$\leq \frac{2}{n} \sum_{k=j}^{n^2} exp(-\pi^2 t(\frac{k}{n})^2)$

$\leq 2 \int_{\frac{j-1}{n}}^{n} exp(-\pi^2 t x^2 d\mu(x)$

$\leq 2 \int_{\frac{j-1}{n}}^{\infty} exp(-\pi^2 t x^2) d\mu(x)$

$= 2 \int_{\pi^2 t(\frac{j-1}{n})^2}^{\infty} \frac{exp(-u)}{2\pi\sqrt{tu}} d\mu(u)$, $(u = \pi^2 t x^2)$

$\leq \frac{1}{\pi\sqrt{t}} \int_{\pi^2 t(\frac{j-1}{n})^2}^{\infty} exp(-u) d\mu(u)$, $(\frac{j-1}{n} \geq \frac{1}{\pi\sqrt{t}})$

$\leq \frac{1}{\pi\sqrt{t}} exp(-\pi^2 t(\frac{j-1}{n})^2)$

$(\sharp\sharp\sharp\sharp)$. If both the conditions $(\sharp\sharp\sharp)$ and $(\sharp\sharp\sharp\sharp)$ are satisfied, we then have;

$(\check{\theta} * g_{\eta,\omega})(t,x) \simeq (\Gamma * g_{\eta,\omega})(t,x)$ $(\sharp\sharp\sharp\sharp\sharp)$

for finite $t \in \overline{\mathcal{R}}_{\eta>0}$, finite $x \in \overline{\mathcal{R}}_\eta$. Finally, using Definition 0.10, we have;

$\Gamma * g_{\eta,\omega}(t,x) = \int_{\overline{\mathcal{R}_\eta}} \Gamma(t, x-y) g_{\eta,\omega}(y) d\lambda_\eta(y)$ (!)

By the growth condition on g, for $x \in \mathcal{R}$, $t \in \mathcal{R}_{>0}$, if $\Psi(t, x-y)$ denotes the standard heat kernel, the function $\Psi(t, x-y)g(y) : \mathcal{R} \rightarrow \mathcal{C}$ is continuous and satisfies the tail estimate $|\Psi(t, x-y)g(y)| \leq \frac{1}{y^2}$ for sufficiently large $|y| \geq A(t)$, $A(t) \in \mathcal{R}$. Using the proof of Theorem 0.17 in [6] and Theorem 3.24 of [7], we obtain that $\Gamma(t, x-y)g_{\eta,\omega}(y)$ is S-integrable and ${}^\circ(\Gamma * g_{\eta,\omega})(t,x) = H(t,x)$, (!!). For finite $x \in \overline{\mathcal{R}_\eta}$, finite $t \in \overline{\mathcal{R}_{\eta,>0}}$, and ${}^\circ t > 0$, we have that $\Gamma(t, x-y)g_{\eta,\omega}(y)$ is S-integrable, (!!!). In order to see this, choose $0 < t_1 < t < t_2$, with $t_1, t_2 \in \mathcal{R}$, and $x_1 < x < x_2$, with $x_1, x_2 \in \mathcal{R}$. We then have;

$\Gamma(t, x-y)|g_{\eta,\omega}(y)| \leq \sqrt{\frac{t_1}{t_2}}\Gamma(t_2, x_1 - y)|g_{\eta,\omega}(y)|$, for $y \leq x_1$

$\Gamma(t, x-y)|g_{\eta,\omega}(y)| \leq \sqrt{\frac{t_2}{t_1}}\Gamma(t_2, x_2 - y)|g_{\eta,\omega}(y)|$, for $y \geq x_2$

$\Gamma(t, x-y)|g_{\eta,\omega}(y)| \leq C(t)$, for $x_1 \leq y \leq x_2$ (!!!!)

where $C(t) \in \mathcal{R}$, and in (!!!!), we have used the fact that g is continuous. Now applying the result of (!!) and using [1] (Corollary 5), we obtain (!!!). Then, again using Theorem 3.24 of [7], we have that ${}^\circ(\Gamma * g_{\eta,\omega})(t,x) = H({}^\circ t, {}^\circ x)$, (!!!!!). Combining $(****)$,$(\sharp\sharp\sharp\sharp\sharp)$ and (!!!!!), we obtain that;

${}^\circ(\check{F_{\omega'}} * f)(t,x) = H({}^\circ t, {}^\circ x)$ (A)

for finite $x \in \overline{\mathcal{R}_\eta}$, finite $t \in \overline{\mathcal{R}}_{\eta>0}$, under the conditions;

$\eta > max(C_4(\omega', \frac{{}^*exp(-B|\omega|^\rho)}{\omega^2\omega'}, \omega'), 25\omega^{5*}exp(B|\omega|^\rho))$ (B)

$\omega' > Cmax({}^*log(\omega)^{\frac{1}{2}}, \omega^{\frac{\rho+1}{2}})$ (C)

By a simple calculation, we can satisfy $(B), (C)$ if;

$\eta > \omega^5\omega'^{4*}exp(B\omega^\rho)$, $\omega' > \omega^2$ (D)

and (D) if;

$\eta > \omega'^6 exp(B\omega')$, $\omega' > \omega^2$ (E)

Therefore, it is sufficient to have;

$\omega' < log(\eta)^{\frac{1}{2}}$, $\omega < \omega^{'\frac{1}{2}}$ (F).

Using (A) and condition (F), we obtain the result.

□

Theorem 0.18. *Let g be as in Definition 0.16, let g_η denote its measurable extension to $\overline{\mathcal{R}_\eta}$, and, let $g_{\eta,\omega}$ be the truncation of g_η, given by;*

$g_{\eta,\omega} = g_\eta \chi_{[-\omega,\omega)}$

for a nonstandard step function $\chi_{[-\omega,\omega)}$, with $\omega \in {}^*\mathcal{N} \setminus \mathcal{N}$, $\eta - \omega$ infinite and $\omega < \omega^{'\frac{1}{2}}$. Then, with $\hat{f}$ determined by Theorem 0.9, with $g_{\eta,\omega}$ as the boundary condition, we have;

$^\circ(\check{}(F_{\omega'}\hat{f}))|_{st^{-1}(\mathcal{T}_{>0}\times\mathcal{R})} = st^*(H_\infty)$

if $\omega' < log(\eta)^{\frac{1}{2}}$, and H_∞ is obtained from the classical solution H of the heat equation, with boundary condition g, given in Definition 0.16.

Proof. With notation as above, we have that;

$\check{F_{\omega'}} * f = \frac{1}{2}(\check{F_{\omega'}} * (\check{\hat{f}}))$ (by Theorem 0.7)

$\check{F_{\omega'}} * (\check{\hat{f}}) = \check{}(2F_{\omega'}\hat{f})$

as, by Theorem 0.7 and Remarks 0.10, for $a, b : \overline{\mathcal{T}}_\eta \times \overline{\mathcal{R}}_\eta \rightarrow {}^*\mathcal{C}$, $\hat{}(\check{a} * \check{b}) = \hat{\check{a}}\hat{\check{b}} = 2a.2b = 4ab$, $(*)$, and, $2(\check{a} * \check{b}) = \check{}\,\hat{}(\check{a} * \check{b}) = \check{}(4ab)$, by $(*)$ and Theorem 0.7. Therefore;

$\check{F_{\omega'}} * f = \check{}(F_{\omega'}\hat{f})$

and the result follows by Theorem 0.17.

□

Remarks 0.19. *Theorem 0.18 gives a solution to the heat equation, obtained by the following steps;*

(i). Truncating the transfer of the boundary data.

(ii). Taking the nonstandard Fourier transform of this data and solving the resulting ODE in Theorem 0.9.

(iii). Truncating the solution again.

(iv). Taking the inverse nonstandard Fourier transform.

(v). Specialising.

By straightforward results on limits in nonstandard analysis, see Theorem 2.22 of [7], *it follows that the above algorithm converges for* $\{m, n, n'\}$, *with* $n < (n')^{\frac{1}{2}}$, $n' < log(m)^{\frac{1}{2}}$, *(replacing* $\{\eta, \omega, \omega'\}$ *respectively), as* $m \rightarrow \infty$ *(noting that, for* η *infinite,* $\eta - log(\eta)^{\frac{1}{4}}$ *is infinite). It seems likely that the algorithm is faster than current methods involving a recursion over both the space and time steps. However, this still has to be decided computationally. This would be a useful result in financial mathematics, as it is well known that the Black Scholes equation, with the boundary condition for call options, can be transformed into the heat equation by a simple change of variables.*

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