DECAY RATES FOR CUSP FUNCTIONS

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Abstract. We make some observations on the decay rates of the Fourier coefficients of cusp functions.

Definition 0.1. *We let $H(0,1)$ denote the restrictions of holomorphic functions on the open disc $D(0,1+\epsilon)$, for some $\epsilon>0$, and $C^\infty(S^1)$, the set of smooth, complex-valued functions on the unit circle S^1. For $f \in H(0;1)$, with power series expansion $f(z)=\sum_{n\in\mathcal{Z}_{\geq 0}} a_n z^n$, we let $\overline{f}(n)=a_n$, for $n \in \mathcal{Z}_{\geq 0}$, and $f_1 = f|_{\partial D(0;1)}$. For $g \in C^\infty(S^1)$, and $n \in \mathcal{Z}$, we let;*

$$\hat{g}(n)=\frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-in\theta}d\theta$$

denote the n'th Fourier coefficient.

Lemma 0.2. *With notation as in 0.1, if $f \in H(0,1)$, and $n \in \mathcal{Z}_{\geq 0}$, we have $\overline{f}(n)=\hat{f}_1(n)$.*

Proof. By the definition of residues for holomorphic function, Cauchy's residue theorem, see [2], and Fourier series, see [3], we have, for $n \in \mathcal{Z}_{\geq 0}$, that;

$$\overline{f}(n)=\frac{1}{2\pi i}\int_{\partial D(0,1)}\frac{f(z)}{z^{n+1}}dz$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(e^{i\theta})}{e^{i(n+1)\theta}}ie^{i\theta}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta})}{e^{in\theta}}d\theta=\hat{f}_1(n)$$

□

Remarks 0.3. *By rescaling, we obtain a similar result for $0<\delta<1$ and $f \in H(0,1-\delta)$, $f_{1-\delta} \in C^\infty(S^1)$, where $f_{1-\delta}(z)=f((1-\delta)z)$, for $|z|=1$. Namely;*

$$\overline{f}(n)=\frac{1}{(1-\delta)^n}\hat{f}_{1-\delta}(n).$$

Definition 0.4. *We let* $\mathcal{H}_+ = \{z \in \mathcal{C} : Im(z) > 0\}$ *denote the upper half plane and* $H(\mathcal{H}_+)$ *denote the set of holomorphic functions on* $\mathcal{H}_+$*. We let* $Cusp(\mathcal{H}_+) \subset H(\mathcal{H}_+)$ *denote holomorphic functions on* $\mathcal{H}_+$*, satisfying the additional symmetry condition;*

(i). $g(z+1) = g(z)$*, for* $z \in \mathcal{C}$*.*

and the cusp condition;

(ii). $lim_{Im(z)\rightarrow\infty} g(z) = 0$ *(we assume the limit is uniform in* $Re(z)$*))*

Remarks 0.5. *Observe that cusp forms , see [1], are special examples of cusp functions, as given in Definition 0.4. The map* $\Phi = exp(2\pi iz) : \mathcal{H}_+ \rightarrow D(0,1)$ *is holomorphic, and, taking a principal branch of the logarithm* $\Gamma = \frac{log(z)}{2\pi i} : (D(0,1) \setminus [0,1)) \rightarrow (\mathcal{H}_+ \cap U)$*, where* $U = \{z \in \mathcal{H}_+ : 0 < z < 1\}$*, we obtain a holomorphic function* $\Gamma^*(g)$ *on* $(D(0,1) \setminus [0,1))$*. The condition (i) ensures that* $\Gamma^*(g)$ *extends uniquely to a holomorphic function* f_0 *on the annulus* $D(0,1) \setminus \{0\}$*, such that* $\Phi^*(f_0) = g$*. Using a Laurent's Theorem and the condition (ii),* f_0 *extends uniquely to a holomorphic function* f *on* $D(0,1)$ *with* $f(0) = 0$*. Taking the power series expansion* $f(z) = \sum_{n\in\mathcal{Z}_{\geq 0}} a_n z^n$ *on* $D(0,1)$*, we obtain a convergent expansion* $g(z) = \sum_{n\in\mathcal{Z}_{\geq 0}} a_n exp(2\pi inz)$ *on* $\mathcal{H}_+$*. Similarly to before, for* $n \in \mathcal{Z}_{\geq 0}$*, we let* $\overline{g}(n) = a_n$*.*

Lemma 0.6. *If* $g \in Cusp(\mathcal{H}_+)$*, and* $\delta > 0$*, then, for any* $m \in \mathcal{Z}_{>0}$*, there exist constants* $A_m \in \mathcal{R}_{\geq 0}$*, such that;*

$|\overline{g}(n)| \leq \frac{A_m}{n^m(1-\delta)^n}$*, for all* $n \geq 1$

Proof. Let f denote the holomorphic function on $D(0,1)$, obtained in Remark 0.5. For any $0 < \delta < 1$, we have that $f \in H(0, 1-\delta)$. By Remarks 0.3, we have that $\overline{f}(n) = \frac{1}{(1-\delta)^n} \hat{f}_{1-\delta}(n)$, $(*)$. As $f_{1-\delta} \in C^\infty(S^1)$, by a simple extension of Corollary 2.4 in [3], we have that, for any $m \in \mathcal{Z}_{>0}$, there exist constants $A_m \in \mathcal{R}_{\geq 0}$, such that;

$|\hat{f}_{1-\delta}(n)| \leq A_m n^{-m}$, for all $n \geq 1$

Then, by $(*)$ and Remarks 0.5, we have;

$|\overline{g}(n)| = |\overline{f}(n)| = \frac{1}{(1-\delta)^n} |\hat{f}_{1-\delta}(n)| \leq \frac{A_m}{n^m(1-\delta)^n}$, for all $n \geq 1$

□

Remarks 0.7. *This result should be compared with that of the Ramunajan-Petersson conjecture, that, for cusp forms of weight* $k \geq 2$, $|\overline{g}(n)| = O(n^{\frac{k-1}{2}+\gamma})$, $\gamma > 0$, $n \in \mathcal{Z}_{\geq 0}$, *see* [1]. *Observe that the bounding constant* A_m *cannot be chosen independently of* m, *so we do not get a better result. The result also easily generalises to holomorphic functions on* $\mathcal{H}_+$, *satisfying just the symmetry condition* (i), *in Definition 0.4, for* $n \in \mathcal{Z}$, *with* $|n| \geq 1$.

References

[1] Etale Cohomology and the Weil Conjecture, Eberhard Freitag, Springer-Verlag, Reinhardt Kiehl, (1987).

[2] Introduction to Complex Analysis, OUP, H.A. Priestley, (1990).

[3] Fourier Analysis, An Introduction, Elias Stein and Rami Shakarchi, Princeton Lectures in Analysis, (2003).

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