A NOTE ON INFLEXIONS OF CURVES

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Remarks 0.1. *We define a real projective algebraic curve $C \subset P^n(\mathcal{R})$ to be an irreducible algebraic scheme over $\mathcal{R}$ of dimension 1, and a real projective algebraic hypersurface $C \subset P^n(\mathcal{R})$ to be an irreducible algebraic scheme over $\mathcal{R}$ of dimension $n-1$. Working in the context of Robinson's theory of enlargements, we can define an infinitesimal neighborhood $\mathcal{V}_x$ of a point $x \in P^m(\mathcal{R})$, to be $P^m({}^*\mathcal{R}) \cap \mu(x)$, where $\mu(x) = \bigcap_{\epsilon \in \mathcal{R}_{>0}} D(x,\epsilon)$. We let L_x denote the Grassmannian of lines through x. We define the intersection multiplicity $I(C_{s_1}, C_{s_2}, x)$ of a real curve and a hypersurface $\{C_{s_1}, C_{s_2}\}$ at x, to be;*

$$max_{(s_1',s_2') \in (\mu(s_1,s_2) \cap l : l \in L_{(s_1,s_2)})} Card(C_{s_1'} \cap C_{s_2'} \cap \mu(x)).\ (\dagger)$$

In Theorem 18.7 of [2], *it is shown this definition coincides with algebraic multiplicity for plane complex algebraic curves. If $I(C_{s_1}, C_{s_2}, x) = m > 0$, then, choose parameters (s_1', s_2') witnessing this, and a line l_0, containing (s_1, s_2) and (s_1', s_2'). Now choose $\delta > 0$ standard, then, given any $\epsilon > 0$, there exists standard parameters $(t_1, t_2) \in (D((s_1, s_2), \epsilon) \cap l_0)$, such that $Card(C_{t_1} \cap C_{t_2} \cap D(\delta, x)) = m$, $(*)$. This follows, by transfer, as $\mu(s_1, s_2) \subset D((s_1, s_2), \epsilon)$ and $\mu(x) \subset D(x, \delta)$. Now, for such a $\delta > 0$, we can find a sequence of standard parameters $\{(s_1^n, s_2)^n : n \in \mathcal{N}\}$, converging to (s_1, s_2) on the line l_0, such that $|C_{s_1^n} \cap C_{s_2^n} \cap D(x, \delta)| = m$, $(**)$. For suppose not, then there exists a disc $D((s_1, s_2), \epsilon)$ for which there are no parameters $(y_1 y_2) \in D((s_1, s_2), \epsilon)$ with $Card(C_{(y_1,y_2)} \cap D(x, \delta)) = m$, contradicting $(*)$, hence, $(**)$ holds. Now let $\psi(y, z, \delta)$ be the formula $[(y, z) \neq (s_1, s_2), (y, z) \in l_0 : |(C_y \cap C_z \cap D(x, \delta))| = m]$, then, $\psi(y, z, \delta)$ is definable in the language of real ordered fields, hence, by $(**)$, contains an interval $U_\delta \subset l$, with $(s_1, s_2) \in \partial U_\delta$. We can assume that $U_\delta \subset l_0^+$, where $l_0^+ \subset l_0$ is a half-line, emanating from (s_1, s_2). As $\delta > 0$ was arbitrary, the sentence $\sigma = (\forall z > 0)(\exists t', t'' > 0)(\forall (t_1, t_2))[(s_1, s_2) < (t_1, t_2) < (s_1 + t', s_2) + t'']|(C_{t_1} \cap C_{t_2}) \cap D(x, z))| = m$ holds in $\mathcal{R}$, therefore, in ${}^*\mathcal{R}$. Hence, the original statement $(\dagger)$ can be formulated as;*

$I(C_{s_1}, C_{s_2}, x) = Card(C_{s'_1} \cap C_{s'_2} \cap \mu(x))$ *for* any$(s'_1, s'_2) \in (l_0^+ \setminus (s_1, s_2))$ $(*)$

It follows, see also [2], *that we can, purely geometrically, define the notion of a branch and the nature of singularities (Cayley's definition) using birationality arguments, see* [1]. *Using Severi's method of resolving singularities, it seems likely that, given a real projective algebraic curve C, we can find a nonsingular curve $C' \subset P^3$, and a birational map $\Phi : C' \leftrightsquigarrow C$. For a point $p \in C$, we can define the branches $\{\gamma_p^1, \ldots, \gamma_p^r\}$, centred at p, to be the neighborhoods $\{C' \cap \mu(p_1), \ldots, C' \cap \mu(p_r)\}$, where $\Gamma_\Phi(p, p_i)$, for $1 \leq i \leq r$. For a line l_p, centred at p, we define;*

$I(C, l_p, \gamma_p^i) = I(C', (\Phi)^{-1}(l_p), p_i)$

It is easily shown, using the observation $(*)$, that this definition is independent of the choice of birational map Φ. For a plane curve C and a branch γ_p^i, we define the tangent line $l_{\gamma_p^i}$ to be the unique line with the property that;

$I(C, l_{\gamma_p^i}, \gamma_p^i) > I(C, l_p, \gamma_p^i)$ *(for all $l_p \neq l_{\gamma_p^i}$)*

For a plane curve C, we can define a nonsingular point x to be an inflexion if $I(C, l_x) = 3$, where l_x is the tangent line. We define a singular point x to be a node, if there exists 2 branches $\{\gamma_x^1, \gamma_x^2\}$, centred at x, with distinct tangent lines $\{l_{\gamma_x^1}, l_{\gamma_x^2}\}$. We define a real plane projective curve C to be nodal, if it has at most nodes as singularities, and the inflexions are distinct from the nodes. It is easily seen that, for a nodal curve C, there exists finitely many points $\{p_1, \ldots, p_r\}$, for which the tangent lines, centred at p_i, $1 \leq i \leq r$, are horizontal or vertical. We can assume that the line l_∞ intersects C transversely, by a suitable choice of coordinates (x, y). By a simple rotation of the axes, we can assume that each p_i is not a node, and the projections $\{pr_x(p_1), \ldots, pr_x(p_r)\}$ and $\{pr_y(p_1), \ldots, pr_y(p_r)\}$ are all distinct, $(**)$.

In a similar way, for an analytic path $\lambda : (S^1, 1) \rightarrow \mathcal{R}^2$, and a point $p \in \mathcal{R}^2$, we define the branches $\{\gamma_p^1, \ldots, \gamma_p^r\}$, centred at p, to be the neighborhoods $\{C' \cap \mu(t_1), \ldots, C' \cap \mu(t_r)\}$, where $\lambda(t_i) = p$, for $1 \leq i \leq r$. For a line l_p, centred at p, we define;

$I(\lambda, l_{s_1}, \gamma_p^i) = max_{(s_1') \in (\mu(s_1) \cap l : l \in L_{(s_1)})} Card(\lambda^{-1}(l_{s_1'}) \cap \mu(t_i))$. (††)

We define the tangent line $l_{\gamma_p^i}$ *to be the unique line with the property that;*

$I(\lambda, l_{\gamma_p^i}, \gamma_p^i) > I(\lambda, l_p, \gamma_p^i)$ *(for all* $l_p \neq l_{\gamma_p^i}$*)*

It is easily shown that $l_{\gamma_p^i}$ *is given by* $\frac{y-\gamma_2(t_i)}{x-\gamma_1(t_i)} = \frac{\gamma_2'(t_i)}{\gamma_1'(t_i)}$, *where* $\gamma(t_i) = p$, *if* $l_{\gamma_p^i}$ *is not given by* $x = \gamma_1(t_i)$. *We call* p *nonsingular, if there exists a unique* $t \in S^1$, *with* $\lambda(t) = x$ *and* $\lambda'(t) \neq 0$.

We define a nonsingular point x *to be an inflexion if* $I(\lambda, l_x) = 3$, *where* l_x *is the tangent line. We define a singular point* x *to be a node, if there exists 2 branches* $\{\gamma_x^1, \gamma_x^2\}$, *centred at* x, *with distinct tangent lines* $\{l_{\gamma_x^1}, l_{\gamma_x^2}\}$, *equivalently, if* $\{\gamma'(t_1), \gamma'(t_2)\}$ *defines a basis of* $\mathcal{R}^2$, *where* $\gamma(t_1) = \gamma(t_2) = p$.

For a real plane nodal projective curve (C, p), *based at* $p \in \mathcal{R}^2$, *satisfying* $(**)$, *in coordinates* (x, y), *we can associate an analytic path* $\lambda : (S^1, 0) \rightarrow (\mathcal{R}^2, p)$, *as follows;*

Let $C' \subset P^3(\mathcal{R})$, *be a nonsingular real projective curve, with* pr_z : $C' \leftrightsquigarrow C$ *birational. Let* $\{x_1 < \ldots < x_r, y_1, \ldots, y_r\}$ *denote the projections of the vertical tangent points* $\{p_1, \ldots, p_r\}$ *of* C, *with corresponding* $\{q_1, \ldots, q_r\}$ *of* C'. . *Choose* $\{a_{1,1}, a_{1,2}, \ldots, a_{i,j}, \ldots, a_{r,1}, a_{r,2}\}$ *distinct, with* $pr_y(a_{i,1}) < y_i < pr_y(a_{i,2})$ *and* $a_{i,k} \in (\mu(p_i) \cap C)$, *for* $1 \leq k \leq 2$,$1 \leq i \leq r$. *Let;*

$\{b_{1,1}, b_{1,2}, \ldots, b_{i,k}, \ldots, b_{r,1}, b_{r,2}\} = pr_z^{-1}(\{a_{1,1}, a_{1,2}, \ldots, a_{i,k}, \ldots, a_{r,1}, a_{r,2}\})$.

Choose $\{d_{i,j} : 1 \leq i \leq r, 2 \leq j \leq w\}$ *distinct in* C, *with* $\{d_{i,j} : 2 \leq j \leq w\} = ((pr_x^{-1}(pr_x(p_i) \cap C) \setminus p_i)$, *and* $\{c_{i,j,k} : 1 \leq i \leq r, 2 \leq j \leq w, 1 \leq k \leq 2\}$ *distinct in* C, *with* $w = deg(C)$, *such that* $pr_x(c_{i,j,1}) < pr_x(b_{i,j}) < pr_x(c_{i,j,2})$ *and* $c_{i,j,k} \in (\mu(b_{i,j}) \cap C)$, *for* $1 \leq k \leq 2$. *Let;*

$\{e_{i,j} : 1 \leq i \leq r, 2 \leq j \leq w\} = pr_z^{-1}(\{d_{i,j} : 1 \leq i \leq r, 2 \leq j \leq w\})$.

Without loss of generality, assume that p *is based at* $d_{4,2}$, *the lower index cases are left as an exercise for the reader. Let;*

$\{f_{i,j,k}:1\leq i\leq r,2\leq j\leq w,1\leq k\leq 2\}$

$=pr_z^{-1}(\{c_{i,j,k}:1\leq i\leq r,2\leq j\leq w,1\leq k\leq 2\})$. $(\dagger\dagger)$

Let $\{\alpha_1,\ldots,\alpha_w\}$ denote the intersections of C with l_∞, the line at ∞ defined by $Z=0$, in coordinates $x=\frac{X}{Z}$, $y=\frac{Y}{Z}$. Let $O=(X=0)\cap(Z=0)$, and assume that $\{\alpha_1,\ldots,\alpha_w\}$ are distinct from O, are not nodes, and the branches $\{\gamma_{\alpha_1},\ldots,\gamma_{\alpha_w}\}$ are all transverse to l_∞, that is the tangent lines $\{l_{\gamma_{\alpha_1}},\ldots,l_{\gamma_{\alpha_w}}\}$, do not pass through O; this is easily achieved by a change of variables. Let $\{\eta_1,\ldots,\eta_w\}$ denote the corresponding points of C'. Choose a homography $K:P^2(\mathcal{R})\rightarrow P^2(\mathcal{R})$, which fixes O, moves l_∞ to finite position, and such that the tangent lines $\{l_{p_1},\ldots,l_{p_r}\}$ also remain in finite position. We can lift the homography K to a homography $K':P^3(\mathcal{R})\rightarrow P^3(\mathcal{R})$, such that $(K\circ pr_z)=(pr_z\circ K')$. Let (x',y') and (x',y',z') be the new coordinates induced by $\{K,K'\}$. Then, in the coordinates (x',y'), induced by K, the points $\{\alpha_1,\ldots,\alpha_w\}$ have coordinates $\{\alpha'_1,\ldots,\alpha'_w\}$, in finite position. Let $\{\beta'_1,\ldots,\beta'_w\}$ be the points of $(C\cap l'_\infty)$, for the new line at ∞, l'_∞, in (x',y'). As the vertical tangents $\{p_1,\ldots,p_r\}$ remain in finite position and O is fixed, the branches of $\{\beta'_1,\ldots,\beta'_w\}$ are transverse to l'_∞. Let $\{\eta'_1,\ldots,\eta'_w\}$ denote the corresponding points of C' to $\{\alpha'_1,\ldots,\alpha'_w\}$ in (x',y',z'). Let $\{\beta_1,\ldots,\beta_w\}$ denote the points $\{\beta'_1,\ldots,\beta'_w\}$ in the old coordinates (x,y). Let $y(\beta_1)<\ldots<y(\beta_w)$ denote the y-projections of $\{\beta_1,\ldots,\beta_w\}$, and assume that $x_1<x(\beta_1)=x_{\infty'}<x_2$, (observe that $x(\beta_1)=x(\beta_j)$, for $2\leq j\leq w$). Choose $\{c_{\infty',j,k}:1\leq j\leq w,1\leq k\leq 2\}$ in C distinct, with $pr_x(c_{\infty',j,1})<pr_x(\beta_1)=x_{\infty'}<pr_x(c_{\infty',j,2})$ and $c_{\infty',j,k}\in(\mu(\beta_j)\cap C)$, for $1\leq k\leq 2$. Let $y'(\alpha'_1)<\ldots<y'(\alpha'_w)$ denote the y-projections of $\{\alpha'_1,\ldots,\alpha'_w\}$, and assume that $x'_1<x'(\alpha'_1)=x'_\infty<x'_2$, (observe that $x'(\alpha'_1)=x'(\alpha'_j)$, for $2\leq j\leq w$.) Choose $\{c'_{\infty,j,k}:1\leq j\leq w,1\leq k\leq 2\}$ in C distinct, with $pr_{x'}(c'_{\infty,j,1})<pr_{x'}(\alpha'_1)=x'_\infty<pr_{x'}(c'_{\infty,j,2})$ and $c'_{\infty',j,k}\in(\mu(\alpha'_j)\cap C)$, for $1\leq k\leq 2$. Let;

$\{e_{\infty',j}:1\leq j\leq w\}=pr_z^{-1}(\{\beta_j:1\leq j\leq w\})$.

$\{f_{\infty',j,k}:1\leq j\leq w,1\leq k\leq 2\}$

$=pr_z^{-1}(\{c_{\infty',j,k}:1\leq j\leq w,1\leq k\leq 2\})$.

$\{e'_{\infty,j}:1\leq j\leq w\}=pr_{z'}^{-1}(\{\alpha'_j:1\leq j\leq w\})$.

$\{f'_{\infty,j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}$

$= pr_{z'}^{-1}(\{c'_{\infty,j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\})$.

be the corresponding points of C' in coordinates $(x,y,z),(x',y',z')$. Introduce $'$-notation for the points defined in (††), in the coordinates $(x',y'),(x',y',z')$ and let $\{f'_{\infty',j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}, \{c'_{\infty',j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}, \{e'_{\infty',j} : 1 \leq j \leq w\}$ be the corresponding points to $\{f_{\infty',j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}, \{c_{\infty',j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}, \{e_{\infty',j} : 1 \leq j \leq w\}$ in these coordinates. Let $\{f_{\infty,j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}, \{c_{\infty,j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}, \{e_{\infty,j} : 1 \leq j \leq w\}$ be the corresponding points to $\{f'_{\infty,j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}, \{c'_{\infty,j,k} : 1 \leq j \leq w, 1 \leq k \leq 2\}, \{e'_{\infty,j} : 1 \leq j \leq w\}$ in the coordinates (x,y,z).

For each i, with $2 \leq i \leq r-1$, $x_i < pr_x(a_{i,k})$, $1 \leq k \leq 2$, we associate open sets $U_{i,k} \subset C'$, given by $Im(h_{i,k})$, where $h_{i,k} : (x_i, x_{i+1}) \rightarrow C'$ is maximal with the property that $(pr_x \circ pr_z \circ h_{i,k}) = Id_{U_{pr_x(a_{i,k})}}$, for an open $U_{pr_x(a_{i,k})} \subset (x_i, x_{i+1})$, with $pr_x(a_{i,k}) \in U_{pr_x(a_{i,k})}$, and $b_{i,k} \in Im(h_{i,k})$, and, similarly, for $3 \leq i \leq r$, with (x_i, x_{i-1}) replacing (x_i, x_{i+1}), if $pr_x(a_{i,k}) < x_i$, ([1]). If $i = 1$, $x_1 > pr_x(a_{1,k})$, for $1 \leq k \leq 2$, we associate the open sets $U_{\infty,1,k} \subset C'$, given by $Im(h_{\infty,1,k})$, where $h_{\infty,1,k} : (-\infty, x_1) \rightarrow C'$ is maximal with the property that $(pr_x \circ pr_z \circ h_{\infty,1,k}) = Id_{U_{pr_x(a_{1,k})}}$, for an open $U_{pr_x(a_{i,k})} \subset (-\infty, x_1)$, with $pr_x(a_{1,k}) \in U_{pr_x(a_{i,k})}$, and $f_{\infty,j,k} \in Im(h_{\infty,1,k})$, for some $1 \leq j \leq w$, $1 \leq k \leq 2$. Similarly, if $i = r$, and $x_r < pr_x(a_{r,k})$, for $1 \leq k \leq 2$, we associate $\{U_{\infty,r,k}, h_{\infty,r,k}, (x_r, \infty)\}$.

For each (i,j), with $1 \leq i \leq r-1$, $2 \leq j \leq w$, and $x_i < (pr_x \circ pr_z)(f_{i,j,2})$, we associate open sets $V_{i,j,2} \subset C'$, given by $Im(g_{i,j,2})$, where $g_{i,j,2} : (x_i, x_{i+1}) \rightarrow C'$ is maximal with the property that $(pr_x \circ pr_z \circ g_{i,j,2}) = Id_{U_{pr_x(c_{i,j,2})}}$, for an open $U_{pr_x(c_{i,j,2})} \subset (x_i, x_{i+1})$, with $pr_x(c_{i,j,2}) = ((pr_x \circ pr_z)(f_{i,j,2})) \in U_{pr_x(c_{i,j,2})}$, and $f_{i,j,2} \in Im(g_{i,j,2})$, and, similarly, for $2 \leq i \leq r$, and $(pr_x \circ pr_z)(f_{i,j,1}) < x_i$, we associate $\{V_{i,j,1}, g_{i,j,1}, (x_i, x_{i-1})\}$,

[1]We have implicitly included $x_1 < x_{\infty'} < x_2$ in the indices. If $i = 1$, with $x_1 < pr_x(a_{1,k})$, $1 \leq k \leq 2$, we associate the open sets $U_{1,\infty',k} \subset C'$, given by $Im(h_{1,\infty',k})$, where $h_{1,\infty',k} : (x_1, x_{\infty'}) \rightarrow C'$ is maximal with the property that $(pr_x \circ pr_z \circ h_{1,\infty',k}) = Id_{U_{pr_x(a_{1,k})}}$, for an open $U_{pr_x(a_{1,k})} \subset (x_1, x_{\infty'})$, with $pr_x(a_{1,k}) \in U_{pr_x(a_{1,k})}$, and $b_{1,k} \in Im(h_{1,\infty',k})$. Similarly, if, $i = 2$, with $x_2 > pr_x(a_{2,k})$, $1 \leq k \leq 2$, we associate $\{U_{2,\infty',k}, h_{2,\infty',k}, (x_2, x_{\infty'})\}$

([2]). If $i=1$, $2\leq j\leq w$, $x_1>(pr_x\circ pr_z)(f_{i,j,k})$, we associate open sets $V_{\infty,1,j}\subset C'$, given by $Im(g_{\infty,1,j})$, where $g_{\infty,1,j}:(x_1,-\infty)\rightarrow C'$ is maximal with the property that, for $k=1$ or $k=2$, $(pr_x\circ pr_z\circ g_{\infty,1,j})=Id_{U_{pr_x(c_{1,j,k})}}$, for an open $U_{pr_x(c_{1,j,k})}\subset(-\infty,x_1)$, with $pr_x(c_{1,j,k})=((pr_x\circ pr_z)(f_{1,j,k}))\in U_{pr_x(c_{1,j,k})}$, and $f_{1,j,k}\in Im(g_{\infty,1,j})$. Similarly, if $i=r$, and $x_r<pr_x(a_{r,k})$, for either $k=1$ or $k=2$, we associate $\{V_{\infty,r,j},g_{\infty,r,j},(x_r,\infty)\}$, for $2\leq j\leq w$. (†††)

Let $\Gamma:\{1,\ldots,i,\ldots,w,\infty'\}\rightarrow\{1,\ldots,i,\ldots,w,\infty\}$ *be defined by;*

$\Gamma(1)=1$, $\Gamma(i)=w-i+2$, $(2\leq i\leq w)$, $\Gamma(\infty')=\infty$.

In the $'$-coordinates introduced above, we have that;

$x'_{\Gamma(1)}<x'_{\Gamma(\infty')}<x'_{\Gamma(2)}<\ldots<x'_{\Gamma(i)}<\ldots x'_{\Gamma(w)}$.

([3]).

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[2]Again, we have implicitly included $x_1<x_\infty<x_2$ in the indices. If $i=1$, $2\leq j\leq w$, with $x_1<pr_x(c_{1,j,2})$, we associate the open set $V_{1,j,\infty',2}\subset C'$, given by $Im(g_{1,j,\infty',2})$, where $g_{1,j,\infty',2}:(x_1,x_{\infty'})\rightarrow C'$ is maximal with the property that $(pr_x\circ pr_z\circ g_{1,j,\infty',2})=Id_{U_{pr_x(c_{1,j,2})}}$, for an open $U_{pr_x(c_{1,j,2})}\subset(x_1,x_{\infty'})$, with $pr_x(c_{1,j,2})\in U_{pr_x(c_{1,j,2})}$, and $f_{1,j,2}\in Im(g_{1,j,\infty',2})$. Similarly, if, $i=2$, with $x_2>pr_x(c_{2,j,1})$, we associate $\{V_{2,j,\infty',1},g_{2,j,\infty',1},(x_2,x_{\infty'})\}$. For $1\leq j\leq w$, with $x_{\infty'}>pr_x(c_{\infty',j,1})$, we associate the open set $V_{\infty',j,1,1}\subset C'$, given by $Im(g_{\infty',j,1,1})$, where $g_{\infty',j,1,1}:(x_{\infty'},x_1)\rightarrow C'$ is maximal with the property that $(pr_x\circ pr_z\circ g_{\infty',j,1,1})=Id_{U_{pr_x(c_{\infty',j,1})}}$, for an open $U_{pr_x(c_{\infty',j,1})}\subset(x_{\infty'},x_1)$, with $pr_x(c_{\infty',j,1})\in U_{pr_x(c_{\infty,j,1})}$, and $f_{\infty',j,1}\in Im(g_{\infty',j,1,1})$. Similarly, for $1\leq j\leq w$, with $x_{\infty'}<pr_x(c_{\infty',j,2})$, we associate $\{V_{\infty',j,2,2},g_{\infty',j,2,2},(x_{\infty'},x_2)\}$

[3]We make the following associations, which relate the maps given above in the unprimed, primed coordinates $(x,y,z),(x',y',z')$;

(i). $\{U_{i,k},h_{i,k},(x_i,x_{i+1}):2\leq i\leq r-1,1\leq k\leq 2\}$

(i)'. $\{U'_{\Gamma(i),k},h'_{\Gamma(i),k},(x'_{\Gamma(i)},x'_{\Gamma(i+1)}):2\leq i\leq r-1,1\leq k\leq 2\}$

(ii). $\{U_{i,k},h_{i,k},(x_i,x_{i-1}):3\leq i\leq r,1\leq k\leq 2\}$

(ii)'. $\{U'_{\Gamma(i),k},h'_{\Gamma(i),k},(x'_{\Gamma(i)},x'_{\Gamma(i-1)}):3\leq i\leq r,1\leq k\leq 2\}$

(iii). $\{U_{1,\infty',k},h_{1,\infty',k},(x_1,x_{\infty'}):1\leq k\leq 2\}$

(iii).' $\{U'_{\infty',\Gamma(1),k},h'_{\infty',\Gamma(1),k},(x'_{\Gamma(1)},-\infty):1\leq k\leq 2\}$

It will become clear in the proof that;

(iv). $\{U_{2,\infty',k}, h_{2,\infty',k}, (x_2, x_{\infty'}) : 1 \leq k \leq 2\}$

(iv)'. $\{U'_{\infty',\Gamma(2),k}, h'_{\infty',\Gamma(2),k}, (x'_{\Gamma(2)}, \infty) : 1 \leq k \leq 2\}$

(v). $\{U_{\infty,1,k}, h_{\infty,1,k}, (x_1, -\infty) : 1 \leq k \leq 2\}$

(v)'. $\{U'_{\Gamma(1),\infty,k}, h'_{\Gamma(1),\infty,k}, (x'_{\Gamma(1)}, x'_{\infty}) : 1 \leq k \leq 2\}$

(vi). $\{U_{\infty,r,k}, h_{\infty,r,k}, (x_r, +\infty) : 1 \leq k \leq 2\}$

(vi)'. $\{U'_{\Gamma(r),\infty,k}, h'_{\Gamma(r),\infty,k}, (x'_{\Gamma(r)}, x'_{\infty}) : 1 \leq k \leq 2\}$

(vii). $\{V_{i,j,2}, g_{i,j,2}, (x_i, x_{i+1}) : 2 \leq i \leq r-1, 2 \leq j \leq w\}$

(vii)'. $\{V'_{\Gamma(i),j,1}, g'_{\Gamma(i),j,1}, (x'_{\Gamma(i)}, x'_{\Gamma(i+1)}) : 2 \leq i \leq r-1, 2 \leq j \leq w\}$

(viii). $\{V_{i,j,1}, g_{i,j,1}, (x_i, x_{i-1}) : 3 \leq i \leq r, 2 \leq j \leq w\}$

(viii)'. $\{V'_{\Gamma(i),j,2}, g'_{\Gamma(i),j,2}, (x'_{\Gamma(i)}, x_{\Gamma(i-1)}) : 3 \leq i \leq r, 2 \leq j \leq w\}$

(ix). $\{V_{1,j,\infty',2}, g_{i,j,\infty',2}, (x_1, x_{\infty'}) : 2 \leq j \leq w\}$

(ix)'. $\{V'_{\infty',\Gamma(1),j}, g'_{\infty',\Gamma(1),j}, (x'_{\Gamma(1)}, -\infty) : 2 \leq j \leq w\}$

(x). $\{V_{2,j,\infty',1}, g_{2,j,\infty',1}, (x_2, x_{\infty'}) : 2 \leq j \leq w\}$

(x)'. $\{V'_{\infty',\Gamma(2),j}, g'_{\infty',\Gamma(2),j}, (x'_{\Gamma(2)}, \infty) : 2 \leq j \leq w\}$

(xi). $\{V_{\infty',j,1,1}, g_{\infty',j,1,1}, (x_{\infty'}, x_1) : 1 \leq j \leq w\}$

(xi).' $\{V'_{\Gamma(1),\infty',j}, g'_{\Gamma(1),\infty',j}, (-\infty, x_{\Gamma(1)}) : 1 \leq j \leq w\}$

(xii). $\{V_{\infty',j,2,2}, g_{\infty',j,2,2}, (x_{\infty'}, x_2) : 1 \leq j \leq w\}$

(xii)'. $\{V'_{\Gamma(2),\infty',j}, g'_{\Gamma(2),\infty',j}, (+\infty, x_{\Gamma(2)}) : 1 \leq j \leq w\}$

(xiii). $\{V_{\infty,1,j}, g_{\infty,1,j}, (x_1, -\infty) : 1 \leq j \leq w\}$

(xiii)'. $\{V'_{\Gamma(1),j,\infty,2}, g'_{\Gamma(1),j,\infty,2}, (x'_{\Gamma(1)}, x'_{\infty}) : 1 \leq j \leq w\}$

(xiv). $\{V_{\infty,r,j}, g_{\infty,r,j}, (x_r, +\infty) : 1 \leq j \leq w\}$

(xiv)'. $\{V'_{\Gamma(r),j,\infty,1}, g'_{\Gamma(r),j,\infty,1}, (x_{\Gamma(r)}, x'_{\infty}) : 1 \leq j \leq w\}$

$\{U_{i,k}:1\leq i\leq r,1\leq k\leq 2\}\cup\{U_{\infty,1,k},U_{\infty,r,k}:1\leq k\leq 2\}$

$\cup\{U_{1,\infty',k},U_{2,\infty',k}:1\leq k\leq 2\}$

$\cup\{V_{i,j,k}:1\leq i\leq r,2\leq j\leq w,1\leq k\leq 2\}$

$\cup\{V_{\infty,1,j},V_{\infty,r,j}:2\leq j\leq w\}\cup\{V_{1,j,\infty',1},V_{1,j,\infty',2}:2\leq j\leq w\}$

$\cup\{V_{\infty',j,1,1},V_{\infty',j,2,2}:2\leq j\leq w\}$

cover $(C'\setminus\{q_i,e_{i,j}:2\leq j\leq w\}\cup\{\eta_j,\beta_j:1\leq j\leq w\})$. *By the IVT;*

$\{h_{i,k}:1\leq i\leq r,1\leq k\leq 2\}\cup\{h_{\infty,1,k},h_{\infty,r,k}:1\leq k\leq 2\}$

$\cup\{h_{1,\infty',k},h_{2,\infty',k}:1\leq k\leq 2\}$

$\cup\{g_{i,j,k}:1\leq i\leq r,2\leq j\leq w,1\leq k\leq 2\}\cup\{g_{\infty,1,j},g_{\infty,r,j}:2\leq j\leq w\}$

$\cup\{g_{2,j,\infty',1},g_{1,j,\infty',2}:2\leq j\leq w\}\cup\{g_{\infty',j,1,1},g_{\infty',j,2,2}:2\leq j\leq w\}$

are analytic. We define a path γ *inductively as follows;*

$\gamma|_{(x_4,x_5)}=g_{4,2,2}$.

Suppose γ *has been defined by on* $\{\bigsqcup_{1\leq t\leq s}\}(x_{i_t},x_{i_{t+1}})$, *where;*

if $1\leq i_t\leq r$, *and* $i_t\notin\{1,2,r,\infty'\}$, $i_{t+1}=i_t+1$ *or* $i_{t+1}=i_t-1$, *and* $\gamma|_{(x_{i_t},x_{i_t+1})}=h_{i_t,k(t)}$, *or* $\gamma|_{(x_{i_t},x_{i_t+1})}=g_{i_t,j(t),k(t)}$, *and* $\gamma|_{(x_{i_t},x_{i_t-1})}=h_{i_t,k(t)}$, *or* $\gamma|_{(x_{i_t},x_{i_t-1})}=g_{i_t,j(t),k(t)}$ *(where the union is disjoint, and the intervals may repeat).*

if $i_t=1$, $i_{t+1}=\infty'$ *or* $i_{t+1}=-\infty$, *(with the convention that* $x_{-\infty}=-\infty$*) and* $\gamma|_{(x_1,x_{\infty'})}=h_{\infty,1,k(t)}$ *or* $\gamma|_{(x_1,x_{\infty'})}=g_{1,j(t),\infty',2}$, *and* $\gamma|_{(x_1,x_{-\infty})}=g_{\infty,1,k(t)}$ *or* $\gamma|_{(x_1,x_{-\infty})}=g_{\infty,1,j(t)}$.

if $i_t=2$, $i_{t+1}=3$ *or* $i_{t+1}=\infty'$ *and* $\gamma|_{(x_2,x_3)}=h_{2,3}$ *or* $\gamma|_{(x_2,x_3)}=g_{2,j(t),2}$, *and* $\gamma|_{(x_2,x_{\infty'})}=h_{2,\infty',k(t)}$ *or* $\gamma|_{(x_2,x_{\infty'})}=g_{2,j(t),\infty',1}$.

if $i_t=r$, $i_{t+1}=+\infty$, *or* $i_{t+1}=r-1$ *(with the convention that* $x_{+\infty}=\infty$*) and* $\gamma|_{(x_r,x_{+\infty'})}=h_{\infty,r,k(t)}$ *or* $\gamma|_{(x_r,x_{+\infty'})}=g_{\infty,r,j(t)}$, *and*

$\gamma|_{(x_r,x_{r-1}} = h_{r,k(t)}$ *or* $\gamma|_{(x_r,x_{r-1}} = g_{r,j(t),k(t)}$.

if $i_t = \infty'$, $i_{t+1} = 2$, *or* $i_{t+1} = 1$ *and* $\gamma|_{(x_{\infty'},x_2)} = g_{\infty',j(t),2,2}$, *and* $\gamma|_{(x_{\infty'},x_1)} = g_{\infty',j(t),1,1}$.

Then let;

$\gamma|_{(x_{i_{s+1}},x_{i_s})} = h_{i_s,(k(s)+1)(mod2)}$, *if* $i_{s+1} = i_s + 1$, *and either* $a_{i_s+1,1} \in Im(h_{i_s,j(s)})$, *or,* $a_{i_s+1,2} \in Im(h_{i_s,k(s)})$.

$\gamma|_{(x_{i_s},x_{i_{s+1}})} = h_{i_s,(k(s)+1)(mod2)}$, *if* $i_{s+1} = i_s - 1$, *and either* $a_{i_s-1,1} \in Im(h_{i_s,j(s)})$, *or,* $a_{i_s-1,2} \in Im(h_{i_s,k(s)})$.

$\gamma|_{(x_{i_{s+1}},x_{i_{s+1}+1})} = g_{i_{s+1},j(s+1),k(s+1)}$, *if* $i_{s+1} = i_s+1$, *and* $f_{i_{s+1},j(s+1),k(s+1)} \in Im(g_{i_s,j(s),k(s)})$.

$\gamma|_{(x_{i_{s+1}},x_{i_{s+1}-1})} = g_{i_{s+1}-1,j(s+1),k(s+1)}$, *if* $i_{s+1} = i_s-1$, *and* $f_{i_{s+1}-1,j(s+1),k(s+1)} \in Im(g_{i_s,j(s),k(s)})$.

provided that γ has not been defined with one of these cases, or its reverse, on an earlier interval, otherwise, terminate the process. Clearly, as the number of possible intervals (x_i, x_{i+1}), $(1 \leq i \leq r)$, and allowable functions $\{h_{ik}, f_{ijk} : 1 \leq i \leq r, 1 \leq j \leq w-1, 1 \leq k \leq 2,\}$ is finite, the process terminates after a finite number of steps. We claim that the final interval in the process is $((x_2, x_1)$, with $\gamma|_{((x_2,x_1)} = h_{1,2}$. In order to see this, suppose the process terminates after s_0 steps, involving the intervals $\{I_1, \ldots, I_s, \ldots, I_{s_0}\}$, with endpoints $\{x_{i(s),s}, x_{i(s)+1,s} : 1 \leq s \leq s_0\}$. Let $S_t = \{\gamma(x_{i(s),s}), \gamma(x_{i(s)+1,s}) : 1 \leq s \leq t\}$, (with repeats, and the obvious ordering). We have $S_1 = p$, $S_{s_1} \subseteq S_{s_2}$, for $1 \leq s_1 \leq s_2 \leq s$. Then, for each $1 \leq s < s_0$, if v_s is the final vertex, v_s occurs once in S_s, $v_s \neq p$, and if $v \in (S_s \setminus \{p, v_s\})$, then v is repeated twice. This is easily shown by induction. Suppose $s - 1 \leq s < s_0$, with v_s the final vertex. If $v_s = p$, then either $v_{s-1} = h_{1,1}(x_2)$, in which case, the reverse of $h_{1,1}$ has been repeated, contradicting the definition of the construction, or $v_{s-1} = h_{1,2}(x_2)$. In this case, the construction terminates at p, contradicting the hypothesis, as there are only two functions emanating from p which have been used (initial step and penultimate step)). Hence, $v_s \neq p$. We have v_{s-1} is joined to $\{v_s, v_{s-2}\}$. If v_s occurs earlier than v_{s-1} in the ordering, then by induction, as $v_{s-1} \neq p$, it occurs twice, therefore occurs three times, implying that an interval is repeated. If $v_s = v_{s-1}$, then only one point. Hence, v_s occurs once and then v_{s-1}

occurs twice. This completes the induction. Now consider the final step s_0*, if* $v_{s_0} \neq p$*, then, we have, by the above, that* $v_{s_0-1} \neq p$*, and* $v_{s_0-1} \neq v_{s_0}$*. By the same argument,* v_{s_0} *cannot occur earlier than* v_{s_0-1}*, hence, it is possible to continue the construction, contradicting the assumption.*

Now let $\gamma = (x, \gamma_1(x), \gamma_2(x))$ *be defined on the intervals* $\{\bigsqcup_{1\leq t\leq s_0}\}(x_{i_t}, x_{i_{t+1}})$*. We claim that for each* $1 \leq t_0 \leq s_0$*,* $\gamma|(x_{i_t}, x_{i_{t+1}})\bigsqcup(x_{i_{t+1}}, x_{i_{t+2}})$ *extends to* $\gamma|_(x_{i_t}, x_{i_{t+2}})$*. Let* C' *be defined by* $F_1(x, y, z)$ *and* $F_2(x, y, z)$*. Then we have, for all* $x_0 \in (x_{i_t}, x_{i_{t+2}})$*,* $x_0 \neq x_{i_{t+1}}$*, that;*

$$F_{1,x}(\gamma(x_0)) + F_{1,y}(\gamma(x_0))\gamma_1'(x_0) + F_{1,z}(\gamma(x_0))\gamma_2'(x_0) = 0$$

$$F_{2,x}(\gamma(x_0)) + F_{2,y}(\gamma(x_0))\gamma_1'(x_0) + F_{2,z}(\gamma(x_0))\gamma_2'(x_0) = 0$$

As C' *is non singular, for* $x_0 \in (x_{i_t}, x_{i_{t+2}})$*, the hyperplanes defined by;*

$$F_{1,x}(\gamma(x_0)) + F_{1,y}(\gamma(x_0))u + F_{1,z}(\gamma(x_0))v = 0$$

$$F_{2,x}(\gamma(x_0)) + F_{2,y}(\gamma(x_0))u + F_{2,z}(\gamma(x_0))v = 0$$

are transverse, hence, determines a continuous function $\theta : (x_{i_t}, x_{i_{t+2}}) \rightarrow \mathcal{R}^2$*, with* $\theta(x_0) = (\gamma_1'(x_0), \gamma_2'(x_0)$*, for* $x_0 \neq x_{i_{t+1}}$*. This implies the result.*
then we have that;

patch the intervals onto $[0, 1]$*...*
take the projection $(pr_z \circ \gamma)$*.*

Definition 0.2. *We define a nodal path to be a function* $\gamma : S^1 \rightarrow \mathcal{R}^2$*, with the following properties;*

(i). γ *is analytic, that is defines an analytic map of real manifolds.*

(ii). γ *is smooth, that is* $\gamma'(t) \neq 0$*, for* $t \in S^1$*.*

(iii). γ *has at most nodes as singularities, that is there exists at most two distinct points* $\{t_1, t_2\} \subset [0, 1)$*, with* $\gamma(t_1) = \gamma(t_2)$*, and, in this case,* $\{\gamma'(t_1), \gamma'(t_2)\}$ *defines a basis of* $\mathcal{R}^2$*.*

We define a node of γ*, to be a point* $p \in \mathcal{R}^2$*, for which there* do *exist two distinct points* $\{t_1, t_2\} \subset [0, 1)$*, with* $\gamma(t_1) = \gamma(t_2) = p$*. We define* *a time inflexion of* γ*, to be a point* $t_0 \in S^1$*, such that, in coordinates* (x, y)*, with* $p = (x(t_0), y(t_0))$*,* $x''(t_0)y'(t_0) = x'(t_0)y''(t_0)$*,* $(*)$*, that is the curvature* $\kappa(t_0) = 0$*, and* $x'''(t_0)y'(t_0) \neq x'(t_0)y'''(t_0)$*, (*[4]*)*

(iv). The nodes and inflexions are distinct from $\gamma(0)$*.*

(v). If t_0 *is a time inflexion of* γ*, then* $\gamma(t_0)$ *is not a node.*

We define an inflexion of γ *to be a point* p*, for which there exists a time inflexion* t_0 *such that* $\gamma(t_0) = p$*.*

Definition 0.3. *We define a smooth closed path to be a function* $\gamma : (S^1, 1) \rightarrow (\mathcal{R}^2, (0, 0))$*, with the following properties;*

(i). γ *is analytic, that is defines an analytic map of real manifolds.*

(ii). γ *is smooth, that is* $\gamma'(t) \neq 0$*, for* $t \in S^1$*.*

We define a vertical tangent point to be t_0*, for which* $\gamma_1'(t_0) = 0$*, and, a horizontal tangent point* t_0'*, for which* $\gamma_2'(t_0') = 0$*. Using results of* [5]*, there exist finitely many points* $\{t_1, \ldots, t_w\}$*, which are horizontal or vertical tangents. We require;*

(iii). The vertical and horizontal tangents are distinct from;

$\gamma(0) = \gamma(1) = (0, 0)$*.*

(iv). We require that on each interval $[t_i, t_j]$*,* $[t_{i'}, t_{j'}]$*, with* $\{t_i, t_{j'}\}$ *horizontal,* $\{t_j, t_{i'}\}$ *vertical,* $1 \leq i < j \leq w$*,* $1 \leq i' < j' \leq w$*, that;*

$$\gamma_1|_{[t_i,t_j]}(e^{2\pi it}) = \gamma_1(t_i) + \tfrac{(\gamma_1(t_j)-\gamma_1(t_i))(t-t_i)}{(t_j-t_i)}$$

$$\gamma_2|_{[t_{i'},t_{j'}]}(e^{2\pi it}) = \gamma_2(t_{i'}) + \tfrac{(\gamma_2(t_{j'})-\gamma_2(t_{i'}))(t-t_{i'})}{(t_{j'}-t_{i'})}$$

(iii). γ *has at most nodes as singularities, that is there exists at most two distinct points* $\{t_1, t_2\} \subset [0, 1)$*, with* $\gamma(t_1) = \gamma(t_2)$*, and, in this case,* $\{\gamma'(t_1), \gamma'(t_2)\}$ *defines a basis of* $\mathcal{R}^2$*.*

[4]It is an interesting point that the condition $(*)$ is implied by $x'(t_0)y(t_0) = x(t_0)y'(t_0)$ as, by differentiating, we have, $x''(t_0)y(t_0) + x'(t_0)y'(t_0) = x'(t_0)y'(t_0) + x(t_0)y''(t_0)$, and $x''(t_0)y(t_0) = x(t_0)y''(t_0)$, so $(\frac{y''(t_0)}{x''(t_0)}) = (\frac{y'(t_0)}{x'(t_0)}) = (\frac{y(t_0)}{x(t_0)})$. However, the converse is not necessarily true, that $(*)$ implies $x'(t_0)y(t_0) = x(t_0)y'(t_0)$.

We define a vertical tangent point to be t_0, for which $\gamma_1'(t_0)=0$, and, a horizontal tangent point t_0', for which $\gamma_2'(t_0')=0$. We define a node of γ, to be a point $p\in\mathcal{R}^2$, for which there do *exist two distinct points, $\{t_1,t_2\}\subset[0,1)$, with $\gamma(t_1)=\gamma(t_2)=p$. We define a time inflexion of γ, to be a point $t_0\in S^1$, such that, in coordinates (x,y), with $p=(x(t_0),y(t_0))$, if t_0 is a vertical tangent point,*

and

$x''(t_0)y'(t_0)=x'(t_0)y''(t_0)$, $(*)$, *that is the curvature $\kappa(t_0)=0$, and* $x'''(t_0)y'(t_0)\neq x'(t_0)y'''(t_0)$, *([5])*

(iv). The nodes and inflexions are distinct from $\gamma(0)$.

(v). If t_0 is a time inflexion of γ, then $\gamma(t_0)$ is not a node.

We define an inflexion of γ to be a point p, for which there exists a time inflexion t_0 such that $\gamma(t_0)=p$.

Remarks 0.4. *We recall the following result from* [5], *Lemma 3.5, that, for a nodal path, there exist finitely many nodes $\{\nu_1,\ldots,\nu_m\}$. In a similar way, one can show that there exist finitely many inflexions $\{i_1,\ldots,i_r\}$.*

Lemma 0.5. *Let $\phi:S^1\rightarrow S^1$ be defined by;*

$$\phi(t)=\frac{\gamma'(t)}{|\gamma'(t)|}$$

Then $\phi'(t_0)=0$ iff $p=(\gamma_1(t_0),\gamma_1(t_0))$ is an inflexion.

Proof. Let $r(t)=[(\gamma_1')^2(t)+(\gamma_2')^2(t)]^{\frac{1}{2}}$. Then;

$$r'(t)=\frac{\gamma_1'\gamma_1''(t)+\gamma_2'\gamma_2''(t)}{r(t)}$$

$$\phi'(t)=\left(\frac{\gamma_1''r-\frac{(\gamma_1')^2\gamma_1''}{r}-\frac{\gamma_1'\gamma_2'\gamma_2''}{r}}{r^2},\frac{\gamma_2''r-\frac{(\gamma_2')^2\gamma_2''}{r}-\frac{\gamma_1'\gamma_2'\gamma_1''}{r}}{r^2}\right)\ (*)$$

If $\phi'(t_0)=0$, we have;

$$(\gamma_1''r^2-(\gamma_1')^2\gamma_1''-\gamma_1'\gamma_2'\gamma_2'')|_{t_0}=0\ (1)$$

[5]It is an interesting point that the condition $(*)$ is implied by $x'(t_0)y(t_0)=x(t_0)y'(t_0)$ as, by differentiating, we have, $x''(t_0)y(t_0)+x'(t_0)y'(t_0)=x'(t_0)y'(t_0)+x(t_0)y''(t_0)$, and $x''(t_0)y(t_0)=x(t_0)y''(t_0)$, so $(\frac{y''(t_0)}{x''(t_0)})=(\frac{y'(t_0)}{x'(t_0)})=(\frac{y(t_0)}{x(t_0)})$. However, the converse is not necessarily true, that $(*)$ implies $x'(t_0)y(t_0)=x(t_0)y'(t_0)$.

$(\gamma_2''r^2 - (\gamma_2')^2\gamma_2'' - \gamma_1'\gamma_2'\gamma_1'')|_{t_0} = 0$ (2)

Then, from (2), we have;

$(\gamma_1'\gamma_2')|_{t_3} = (\frac{\gamma_2''}{\gamma_1''}(r^2 - (\gamma_2')^2))|_{t_0}$

Then, substituting into (1), we obtain;

$(\gamma_1'')^2(r^2 - (\gamma_1')^2)|_{t_0} = (\gamma_2'')^2(r^2 - (\gamma_2')^2)|_{t_0}$

and using $r^2 = (\gamma_1')^2 + (\gamma_2')^2$, we obtain that;

$((\gamma_1'')^2)|_{t_0} = ((\gamma_2'')^2\frac{(\gamma_1')^2}{(\gamma_2')^2})|_{t_0}$

$(\gamma_1''\gamma_2')|_{t_0} = (\gamma_2''\gamma_1')|_{t_0}$.

implying that $p = (\gamma_1(t_0), \gamma_2(t_0))$ is an inflexion, by Definition 0.3.

Conversely, if $p = (\gamma_1(t_0), \gamma_2(t_0))$ is an inflexion, then, again, by Definition 0.3, we have that $(\gamma_1''\gamma_2')|_{t_0} = (\gamma_2''\gamma_1')|_{t_0}$. Reversing the steps of the above argument, we obtain that $\phi'(t_0) = 0$.

□

Lemma 0.6. *Let γ be a nodal path, then the number r of inflexions is even.*

Proof. Let $0 \leq t_1 < t_2 < 1$, with the property that there does not exist t_3, with $t_1 < t_3 < t_2$ such that $\gamma(t_3)$ is an inflexion, and $\{\gamma(t_1), \gamma(t_2)\}$ are inflexions. Letting $\phi : [t_1, t_2] \rightarrow S^1$ be defined, as above, we have that, if $\phi(t) = (\phi_1(t), \phi_2(t))$, then $\phi'(t).\phi(t) = 0$, so $\phi_1'\phi_1 + \phi_2'\phi_2 = 0$ (1). If $\theta(t) = tan^{-1}(\frac{\phi_1(t)}{\phi_2(t)})$, then;

$\frac{d\theta}{dt} = \frac{\phi_1'\phi_2(t) - \phi_2'\phi_1(t))}{((\phi_1)^2(t) + (\phi_2)^2(t))}$ $(*)$

Suppose $t_1 < t_3 < t_2$ and $(\frac{d\theta}{dt})|_{t_3} = 0$, then, we obtain, by $(*)$, that $(\phi_1'\phi_2(t) - \phi_2'\phi_1(t))|_{t_3} = 0$, (2). Combing (1), (2), we obtain $(\phi_1)^2(\frac{\phi_2'}{\phi_2}) + (\phi_2)^2(\frac{\phi_2'}{\phi_2})|_{t_3} = (\frac{\phi_2'}{\phi_2})|_{t_3} = 0$. Therefore, $\phi'(t_3) = (\phi_1'(t_3), \phi_2'(t_3)) = 0$. By Lemma 0.5, we would have that $\gamma(t_3)$ is an inflexion, hence, $(\frac{d\theta}{dt})|_{(t_1,t_2)} \neq 0$. Again, by Lemma 0.5, we have $\phi'(t_2) = \phi'(t_3) = 0$, (†), hence, by $(*)$, $\frac{d\theta}{dt}(t_2) = \frac{d\theta}{dt}(t_3) = 0$. We claim that $\frac{d^2\theta}{dt^2}(t_2) \neq 0$, $(**)$. If $(**)$ fails, then, by $(*)$, we have that $\phi_1''\phi_2(t_2) - \phi_2''\phi_1(t_2) = 0$, (††). By the Fundamental Theorem of Calculus, using the fact that $\phi'(t_2) = (\phi_1'(t_2), \phi_2'(t_2)) = 0$, by (†), we have;

$\phi'(t_2+\epsilon)=\int_{t_2}^{t_2+\epsilon}\phi''(t)dt$ (††)

As γ is analytic, if $\phi''(t_2)\neq 0$, (†††), then if $\alpha(t)=cos^{-1}(\frac{\phi'(t).l_{\phi(t)}}{|\phi'(t)|})$ measures the angle between the velocity vector $\phi'(t)$ and the tangent line $l_{\phi(t)}$ to S^1, we have, for sufficiently small ϵ, that $\alpha(t_2+\epsilon)\neq 0$, by (††) and the fact that, for $\beta(t)=cos^{-1}(\frac{\phi''(t).l_{\phi(t)}}{|\phi'(t)|})$, we have $\beta(t_2)=\frac{\pi}{2}\neq 0$. This clearly contradicts the fact that, for all t, $\alpha(t)=0$, as $\phi'(t)||l_{\phi(t)}$. Hence, (†††) fails and $\phi''(t_2)=0$. By $(*)$ of 0.5, and $\phi'(t_2)=\phi''(t_2)=0$, we have that;

$(r^3\phi_1')'|_{t_2}=(\gamma_1''r^2-(\gamma_1')^2\gamma_1''-\gamma_1'\gamma_2'\gamma_2'')'|_{t_2}$

$=(\gamma_1'''r^2+2\gamma_1''rr'-2\gamma_1'(\gamma_1'')^2-(\gamma_1')^2\gamma_1'''-\gamma_1''\gamma_2'\gamma_2''-\gamma_1'(\gamma_2'')^2-\gamma_1'\gamma_2'\gamma_2''')|_{t_2}=0$

and, similarly;

$(\gamma_2'''r^2+2\gamma_2''rr'-2\gamma_2'(\gamma_2'')^2-(\gamma_2')^2\gamma_2'''-\gamma_2''\gamma_1'\gamma_1''-\gamma_2'(\gamma_1'')^2-\gamma_2'\gamma_1'\gamma_1''')|_{t_2}=0$

Using the fact, by Lemma 0.5, that $(\gamma_1''\gamma_2')|_{t_2}=(\gamma_2''\gamma_1')|_{t_2}$, we obtain that $(\gamma_1'''\gamma_2')|_{t_2}=(\gamma_1'\gamma_2''')|_{t_2}$, contradicting Definition 0.3. Hence, $(**)$ holds, that is $\frac{d^2\theta}{dt^2}(t_2)\neq 0$, and, similarly $\frac{d^2\theta}{dt^2}(t_3)\neq 0$. Enumerating the inflexions $\{i_1,\ldots,i_r\}$, with corresponding $\{t_1,\ldots,t_r\}$, we have, by definition of a maximum/minimum for θ, that the angle θ is increasing/decreasing in the intervals (t_i,t_{i+1}), and changes direction at each t_i, for $1\leq i\leq r$. If the number of inflexions were odd, then clearly θ would be both increasing and decreasing on each interval (t_i,t_{i+1}), implying that θ is constant. This clearly implies that γ is contained in a line l, with no inflexions. Otherwise, we obtain that the number of inflexions is even as required. □

Remarks 0.7. *We define a real projective algebraic curve $C\subset P^n(\mathcal{R})$ to be an irreducible algebraic scheme over $\mathcal{R}$ of dimension 1. Working in the context of Robinson's theory of enlargements, we can define an infinitesimal neighborhood $\mathcal{V}_x$ of a point $x\in P^m(\mathcal{R})$, to be $P^m({}^*\mathcal{R})\cap\mu(x)$, where $\mu(x)=\bigcap_{\epsilon\in\mathcal{R}_{>0}}D(x,\epsilon)$. We let L_x denote the Grassmannian of lines through x. We define the intersection multiplicity $I(C_{s_1},C_{s_2},x)$ of two real plane curves $\{C_{s_1},C_{s_2}\}$ at x, to be;*

$max_{(s_1',s_2')\in(\mu(s_1,s_2)\cap l:l\in L_{(s_1,s_2)})}Card(C_{s_1'}\cap C_{s_2'}\cap\mu(x))$. (†)

In Theorem 18.7 of [2], *it is shown this definition coincides with algebraic multiplicity for complex algebraic curves. If* $I(C_{s_1},C_{s_2},x)=m>0$, *then, choose parameters* (s_1',s_2') *witnessing this, and a line* l_0, *containing* (s_1,s_2) *and* (s_1',s_2'). *Now choose* $\delta>0$ *standard, then, given any* $\epsilon>0$, *there exists standard parameters* $(t_1,t_2)\in(D((s_1,s_2),\epsilon)\cap l_0)$, *such that* $Card(C_{t_1}\cap C_{t_2}\cap D(\delta,x))=m$, $(*)$. *This follows, by transfer, as* $\mu(s_1,s_2)\subset D((s_1,s_2),\epsilon)$ *and* $\mu(x)\subset D(x,\delta)$. *Now, for such a* $\delta>0$, *we can find a sequence of standard parameters* $\{(s_1^n,s_2)^n:n\in\mathcal{N}\}$, *converging to* (s_1,s_2) *on the line* l_0, *such that* $|C_{s_1^n}\cap C_{s_2^n}\cap D(x,\delta)|=m$, $(**)$. *For suppose not, then there exists a disc* $D((s_1,s_2),\epsilon)$ *for which there are no parameters* $(y_1y_2)\in D((s_1,s_2),\epsilon)$ *with* $Card(C_{(y_1,y_2)}\cap D(x,\delta))=m$, *contradicting* $(*)$, *hence,* $(**)$ *holds. Now let* $\psi(y,z,\delta)$ *be the formula* $[(y,z)\neq(s_1,s_2),(y,z)\in l_0:|(C_y\cap C_z\cap D(x,\delta))|=m]$, *then,* $\psi(y,z,\delta)$ *is definable in the language of real ordered fields, hence, by* $(**)$, *contains an interval* $U_\delta\subset l$, *with* $(s_1,s_2)\in\partial U_\delta$. *We can assume that* $U_\delta\subset l_0^+$, *where* $l_0^+\subset l_0$ *is a half-line, emanating from* (s_1,s_2). *As* $\delta>0$ *was arbitrary, the sentence* $\sigma=(\forall z>0)(\exists t',t''>0)(\forall(t_1,t_2))[(s_1,s_2)<(t_1,t_2)<(s_1+t',s_2)+t'']|(C_{t_1}\cap C_{t_2})\cap D(x,z))|=m$ *holds in* $\mathcal{R}$, *therefore, in* ${}^*\mathcal{R}$. *Hence, the original statement* (†) *can be formulated as;*

$$I(C_{s_1},C_{s_2},x)=Card(C_{s_1'}\cap C_{s_2'}\cap\mu(x))\ \textit{for}\ \text{any}(s_1',s_2')\in(l_0^+\setminus(s_1,s_2))$$

It follows, see also [2], *that we can, purely geometrically, define the notion of a branch and the nature of singularities (Cayley's definition) using birationality arguments, see* [1]. *For a plane curve* C, *we can define a nonsingular point* x *to be an inflexion if* $I(C,l_x)=3$, *where* l_x *is the tangent line.*

Lemma 0.8. *Let* C *be a real projective algebraic curve in the sense of Definition 0.7, defined by a polynomial* $F(x,y)$. *Let* $p=(x_0,y_0)$ *be a nonsingular point of* C, *with the property that* $\frac{\partial F}{\partial x}|_{(x_0,y_0)}\neq 0$. *Without loss of generality, assume that* p *is located at the origin* $(0,0)$, *then if* $(t,y(t))$ *is a power series representation of* C *at* p, *such that* $y(0)=0$, *then, if* p *satisfies the definition of an inflexion in Remarks 0.7, property* (i) *for an inflexion in Definition 0.3 holds, that is* $y''(0)=0$.

Proof. Let l_p be the tangent line to C, at p. Then l_p is defined by the equation $F_x|_{(0,0)}x+F_y|_{(0,0)}y=0$ or $y'|_{(0)}x-y=0$. Consider the family of lines defined by $\{l(x,y,s) : y = (y'(0)+s)x, s \in \mathcal{R}\}$, so $l(x,y,0) = l_p$. Then, we have $l(t,y(t),s) = 0$ iff $y(t) = (y'(0)+s)t$, ([6])

Using the methods of [1] and Remarks 0.7, one can show that the intersection multiplicity $I(C, l_p, (0,0))$ is given by the "real geometric multiplicity" of the cover $\mathcal{R}[x,y,s]/ < F(x,y), l(x,y,s) >$, $(**)$. We adapt the definition of "real geometric multiplicity" in line with (†) of Remarks 0.7, and, use methods from [3] and [1], to show that it is sufficient to vary the line l_p by rotating it about $(0,0)$. Using the method of [4], one can compute this multiplicity in $(**)$ as the multiplicity of $\mathcal{R}[[x]][s]/ < y(x) - (y'(0)+s)x >= \mathcal{R}[[x]][s]/ < x(h(x) - s) >$, $(***)$ where $h(x) = \frac{y(x)}{x} - y'(0) = y''(0)x + o(x^2)$. ([7]). This is a reducible cover, with multiplicity $b = m+1$, where m is the "real geometric multiplicity" of $\mathcal{R}[[x]][s]/ < (h(x)-s) >$, $(****)$. If $y''(0) \neq 0$, (††), then we compute the multiplicity m of $\mathcal{R}[[x]][s]/ < xu(x) - s >$, where $u(x) = y''(0) + o(x)$ is a unit in $\mathcal{R}[[x]]$. Using the method of [4], we can factor this is as $\mathcal{R}[s] \rightarrow_k \mathcal{R}[x,s]/ < (x-s) > \rightarrow_g \mathcal{R}[x,s]/ < (xu(x) - s) >$, where g is etale and k clearly has multiplicity $m = 1$ Hence, $b = m+1 = 2$, contradicting the assumption. Therefore, (††) fails, and $y''(0) = 0$ as required.

□

Lemma 0.9. *Let C be a nonsingular real plane algebraic curve, defined by $G(x,y)$, with $p = (0,0) \in C$, and $\frac{\partial G}{\partial x}|_{(0,0)} \neq 0$, $\frac{\partial G}{\partial y}|_{(0,0)} = 0$. Let O be the point $(-a,0)$, with $a > 0$, and assume that $O \notin C$. Clearly, the tangent line l_p, $(x=0)$, of C at p passes through O. Let $y(t)$ be an analytic power series, with $y(0) = 0$, such that $G(t,y(t)) = 0$, and let $\gamma : \mathcal{R} \rightarrow \mathcal{R}^2$ be defined by $\gamma(t) = (t, y(t))$. Let $p_\phi(t) = tan^{-1}(\frac{y(t)}{a+t})$ measure the angle ϕ of the position of γ at time t, and let $v_\psi(t) = tan^{-1}(y'(t))$ measure the angle of the velocity ψ of γ at time t. Then, if $\frac{dv_\psi}{dt}|_{(0)} = 0$ and $\frac{d^2 v_\psi}{dt^2}|_{(0)} \neq 0$, we have that $\frac{dp_\phi}{dt}|_{(0)} = \frac{d^2 p_\phi}{dt^2}|_{(0)} = 0$, and $\frac{d^3 p_\phi}{dt^3}|_{(0)} \neq 0$. In particular, $(0,0)$ is an inflexion in the sense of Remarks 0.7.*

Proof. As O lies on l_p, we have that $p_\phi(0) = v_\psi(0) = 0$, (†). Moreover, as $y(t) = tan(p_\phi)(a+t)$, we have that;

[6]Considering $l_p(x,y) : C \rightarrow \mathcal{R}$, and the analytic power series $g(t) : \mathcal{R} \rightarrow R$, defined by $l_p(t,y(t)) = y'(0)t - y(t)$, we have that $g(0) = 0$, $\frac{dg}{dt}|_{(0)} = y'(0) - y'(0) = 0$ and $\frac{d^2 g}{dt^2}|_{(0)} = y''(0)$, $(*)$. The condition that $ord_{(0)} g \geq 3$ is given by $(*)$.

[7]Namely, show that $\mathcal{R}[[x]][y][s]/ < y - y(x), y - (y'(0)+s)x >=$ is an etale cover of $\mathcal{R}[[x]][s]/ < y(x) - (y'(0)+s)x >$, and show that multiplicity is preserved

$tan(v_\psi(t)) = y'(t)$

$= [tan(p_\phi)(a+t)]'$

$= tan(p_\phi(t)) + (a+t)(1+tan^2(p_\phi(t)))p'_\phi(t)$ $(*)$

Let $S(z)$ be the analytic power series expansion of $tan(z)$, $S(z) = zd(z)$, with $d(0) = f \neq 0$, ([8]). Let $\{a(t), b(t)\}$ be the power series expansion of $\{v_\psi(t), p_\phi(t)\}$, By (†) and the assumptions $\frac{dv_\psi}{dt}|_{(0)} = 0$ and $\frac{d^2v_\psi}{dt^2}|_{(0)} \neq 0$, we have that $a(t) = t^2(c + w(t))$, where $c = \frac{d^2v_\psi}{dt^2}|_{(0)} \neq 0$, and $w(0) = 0$. Let $ord_t b(t) = m$, then;

$ord_t(tan(b(t)) + tan(a+t)(1+tan^2(b(t)))b'(t)) = m-1$

$tan(a(t)) = cft^2 + o(t^3)$, so $ord_t(tan(a(t)) = 2$

Therefore, $m = 3$, and the lemma is shown. It follows that $p_\phi(t)$ is an odd function, with $p_\phi(0) = (0,0)$. Letting $p_\phi(t) = b(t) = t^3u(t)$, with $u(0) \neq 0$, ..we can find an analytic function.. $t : [-\epsilon, \epsilon] \rightarrow \mathcal{R}$ with $p_\phi(t(p_\phi)) = p_\phi$, and $p_\phi(0) = 0$.. Let $\Gamma : [-\epsilon, \epsilon] \rightarrow \mathcal{R}$ be defined by $\Gamma(p_\phi) = \frac{-y(t)}{a+t}|_{t(p_\phi)}$. Then , assuming that $p_\phi|_{[0,\epsilon)} \geq 0$, and the fact that as $x(t) - a$ is odd, we have Γ is a positive even analytic function with $\Gamma(0) = 0$. Then, choosing $\delta > 0$, and $\epsilon_1 < 0 < \epsilon_2$, with $\Gamma(\epsilon_1) = \Gamma(\epsilon_2) = \delta$, we have, letting $p_1 = (t(\epsilon_1), y(t(\epsilon_1)))$, $p_2 = (t(\epsilon_2), y(t(\epsilon_2)))$, that the line $l_{(0,0),p_1} = l_{(0,0),p_2}$ passes through $\{(0,0), p_1, p_2\}$, hence $I_{it}(C, l_p) = 3$. □

Lemma 0.10. *Let C satisfy the conditions of Lemma 0.9. Then, if;*

(i). $y''|_{t=0} = 0$.

(ii). $y'''|_{t=0} \neq 0$.

$(0,0)$ is an inflexion in the sense of Remarks 0.7. Conversely, if $(0,0)$ is an inflexion in the sense of Remarks 0.7, and $y'''(0) \neq 0$, then conditions (i) and (ii) hold.

Proof. With notation as in 0.9, we have that;

[8] $S(z) = \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(-1)^k 4^k (1-4^k) x^{2k-1}$, where, for $k \in \mathcal{N}$, B_{2k} denotes the $2k$'th Bernouilli number.

$\frac{dv_\psi}{dt}|_{t=0} = \frac{d}{dt}|_{t=0}(tan^{-1}(y'(t)) = \frac{y''}{1+(y')^2}|_{t=0} = y''(0) = 0$

$\frac{d^2v_\psi}{dt^2}|_{t=0} = \frac{y'''(1+(y')^2)-2y'(y'')^2}{[1+(y')^2]^2}|_{t=0} = y'''|_{t=0} \neq 0$

Hence, by Lemma 0.9, $(0,0)$ is an inflexion in the sense of Remarks 0.7. Conversely, by Lemma 0.8, we have, if $(0,0)$ is an inflexion in the sense of Remarks 0.7, then condition (i), $y''(0) = 0$ holds. If, in addition $y'''(0) \neq 0$, then, clearly, condition (ii) holds as well.

□

Lemma 0.11. *Let C be a nonsingular real plane algebraic curve, defined by $G(x,y)$, with $p = (x_0,y_0) \in C$, and $\frac{\partial G}{\partial x}|_{(x_0,y_0)} \neq 0$. Let $\{x(t),y(t)\}$ be an analytic power series, with $y(0) = x_0$, and $x(t) = x_0 + t$, such that $G(x(t),y(t)) = 0$, then, if;*

(i). $y''|_{t=0} = 0$.

(ii). $y'''|_{t=0} \neq 0$.

(x_0,y_0) is an inflexion in the sense of Remarks 0.7. Conversely, if (x_0,y_0) is an inflexion in the sense of Remarks 0.7, and, $y'''|_{t=0} \neq 0$, then conditions (i) and (ii) hold.

Proof. Let $x^1 = x - x_0$, $y^1 = y - y_0$ be new coordinates, obtained by translating the point (x_0,y_0) to $(0,0)$. Let $x^1(t) = (x_0+t) - x_0 = t$, $y^1(t) = y(t) - y_0$ be new analytic power series with $(x^1(0), y^1(0)) = (0,0)$, parametrising $G(x^1+x_0, y^1+y_0) = 0$, in the new coordinates. The tangent line $l_{(0,0)}$ to C in the new coordinates, is given by $y = (y^1)'(0)x = y'(0)x$. Let $\phi = tan^{-1}(y'(0))$, be the angle of $l_{(0,0)}$, and let $\Gamma_{-\phi}$ be the rotation of $-\Phi$ about $(0,0)$, given by;

$$\Gamma_{-\phi} = \begin{pmatrix} cos(\phi) & sin(\phi) \\ -sin(\phi) & cos(\phi) \end{pmatrix}$$

Let;

$x^2 = cos(\phi)x^1 + sin(\phi)y^1$

$= cos(tan^{-1}(y'(0)))(x - x_0) + sin(tan^{-1}(y'(0)))(y - y_0)$

$y^2 = -sin(\phi)x^1 + cos(\phi)y^1$

$= -sin(tan^{-1}(y'(0)))(x - x_0) + cos(tan^{-1}(y'(0)))(y - y_0)$

be the new coordinates, obtained after rotating by $\Gamma_{-\phi}$. Clearly the tangent line $l_{(0,0)}$ to C in the coordinates (x^2, y^2) is then given by $y^2 = 0$. Using the identities $cos(tan^{-1}(\phi)) = \frac{1}{(1+\phi^2)^{\frac{1}{2}}}$, $sin(tan^{-1}(\phi)) = \frac{\phi}{(1+\phi^2)^{\frac{1}{2}}}$, we let;

$$x^2(t) = \frac{t+y'(0)(y(t)-y_0)}{(1+(y'(0))^2)^{\frac{1}{2}}}$$
$$y^2(t) = -\frac{(y(t)-y_0)-y'(0)t}{(1+(y'(0))^2)^{\frac{1}{2}}}$$

be new analytic power series with $(x^2(0), y^2(0)) = (0, 0)$, parametrising $G(\Gamma_\phi(x^2, y^2)+(x_0, y_0)) = 0$, in the new coordinates. Observe that C in the new coordinates satisfies the conditions of Lemma 0.9. Moreover;

$$(y^2)''|_{t=0} = -\frac{y''(0)}{(1+(y'(0))^2)^{\frac{1}{2}}}$$
$$(y^2)'''|_{t=0} = -\frac{y'''(0)}{(1+(y'(0))^2)^{\frac{1}{2}}}$$

Hence, if conditions $(i), (ii)$ are satisfied, then $(y^2)''(0) = 0$ and $(y^2)'''(0) \neq 0$. By Lemma 0.10, we obtain $(0, 0)$ of C in the new coordinates (x^2, y^2) is an inflexion in the sense of Remarks 0.7. It is then elementary to see that (x_0, y_0) is also an inflexion in this sense, as $I(C, l, p)$ is preserved by translations and rotations, $(*)$. Conversely, if $(0, 0)$ is an inflexion in the sense of Remarks 0.7, and $y'''(0) \neq 0$, then conditions (i) and (ii) hold. Conversely, if (x_0, y_0) is an inflexion in the sense of Remarks 0.7, then, by $(*)$, we have that $(0, 0)$ is an inflexion of C in the new coordinates (x^2, y^2). By Lemma 0.10, we have that $(y^2)''|_{t=0}$, hence, $y''(0) = 0$. As, by assumption, $y'''(0) \neq 0$, conditions (i) and (ii) hold. □

Lemma 0.12. *Let C satisfy the conditions of Lemma 0.9, with the additional property that there exists $K > 0$, such that $y'|_{(0,K)}$ and $y''_{(0,K)} > 0$. Then, if $x_0 \in (0, K)$, we have that the tangent line $l_{y(x_0)}$ to C, intersects the line $y = 0$ at $x_3 > 0$. In particular, if $u > 0$, and $l_{u,v}$ denotes the line $y = u + vx$, we have that, if $(x_0, y(x_0)) \in (l_{u,v} \cap C)$, then $y'(x_0) > v$.*

Proof. Let $x_1 = \mu_x(x > 0 \wedge (x, y(x)) \in l_{O,x_0}$, $(*)$, where $l_{O,x_0} = l_{(0,0),(x_0,y(x_0))}$, then $0 < x_1 \leq x_0$. The line l_{O,x_0} intersects C at $\{0, x_1\}$, hence, if $z(x) = l_{x_0}(x) - y(x)$, we have that $z(0) = z(x_1) = 0$. By Rolle's Theorem, there exists $0 < x_2 < x_1 \leq x_0$, such that $z'(x_2) = 0$,

$y'(x_2) = l'_{x_0}(x_2) = \frac{y(x_0)}{x_0}$. Then, as $y'|_{[x_2,x_0]}$ is increasing, we have that $y'(x_0) > \frac{y(x_0)}{x_0}$. It follows immediately, as $O \in l_{O,x_0}$, that the tangent line l_{x_0} to C, at x_0, intersects the line $y = 0$ at $x_3 > 0$. Moreover, if $l_{u,v}$ passes through $(x_0, y(x_0))$, then, as $u > 0$, we have that $\delta < \frac{y(x_0)}{x_0 - x_3}$, hence, $y'(x_0) > v$. □

Lemma 0.13. *Let C satisfy the conditions of Lemma 0.11, and let $\{x(t), y(t)\}$ be analytic power series parametrising C at $p = (x_0, y_0)$, with $x(t) = x_0 + t$. Then, $I(C, l_p, p) \leq 3$, where l_p is the tangent line.*

Proof. Suppose that $I(C, l_p, p) \geq 4$. Using the remarks at the end of Lemma 0.11, we can assume that $(0,0) \in C$, l_p is the tangent line $y = 0$, $x(t) = t$ and $y(0) = 0$, $y'(0) = 0$. By definition, we can find $l_{\epsilon,\lambda\epsilon}$, (denoting the line $y = \lambda\epsilon + \epsilon x$), with $(\epsilon, \lambda\epsilon) \in \mu(0,0)$ and $\lambda \in \mathcal{R}$, and, $\{x_i : 1 \leq i \leq 4\} \subset \mu(0)$ distinct, with $\{(x_i, y(x_i)) : 1 \leq i \leq 4\} \subset (C \cap l_{\epsilon,\lambda\epsilon} \cap \mu(O))$. Without loss of generality, we can assume that there exists $K > 0$, with $y''|_{(0,K)} > 0$, then, for $0 < x < K$, we have that $y'(x) = \int_0^x y''(s)ds > 0$, $y(x) = \int_0^x y'(s)ds > 0$, hence, $y|_{(0,K)} > 0$ and $y'|_{(0,K)} > 0$. Assuming $\epsilon > 0$, $\lambda > 0$ and $0 < x_2 < x_3$. Transferring the statement about infinitesimals, we obtain, given $\{C, E\} \subset \mathcal{R}_{>0}$, that;

$$\mathcal{R} \models \exists x \exists w \exists z [(0 < x < C) \wedge (0 < z < E) \wedge (0 < x < w < C) \wedge \{(x, y(x)), (w, y(w))\} \subset (C \cap l_{z,\lambda z})]$$

Choosing $C < K$, we obtain points $0 < a_0 < a_1 < K$, and $\epsilon > 0$, with $\{(a_0, y(a_0)), (a_1, y(a_1))\} \subset (C \cap l_{\epsilon,\lambda\epsilon})$. By Lemma 0.12, we obtain that $y'(a_0) > \lambda\epsilon$, and, $y'|_{[a_0,a_1]} > \lambda\epsilon$, as $y'|_{[a_0,a_1]}$ is increasing, but, by Rolle's Theorem, there exists $a_2 \in (a_0, a_1)$, with $z'(a_2) = 0$, where $z(x) = l_{\epsilon,\lambda\epsilon}(x) - y(x)$, that is $y'(a_2) = \lambda\epsilon$, a contradiction. If $\epsilon < 0$, $\lambda < 0$, we have that $x_2 > 0$, and $x_2 = \frac{y(x_2)+\epsilon}{\lambda\epsilon} = \frac{1}{\lambda} + \frac{y(x_2)}{\lambda\epsilon} > \frac{1}{\lambda}$, contradicting the fact that x_2 is infinitesimal. The other cases $\epsilon > 0$, $\lambda < 0$, and $\epsilon < 0$, $\lambda > 0$ are easier, and left to the reader.

□

Lemma 0.14. *Let C satisfy the conditions of Lemma 0.9, and let $\{x(t), y(t)\}$ be analytic power series parametrising C at $p = (0,0)$, with $x(t) = t$. Then, p is an inflexion in the sense of Lemma 0.7 iff;*

$$ord_t(y(t)) = 2m + 1.$$

where $m \in \mathcal{N}$.

Proof. We divide the proof into cases. Observe that if $ord_t(y(t)) = 1$, then $y(t) = tu(t)$, with $u(0) \neq 0$. Then $y'(0) = u(0) + 0u'(0) \neq 0$, contradicting the assumption that the tangent line l_O is the line $y = 0$.

Case 1. $ord_t(y(t)) = 2m + 1$, where $m \in \mathcal{N}$.

Let $y(t) = t^{2m+1}u(t)$, with $u(0) \neq 0$, and $m \geq 1$. Choose $\epsilon > 0$ infinitesimal. We have that $y(\epsilon) = \epsilon^{2m+1}u(\epsilon)$, and $l_{0,\epsilon}$ is given by the equation $y = \epsilon^{2m}u(\epsilon)x$. Without loss of generality, assume $\epsilon > 0$, and $u(\epsilon) > 0$. Let $u(\epsilon) = c \notin \mu(0)$. Similarly, $u(\epsilon)^{\frac{1}{2m}} = c^{\frac{1}{2m}} \notin \mu(0)$ (positive root). Let $g(t) = u(t)^{\frac{1}{2m}}$ be a positive analytic root of $u(t)$. We claim that there exists a solution to $tg(t) = -\epsilon u(\epsilon)^{\frac{1}{2m}} = -\epsilon c^{\frac{1}{2m}}$, for $t_0 \in \mu(0)$, $t_0 < 0$, $(*)$, in which case, $(t_0, y(t_0))$ is a solution to $C \cap l_{0,\epsilon}$. Using Lemma 0.13, we then obtain $I(C, l_p, p) = 3$ as required. Observing that $\delta = -\epsilon c^{\frac{1}{2m}} \in \mu(0)$, and $h(0) = ty(t)|_{t=0} = 0$, $(*)$ follows elementarily, by transfer, from the fact that there exists $K, L > 0$, with $max_{x\in[-K,0]}h'(x) \leq L$, $h'|_{[-K,0)} < 0$ and $h|_{[-K,0)} < 0$, so, for any $\epsilon > 0$, there exists $\frac{-\epsilon}{L} < x < 0$, with $h(x) = -\epsilon$.

Case 2. $ord_t y(t) = 2m$, where $m \in \mathcal{N}$.

Let $y(t) = t^{2m}u(t)$, with $u(0) \neq 0$. Wlog, we can assume that $u(0) > 0$. Suppose $I(C, l_p, p) = 3$, then there exists $(\epsilon, \lambda\epsilon) \in \mu(0, 0)$, we can assume that $\epsilon > 0$, and $\{x_1, x_2, x_3\} \subset \mu(0)$, with $\{(x_1, y(x_1)), (x_2, y(x_2)), (x_3, y(x_3))\} \subset C \cap l_{(\epsilon,\lambda\epsilon)} \cap \mu((0, 0))$. If $\lambda = 0$, then, we can assume either that $0 < x_2 < x_3$, $(*)$, or $x_2 < 0 < x_3$, $(**)$. If $(*)$ holds, then repeating the argument of Lemma 0.13, we obtain a contradiction. Observe that we can find $K > 0$, such that $y|_{(-K,K)} > 0$. If $(**)$ holds, then, by transfer, we can find $-K < a_2 < 0 < a_3 < K$, with $\{(a_2, a_2\epsilon), (a_3, a_3\epsilon)\} \subset C$, a contradiction. If $\lambda \neq 0$, (we can assume $(\lambda > 0)$, and then $(0, 0) \notin (C \cap l_p)$), we can then assume that $0 < x_2 < x_3$, $(*)$, holds again, obtaining a contradiction. It is, then, trivial to see that $I(C, l_p) = 2$, by choosing a line $l_{0,\epsilon}$, passing through O and $(\epsilon, y(\epsilon))$

□

Lemma 0.15. *Let C satisfy the conditions of Lemma 0.11, and let $\{x(t), y(t)\}$ be analytic power series parametrising C at $p = (x_0, y_0)$, with $x(t) = x_0 + t$ and $y(0) = y_0$. Then, p is an inflexion in the sense of Lemma 0.7 iff;*

$ord_t(y(t)-y_0-y'(0)t)=2m+1$.

where $m\in\mathcal{N}$.

Proof. Following the proof of Lemma 0.11, let $x^2(t)$ and $y^2(t)$ be the new analytic power series, parametrising C at $(0,0)$ in the new coordinates, after translating the axes by (x_0,y_0) and applying the rotation $\Gamma_{-\Phi}$, where $\Phi=tan^{-1}(\frac{y_0}{x_0})$. Then, as $y'(0)^2\geq 1$, we have that $ord_t x^2(t)=1$, hence, we can write $x^2(t)=tu(t)$ with $u(0)\neq 0$. By the inverse function theorem, we can find an analytic power series $\lambda(t)$, with $ord_t\lambda(t)=1$, such that $tx^2(\lambda(t))=t$. Clearly, $(t,y^2(\lambda(t)))$ parametrises the curve C at $(0,0)$ in the new coordinates (x^2,y^2). By Lemma 0.14, and the remarks at the end of Lemma 0.11, we have that p is an inflexion in the sense of Lemma 0.7 iff $(0,0)$ is an inflexion in the sense of Lemma 0.7 of C (in the new coordinates) iff $ord_t(y^2(\lambda(t)))=2m+1$, where $m\in\mathcal{N}$, iff $ord_t(y^2(t))=1$, iff $ord_t(y(t)-y_0-y'(0)t)=2m+1$, by the definition of $y^2(t)$.

□

Lemma 0.16. *Let C be a nonsingular real plane algebraic curve, defined by $G(x,y)$, with $p=(0,0)\in C$, and $\frac{\partial G}{\partial x}|_{(0,0)}\neq 0$, $\frac{\partial G}{\partial y}|_{(0,0)}=0$. Clearly, the tangent line l_p, $(y=0)$, of C at p passes through O. Let $y(t)$ be an analytic power series, with $y(0)=0$, such that $G(t,y(t))=0$, and let $\gamma:\mathcal{R}\rightarrow\mathcal{R}^2$ be defined by $\gamma(t)=(t,y(t))$. Let $v_\psi(t)=tan^{-1}(y'(t))$ measure the angle of the velocity ψ of γ at time t. Then, if $y''(0))=0$, $(0,0)$ is an inflexion of C in the sense of Remarks 0.7 iff $(0,0)$ is not an inflexion of the the curve C' defined by $y=v_\psi(x)$, and, $y''(0)\neq 0$ (not an inflexion) iff $\frac{dv_\psi}{dt}|_{t=0}\neq 0$.*

Proof. As $(0,0)$ lies on l_p, we have that $v_\psi(0)=0$, (†). Moreover, we have that;

$$tan(v_\psi(t))=y'(t)\ (*)$$

Let $S(z)$ be the analytic power series expansion of $tan(z)$, $S(z)=zd(z)$, with $d(0)=f\neq 0$, ([9]). If $(0,0)$ is an inflexion of C, in the sense of Remarks 0.7, then, by Lemma 0.14, we have that $ord_t(y(t))=2m+1$, where $m\in\mathcal{N}$. Then $ord_t(y'(t))=2m$, and, by $(*)$, $ord_t(v_\psi(t))=ord_t(tan(v_\psi(t)))=ord_t(y'(t))=2m$, so, again, by Lemma 0.14, as $v'_\psi(t)=0$, $(0,0)$ is not an inflexion of the curve C' defined by $y=$

[9] $S(z)=\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(-1)^k4^k(1-4^k)x^{2k-1}$, where, for $k\in\mathcal{N}$, B_{2k} denotes the $2k$'th Bernouilli number.

$v_\psi(x)$. Conversely, if $(0,0)$ is not an inflexion of C, in the sense of Remarks 0.7, then, by Lemma 0.14, we have that $ord_t(y(t)) = 2m$, where $m \in \mathcal{N}$. Then $ord_t(y'(t)) = 2m-1$, and, by $(*)$, $ord_t(v_\psi(t)) = ord_t(tan(v_\psi(t))) = ord_t(y'(t)) = 2m-1$, so, again, by Lemma 0.14, $(0,0)$ is not an inflexion of the the curve C' defined by $y = v_\psi(x)$, unless $m = 1$, in which case we obtain the final claim of the Lemma.

□

Lemma 0.17. *Let C satisfy the conditions of Lemma 0.11, and let $\{x(t), y(t)\}$ be analytic power series parametrising C at $p = (x_0, y_0)$, with $x(t) = x_0 + t$ and $y(0) = y_0$. Let $\gamma : \mathcal{R} \rightarrow \mathcal{R}^2$ be defined by $\gamma(t) = (x(t), y(t))$. Let $v_\psi(t) = tan^{-1}(y'(t)) - tan^{-1}(y'(0))$ measure the angle of the velocity ψ of γ at time t, relative to the tangent line l_p. Then, if $y''(0)) = 0$, $(0,0)$ is an inflexion of C in the sense of Remarks 0.7 iff $(0,0)$ is not an inflexion of the the curve C' defined by $y = v_\psi(x)$, and, $y''(0) \neq 0$ (not an inflexion) iff $\frac{dv_\psi}{dt}|_{t=0} \neq 0$.*

Proof. As $p = (x_0, y_0)$ lies on l_p, we have that $v_\psi(0) = 0$, $(\dagger)$. Let $y(t) = ty'(0) + y_2(t)$, then $y'(t) = y'(0) + y_2'(t)$, and;

$$tan(v_\psi(t)) = y_2'(t) \ (*)$$

Let $S(z)$ be the analytic power series expansion of $tan(z)$, $S(z) = zd(z)$, with $d(0) = f \neq 0$, ([10]). If $(0,0)$ is an inflexion of C, in the sense of Remarks 0.7, then, by Lemma 0.15, we have that $ord_t(y_2(t)) = 2m+1$, where $m \in \mathcal{N}$. Then $ord_t(y_2'(t)) = 2m$, and, by $(*)$, $(\dagger)$, $ord_t(v_\psi(t)) = ord_t(tan(v_\psi(t))) = ord_t(y_2'(t)) = 2m$, so, by Lemma 0.14, as $v_\psi'(t) = 0$, $(0,0)$ is not an inflexion of the curve C' defined by $y = v_\psi(x)$. Conversely, if $(0,0)$ is not an inflexion of C, in the sense of Remarks 0.7, then, by Lemma 0.15, we have that $ord_t(y_2(t)) = 2m$, where $m \in \mathcal{N}$. Then $ord_t(y_2'(t)) = 2m-1$, and, by $(*)$, $ord_t(v_\psi(t)) = ord_t(tan(v_\psi(t))) = ord_t(y'(t)) = 2m-1$, so, again, by Lemma 0.14, $(0,0)$ is not an inflexion of the the curve C' defined by $y = v_\psi(x)$, unless $m = 1$, in which case, again, we obtain the final claim.

□

Lemma 0.18. *Let γ be a nodal path, then the number r of inflexions is even.*

Proof. Let $0 \leq t_1 < t_2 < 1$, with the property that there does not exist t_3, with $t_1 < t_3 < t_2$ such that $\gamma(t_3)$ is an inflexion, and $\{\gamma(t_1), \gamma(t_2)\}$

[10] $S(z) = \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(-1)^k 4^k (1-4^k) x^{2k-1}$, where, for $k \in \mathcal{N}$, B_{2k} denotes the $2k$'th Bernouilli number.

are inflexions. Letting $\phi:[t_1,t_2]\rightarrow S^1$ be defined, as above, we have that, if $\phi(t)=(\phi_1(t),\phi_2(t))$, then $\phi'(t).\phi(t)=0$, so $\phi_1'\phi_1+\phi_2'\phi_2=0$ (1). If $\theta(t)=tan^{-1}(\frac{\phi_1(t)}{\phi_2(t)})$, then;

$\frac{d\theta}{dt}=\frac{\phi_1'\phi_2(t)-\phi_2'\phi_1(t))}{((\phi_1)^2(t)+(\phi_2)^2(t))}$ $(*)$

Suppose $t_1<t_3<t_2$ and $(\frac{d\theta}{dt})|_{t_3}=0$, then, we obtain, by $(*)$, that $(\phi_1'\phi_2(t)-\phi_2'\phi_1(t))|_{t_3}=0$, (2). Combing $(1),(2)$, we obtain $(\phi_1)^2(\frac{\phi_2'}{\phi_2})+(\phi_2)^2(\frac{\phi_2'}{\phi_2})|_{t_3}=(\frac{\phi_2'}{\phi_2})|_{t_3}=0$. Therefore, $\phi'(t_3)=(\phi_1'(t_3),\phi_2'(t_3))=0$. By Lemma 0.5, we would have that $\gamma(t_3)$ is an inflexion, hence, $(\frac{d\theta}{dt})|_{(t_1,t_2)}\neq 0$. Again, by Lemma 0.5, we have $\phi'(t_2)=\phi'(t_3)=0$, $(\dagger)$, hence, by $(*)$, $\frac{d\theta}{dt}(t_2)=\frac{d\theta}{dt}(t_3)=0$. We claim that $\frac{d^2\theta}{dt^2}(t_2)\neq 0$, $(**)$. If $(**)$ fails, then, by $(*)$, we have that $\phi_1''\phi_2(t_2)-\phi_2''\phi_1(t_2)=0$, $(\dagger\dagger)$. By the Fundamental Theorem of Calculus, using the fact that $\phi'(t_2)=(\phi_1'(t_2),\phi_2'(t_2))=0$, by $(\dagger)$, we have;

$\phi'(t_2+\epsilon)=\int_{t_2}^{t_2+\epsilon}\phi''(t)dt$ $(\dagger\dagger)$

As γ is analytic, if $\phi''(t_2)\neq 0$, $(\dagger\dagger\dagger)$, then if $\alpha(t)=cos^{-1}(\frac{\phi'(t).l_{\phi(t)}}{|\phi'(t)|})$ measures the angle between the velocity vector $\phi'(t)$ and the tangent line $l_{\phi(t)}$ to S^1, we have, for sufficiently small ϵ, that $\alpha(t_2+\epsilon)\neq 0$, by $(\dagger\dagger)$ and the fact that, for $\beta(t)=cos^{-1}(\frac{\phi''(t).l_{\phi(t)}}{|\phi'(t)|})$, we have $\beta(t_2)=\frac{\pi}{2}\neq 0$. This clearly contradicts the fact that, for all t, $\alpha(t)=0$, as $\phi'(t)||l_{\phi(t)}$. Hence, $(\dagger\dagger\dagger)$ fails and $\phi''(t_2)=0$. By $(*)$ of 0.5, and $\phi'(t_2)=\phi''(t_2)=0$, we have that;

$$(r^3\phi_1')'|_{t_2}=(\gamma_1''r^2-(\gamma_1')^2\gamma_1''-\gamma_1'\gamma_2'\gamma_2'')'|_{t_2}$$

$$=(\gamma_1'''r^2+2\gamma_1''rr'-2\gamma_1'(\gamma_1'')^2-(\gamma_1')^2\gamma_1'''-\gamma_1''\gamma_2'\gamma_2''-\gamma_1'(\gamma_2'')^2-\gamma_1'\gamma_2'\gamma_2''')|_{t_2}=0$$

and, similarly;

$$(\gamma_2'''r^2+2\gamma_2''rr'-2\gamma_2'(\gamma_2'')^2-(\gamma_2')^2\gamma_2'''-\gamma_2''\gamma_1'\gamma_1''-\gamma_2'(\gamma_1'')^2-\gamma_2'\gamma_1'\gamma_1''')|_{t_2}=0$$

Using the fact, by Lemma 0.5, that $(\gamma_1''\gamma_2')|_{t_2}=(\gamma_2''\gamma_1')|_{t_2}$, we obtain that $(\gamma_1'''\gamma_2')|_{t_2}=(\gamma_1'\gamma_2''')|_{t_2}$, contradicting Definition 0.3. Hence, $(**)$ holds, that is $\frac{d^2\theta}{dt^2}(t_2)\neq 0$, and, similarly $\frac{d^2\theta}{dt^2}(t_3)\neq 0$. Enumerating the inflexions $\{i_1,\ldots,i_r\}$, with corresponding $\{t_1,\ldots,t_r\}$, we have, by definition of a maximum/minimum for θ, that the angle θ is increasing/decreasing in the intervals (t_i,t_{i+1}), and changes direction at each

t_i, for $1 \leq i \leq r$. If the number of inflexions were odd, then clearly θ would be both increasing and decreasing on each interval (t_i, t_{i+1}), implying that θ is constant. This clearly implies that γ is contained in a line l, with no inflexions. Otherwise, we obtain that the number of inflexions is even as required. $\square$

Definition 0.19. *Let $V_m = \{\nu_1, \ldots, \nu_m\}$, $m \geq 1$, indexed by the ordered set M, with $|M| = m$, be a set of nodes, $I_r = \{i_1, \ldots, i_r\}$, $r \geq 0$, indexed by the ordered set I, with $|I| = r$, be a set of inflexions. We let $W_{I_r}^{V_m} = \{S_v : 1 \leq v \leq \frac{(2m+r)!}{2}\}$ consist of the ordered sets of cardinality $2m + r$, that are made up of the inflexions and repeats of the nodes $\{\nu_j^i : 1 \leq i \leq 2, 1 \leq j \leq m\}$, with the single requirement that $\nu_j^1 < \nu_j^2$, for $1 \leq j \leq m$. If $r = 0$, and $S_v \in W_{I_0}^{V_m}$, we define $Loop(S_v) = \{\nu_j^1, \nu_j^2 : 1 \leq j \leq m, \{\nu_j^1, \nu_j^2\}$ occur in adjacent positions$\}$ Given such a set S_v, we define a sequence of sets $\{S_{v,z} : 1 \leq z \leq m\}$ inductively, by setting $S_{v,1} = S_v$, and, $S_{v,z+1} = (S_{v,z} \setminus Loop(S_{v,z}))$. We call S_v a source if $S_{v,m} = \emptyset$. If γ is a nodal path, with m nodes, $(m \geq 1)$, and r inflexions, then γ determines a set $S_{\gamma,v} \in W_{I_r}^{V_m}$, by ordering the times $\{t_k : 1 \leq k \leq (2m + r)\}$, for which $\gamma(t_k)$ is an inflexion or a node. We let $X_{I_r}^{V_m} = \{S_{\gamma,v} : \gamma$ a nodal path$\}$, $X_{I_r}^{V_m} \subset W_{I_r}^{V_m}$.*

Lemma 0.20. *Let γ be a nodal path with no inflexions. Then $S_{\gamma,v}$ is a source, and, conversely, every source $S_v \in W_{I_0}^{V_m}$ is realised by a nodal path γ with no inflexions. In particular, $|X_{I_0}^{V_m}| = 2^m$ and $|W_{I_0}^{V_m}| = \frac{(2m)!}{2}$, for $m \geq 2$, $|X_{I_0}^{V_1}| = |W_{I_0}^{V_m}| = 1$.*

Proof. We prove this by induction on m. The case $m = 1$ is clear, the path γ defined by;

$$x(t) = cos^3(\tfrac{2\pi t}{3})cos(2\pi t)$$

$$y(t) = cos^3(\tfrac{2\pi t}{3})sin(2\pi t)$$

for $t \in [0, 1)$, (Cayley's sextic), is a nodal path with no inflexions, the single node being located at $(-cos^3(\frac{\pi}{3}), 0)$.

Suppose the result is true for m. Let γ be a nodal path, with $m + 1$ nodes. Suppose $S_{\gamma,t}$ has an adjacent nodal pair $\{\nu_{j_0}^1, \nu_{j_0}^2\}$, (†) for some $1 \leq j_0 \leq m + 1$, and times $\{t_{j_0}^i : 1 \leq i \leq 2\}$ such that $\gamma(t_{j_0}^i) = \nu_{j_0}$. Choose $\epsilon > 0$, with $t_{j_0'}^{i_0'} < t_{j_0}^1 - \epsilon < t_{j_0}^1 < t_{j_0}^2 < t_{j_0}^2 + \epsilon < t_{j_0''}^{i_0''} < 1$, where $\{\nu_{j_0'}, \nu_{j_0''}\}$ are adjacent nodes to ν_{j_0}. Define a nodal path γ_1, by setting $\gamma_1|_{[0,t_{j_0}^1-\epsilon)} = \gamma|_{[0,t_{j_0}^1-\epsilon)}$, $\gamma_1|_{[t_{j_0}^2+\epsilon,1)} = \gamma|_{[t_{j_0}^2+\epsilon,1)}$, and $\gamma_1 : |_{[t_{j_0}^1-\epsilon,t_{j_0}^2+\epsilon)} = \gamma_2 :$

$|_{[t_{j_0}^1-\epsilon, t_{j_0}^2+\epsilon)}$, where $\gamma_2 : [t_{j_0}^1-\epsilon, t_{j_0}^2+\epsilon) \rightarrow \mathcal{R}^2$ is a path with the property that $(\gamma \cap \gamma_2) = \emptyset$, γ_2 has no nodes or inflexions, satisfies properties (i) and (ii), and the concatenated path $\gamma_1|_{[0,t_{j_0}^1-\epsilon)} \cdot \gamma_2 \cdot \gamma_1|_{[t_{j_0}^2+\epsilon,1)}$ is analytic at the points $\{t_{j_0}^1-\epsilon, t_{j_0}^2+\epsilon\}$. Then γ_1 is a nodal path, based on the set of nodes $V_m = (V_{m+1} \setminus \{v_{j_0}\})$. By induction $S_{\gamma_1,v}$ is a source. Hence, by definition, so is $S_{\gamma,v}$. Hence, we can assume that (†) fails, and $S_{\gamma,v}$ has no adjacent nodal pair. Let $\{\nu_{j_0}^2, \nu_{j_1}^2\}$ be the final two elements of $S_{\gamma,v}$, with corresponding $\{t_{j_0}^2, t_{j_1}^2\} \subset [0,1)$, $j_0 \neq j_1$, (as (†) fails). Let $\gamma_3 = \gamma|_{[t_{j_0}^1, t_{j_0}^2]}$. considering the path $\gamma_4 = \gamma|_{(t_{j_0}^2,1]}$, we have that $(\gamma_3 \cap \gamma_4) = \emptyset$, as otherwise

□

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