INFINITESIMALS IN A RECURSIVELY ENUMERABLE PRIME MODEL

TRISTRAM DE PIRO

Abstract. Using methods developed by Robinson, we find a complete theory suitable for a first order description of infinitesimal neighborhoods. We use this to construct a specialisation having universal properties and to find a recursively enumerable model in which the algebraic version of Bezout's theorem is provable by non-standard methods.

1. Specialisations and Valuations

Let L and K be fields with an imbedding $i : L^* \rightarrow K^*$. In the case when L and K have the same characteristic, we will consider L as a subfield of K, otherwise we will by some abuse of notation refer to the embedded set $i(L^*) \cup \{0\}$ as L. Let $P(K) = \bigcup_{n\geq 1} P^n(K)$ and $P(L) = \bigcup_{n\geq 1} P^n(L)$. By a closed algebraic subvariety of $P^n(K)$, we mean a set $W(K)$ where W is defined by homogeneous polynomial equations with coefficients in K. We say that $W(K)$ is defined over L if we can take the coefficients to lie in L. Let $W_n^m(K)$ denote the $m'th$ Cartesian product of $P^n(K)$. By a closed algebraic subvariety of $W_n^m(K)$, we mean a set $W(K)$ defined by multi-homogeneous polynomial equations with coefficients in K, similarly we can make sense of the notion of being defined over L. Note that if K is not algebraically closed, it is not necessarily true that the projection maps $pr_{k,m} : W_n^k(K) \rightarrow W_n^m(K)$ preserve closed algebraic subvarieties.

Definition 1.1. *A specialisation is a map* $\pi = \bigcup_{n\geq 1} \pi_n : P(K) \rightarrow P(L)$*, such that each* $\pi_n : P^n(K) \rightarrow P^n(L)$ *has the following property;*

Let $W_n^m(K)$ *denote the m'th Cartesian product of* $P^n(K)$*. Then, if* $V \subset W_n^m(K)$ *is a closed algebraic subvariety defined over L and* $\bar{a}$ *is an m-tuple of elements from* $W_n(K)$*, such that* $V(\bar{a})$ *holds, then* $V(\pi_n(\bar{a}))$ *holds as well.*

The author was supported by a Marie Curie research fellowship.

The following compatibility requirement must also hold between the π_n*;*

Fix the following chain of embeddings i_n *of* $P^n(K)$ *and* $P^n(L)$ *into* $P^{n+1}(K)$ *and* $P^{n+1}(L)$ *for* $n \geq 1$.

$i_n : [x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_n : 0]$.

Then we require that $\pi_{n+1} \circ i_n = i_{n+1} \circ \pi_n$.

Definition 1.2. *A Krull valuation* v *is a map* $v : K \to \Gamma \cup \infty$ *where* Γ *is an ordered abelian group with the following properties;*

(i). $v(x) = \infty$ *iff* $x = 0$.
(ii). $v(xy) = v(x) + v(y)$
(iii). $v(x + y) \geq min\{v(x), v(y)\}$

Here, we adopt the convention that $\gamma < \infty$ *for* $\gamma \in \Gamma$ *and extend* $+$ *naturally to* $\Gamma \cup \infty$.

We let $\mathcal{O}_v = \{x \in K : v(x) \geq 0\}$ *be the valuation ring of* v *and* $\mathcal{M}_v = \{x \in K : v(x) > 0\}$ *the unique maximal ideal. We also require;*

(iv). The inclusion $i : L^* \cup 0 \to \mathcal{O}_v^* \cup 0$ *maps* L *isomorphically onto* $\mathcal{O}_v/\mathcal{M}_v$*, the residue field of* v.

Definition 1.3. *We say that two Krull valuations* v_1 *and* v_2 *are equivalent, denoted by* $v_1 \sim v_2$ *if* $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

Lemma 1.4. v_1 *and* v_2 *are equivalent iff there exists* $\Theta : \Gamma_1 \to \Gamma_2$ *such that* $\Theta \circ v_1 = v_2$.

In order to see this, define $\Theta(v_1(x)) = v_2(x)$, this is well defined as if $v_1(x) = v_1(x')$, then $v_1(x/x') = 0$, hence x/x' and x'/x belong to $\mathcal{O}_{v_1}$. If $v_1 \sim v_2$, then x/x' and x'/x belong to $\mathcal{O}_{v_2}$ as well, which gives that $v_2(x) = v_2(x')$. One can easily check that Θ is an isomorphism of ordered abelian groups as required.

Our main result in this section is the following;

Theorem 1.5. *Let* $X := \{\pi : P(K) \to P(L)\}$ *be the set of specialisations and* $Y := \{v/\sim : v : K \to \Gamma\}$ *be the set of equivalence classes of*

Krull valuations. Then there exists a natural bijection between X and Y. Specifically, there exists maps Φ and Ψ;

$\Phi : X \rightarrow Y$

$\Psi : Y \rightarrow X$

with $\Psi \circ \Phi = Id_X$ *and* $\Phi \circ \Psi = Id_Y$

We first show;

Theorem 1.6. *There exists* $\Psi : Y \rightarrow X$

Proof. Let $[v]$ denote a class of Krull valuations on K. We define a specialisation map $\pi_{[v]}$ as follows;

Let $(x_0 : x_1 : \ldots : x_n)$ denote an element of $P^n(K)$ written in homogeneous coordinates. For some $\lambda \in K$, the elements $\{\lambda x_0, \ldots, \lambda x_n\}$ will lie in $\mathcal{O}_v$ and not all of them will lie in $\mathcal{M}_v$. Let $\pi : \mathcal{O}_v \rightarrow L$ denote the unique ring morphism such that $\pi \circ i = Id_L$ where i is the inclusion map from L into $\mathcal{O}_v$. Then $(\pi(\lambda x_0) : \pi(\lambda x_1) : \ldots : \pi(\lambda x_n))$ defines an element of $P^n(L)$. As is easily checked, the mapping is independent of the choice of λ and depends only on $\mathcal{O}_v$, hence we obtain $\pi_{n,[v]} : P^n(K) \rightarrow P^n(L)$. We need to check that each $\pi_{n,[v]}$ satisfies the property required of a specialisation. We will just verify this in the case when $m \leq 2$ for each $n \geq 1$, the other cases are straightforward generalisations;

For $m = 1$, let $V \subset P^n(K)$ be a closed subvariety defined over L, then V is defined by a system of homogeneous equations in the variables $\{x_0, \ldots, x_n\}$ with coefficients in L. Taking a tuple $\bar{a}$ belonging to V, we can assume that the elements $\{a_0, a_1, \ldots, a_n\}$ belong to $\mathcal{O}_v$. Now, using the fact that the residue map π is a ring homomorphism fixing L, the reduced elements $\{\pi(a_0), \pi(a_1), \ldots, \pi(a_n)\}$ also satisfy the same homogeneous equations as required.

For the case when $m = 2$, we use the Segre embedding which is defined by;

$Segre : P^n(K) \times P^n(K) \rightarrow P^{n(n+2)}(K)$

$$((x_0 : \ldots : x_n), (y_0 : \ldots : y_n)) \mapsto (x_0y_0 : \ldots : x_0y_n : x_1y_0 : \ldots : x_ny_n)$$

The following diagram is easily checked to commute:

$$\begin{array}{ccc} P^n(K) \times P^n(K) & \xrightarrow{Segre} & P^{n(n+2)}(K) \\ \downarrow{\scriptstyle \pi_{n,[v]} \times \pi_{n,[v]}} & & \downarrow{\scriptstyle \pi_{n(n+2),[v]}} \\ P^n(L) \times P^n(L) & \xrightarrow{Segre} & P^{n(n+2)}(L) \end{array}$$

.

Therefore, it is sufficient to prove that the property holds for $\pi_{n(n+2),[v]} : P^{n(n+2)}(K) \rightarrow P^{n(n+2)}(L)$ when $m = 1$.This is the case covered above.

Finally, we need to check the compatibility requirement for the $\pi_{n,[v]}$, this is a trivial calculation.

Denote the specialisation map we have obtained by $\pi_{[v]}$ and let $\Psi([v]) = \pi_{[v]}$.

□

We now show;

Theorem 1.7. *There exists* $\Phi : X \rightarrow Y$

Proof. Suppose that we are given a specialisation π. In particular we have a map $\pi_1 : P^1(K) \rightarrow P^1(L)$ satisfying the requirements above. We want to show how to recover a Krull valuation on K.

Let $\gamma : K \rightarrow P^1(K)$ be the map $\gamma : k \mapsto [k : 1]$, so $\pi_1 \circ \gamma : K \rightarrow P^1(L)$. Let $U \subset P^1(L)$ be the open subset defined by $P^1 \setminus [1 : 0]$. Let $\mathcal{O}_K = (\pi_1 \circ \gamma)^{-1}(U)$ and $\mathcal{M}_K = (\pi_1 \circ \gamma)^{-1}([0 : 1])$. We now claim the following;

Lemma 1.8. *$\mathcal{O}_K$ is a subring of K with $Frac(\mathcal{O}_K) = K$ and $\mathcal{M}_K$ is an ideal of $\mathcal{O}_K$.*

Proof. Suppose that $\{x, y\} \subset \mathcal{O}_K$, then both $\pi_1([x : 1])$ and $\pi_1([y : 1])$ are in U. Let $C \subset P^1(K) \times P^1(K) \times P^1(K)$ be the closed set defined in coordinates $([u : v], [w : x], [y : z])$ by the equation $uwz = yvx$. As is easily checked, we have that $C([x : 1], [y : 1], [xy : 1])$. By the defining property of π_1, $C(\pi_1([x : 1]), \pi_1([y : 1]), \pi_1([xy : 1]))$ also holds. Therefore, $C([\lambda:1],[\mu:1],[\alpha,\beta])$ where λ,μ,α,β are in L. By definition of C, we have $\lambda\mu\beta=\alpha$ which forces $\beta\neq 0$. Hence, $\pi_1([xy:1])\in U$ and therefore $xy\in\mathcal{O}_K$. Let $D\subset P^1(K)\times P^1(K)\times P^1(K)$ be defined using the same choice of coordinates by the equation $uxz+wvz=yvx$. Then we have that $D([x:1],[y:1],[x+y:1])$ and therefore $D(\pi_1([x:1]),\pi_1([y:1]),\pi_1([x+y:1]))$. Again, we must have $D([\lambda:1],[\mu,1],[\delta,\epsilon])$ where $\lambda,\mu,\delta,\epsilon$ are in L. This forces $(\lambda+\mu)\epsilon=\delta$ and therefore $\epsilon\neq 0$, so $x+y\in\mathcal{O}_K$. Clearly, $1\in\mathcal{O}_K$ which shows that $\mathcal{O}_K$ is a subring of K as required. In order to see that $\mathcal{M}_K$ is an ideal of $\mathcal{O}_K$, let $x\in\mathcal{O}_K$ and $y\in\mathcal{M}_K$. We have that $C([\lambda:1],[0:1],[\alpha,\beta])$ where $\pi_1([xy:1])=[\alpha,\beta]$. Then $\lambda.0.\beta=1.1.\alpha$ forcing $\alpha=0$ and $\beta=1$, so $xy\in\mathcal{M}_K$. If $x\in\mathcal{M}_K$ and $y\in\mathcal{M}_K$ we obtain $D([0:1],[0:1],[\delta,\epsilon])$ where $\pi_1([x+y:1]=[\delta,\epsilon]$. Then $0.1.\beta+1.0.\epsilon=1.1.\delta$, so $\delta=0$ and $\epsilon=1$, hence $x+y\in\mathcal{M}_K$ as required. Finally, we show that $Frac(\mathcal{O}_K)=K$. Suppose $x\notin\mathcal{O}_K$, then $\pi_1([x:1])=[1:0]$. We have that $C([x:1],[1/x:1],[1:1])$, hence $C([1:0],[\alpha,\beta],[1:1])$ where $\pi_1([1/x:1])=[\alpha,\beta]$. This forces $1.\alpha.1=0.\beta.1$, hence $\alpha=0$ and $\beta=1$. Therefore $1/x\in\mathcal{O}_K$ as required. □

We now further claim the following;

Lemma 1.9. *If π_1 is non-trivial, that is π_1 is not a bijection between $P^1(K)$ and $P^1(L)$, then $\mathcal{O}_K$ is a proper subring of K*

Proof. By the same argument as above we have that $\pi_1\circ\gamma(1/\mathcal{M}_K)=[1:0]$, hence *if* $\mathcal{O}_K=K$, using the previous lemma, we must have that $\mathcal{M}_K=0$. If π_1 is non-trivial, we can find $x\in K$ and $y\in K$ distinct such that $\pi_1([x:1])=\pi_1([y:1])$. By the usual arguments, we then have that $\pi_1([x-y:1])=[0:1]$, so $x-y\in\mathcal{M}_K$ contradicting the fact that $\mathcal{M}_K=\{0\}$.

□

We can now construct a Krull valuation on K by a standard method. Let $\Gamma=K^*/\mathcal{O}_K^*$ and define $v:K\rightarrow\Gamma$ by $v(x)=x\ mod\ \mathcal{O}_K^*$ and $v(0)=\infty$. Define an ordering on the abelian group Γ by declaring $v(x)\leq v(y)$ iff $y/x\in\mathcal{O}_K$. This is well defined as if $v(x)=v(x')$ and $v(y)=v(y')$, then y'/y, y/y',x/x' and x'/x are all in $\mathcal{O}_K$. We have that $y'/x'=y/x.y'/y.x/x'$ and $y/x=y'/x'.y/y'.x'/x$, therefore $y'/x'\in\mathcal{O}_K$ iff $y/x\in\mathcal{O}_K$ as required. Transitivity of the ordering follows from the fact that $\mathcal{O}_K$ is a subring of K. $\leq$ is a linear ordering as if $x\in K^*$ and $y\in K^*$ then either x/y or y/x lies in $\mathcal{O}_K$. Finally, we clearly have that if $y/x\in\mathcal{O}_K$ then $yz/xz\in\mathcal{O}_K$, hence $v(x)\leq v(y)$ implies

$v(x)+v(z)\leq v(y)+v(z)$. This turns Γ into an ordered abelian group. Properties (i) and (ii) of the axioms for a Krull valuation are trivial to check. Suppose property (iii) fails, then we can find x,y with $v(x+y)<v(x)$ and $v(x+y)<v(y)$. Therefore $(x+y)/x\notin\mathcal{O}_K$ and $(x+y)/y\notin\mathcal{O}_K$. As $1\in\mathcal{O}_K$, we have that $x/y\notin\mathcal{O}_K$ and $y/x\notin\mathcal{O}_K$ which is a contradiction. Finally, we check property (iv). By definition of π_1, we have that $L^*\subset\mathcal{O}_K^*$, hence $v|L$ is trivial. If $k\in\mathcal{O}_K^*$, we can find $l\in L^*$ such that $\pi_1([k:1]=[l:1]$, then $\pi_1([k-l:1])=[0:1]$ and $k-l\in\mathcal{M}_K$. It follows that L maps onto $\mathcal{O}_K/\mathcal{M}_K$, and $\mathcal{O}_K/\mathcal{M}_K\cong L$ as required. Denote the valuation we have obtained by v_π and set $\Phi(\pi)=[v_\pi]$. This ends the proof of Theorem 1.7

□

We now complete the proof of Theorem 1.5;

Proof. $\Phi\circ\Psi=Id_Y$.

Let $[v]$ be a class of Krull valuations on K with corresponding specialisation $\pi_{[v]}$ provided by Ψ. Let $\pi_{1,[v]}$ be the restriction to $P^1(K)$. By definition, if $k\in\mathcal{O}_v$ then $\pi_{1,[v]}([k:1])=[\pi(k),1]$ where π is the residue map for v. If $k\notin\mathcal{O}_v$, then $\pi_{1,[v]}([k:1])=[0,1]$, so we see that $\mathcal{O}_K$ as defined above is exactly $\mathcal{O}_v$. The valuation $v_{\pi_{[v]}}$ constructed from $\pi_{[v]}$ therefore has the same valuation ring $\mathcal{O}_v$, so $v\sim v_{\pi_{[v]}}$ which gives the result.

$\Psi\circ\Phi=Id_X$.

Let π be a given specialisation and $[v_\pi]$ the corresponding class of Krull valuations. Let π_1 be the restriction of π to $P^1(K)$ and π_{1,v_π} the specialisation constructed from v_π restricted to $P^1(K)$. We have;

(i). $\pi_{1,v_\pi}([k:1])=[0:1]$ iff $v_\pi(k)>0$ iff $k\in\mathcal{M}_{v_\pi}$ iff $k\in\mathcal{M}_K$ as defined above iff $\pi_1([k:1])=[0:1]$

(ii). $\pi_{1,v_\pi}([k:1])=[1:0]$ iff $v_\pi(k)<0$ iff $k\notin\mathcal{O}_{v_\pi}$ iff $k\notin\mathcal{O}_K$ as defined above iff $\pi_1([k:1])\notin U$ iff $\pi_1([k:1])=[1:0]$

(iii). $\pi_{1,v_\pi}([1:0])=\pi_1([1:0])=[1:0]$ trivially.

If $k\in\mathcal{O}_{v_\pi}$, then $\pi_{1,v_\pi}([k:1])=[\alpha(k):1]$ where α is the residue mapping associated to v_π. We also have that $\pi_1([k:1])\in U$, hence as π_1 is a specialisation that $\pi_1([k:1])=[\beta(k):1]$ where β is a homomorphism from $\mathcal{O}_{v_\pi}$ to L. We thus obtain two homomorphisms $\alpha,\beta:\mathcal{O}_{v_\pi}\rightarrow L$ such that (by (i)) $Ker(\alpha)=Ker(\beta)=\mathcal{M}_{v_\pi}$ and with the property that $\alpha\circ i=\beta\circ i=Id_L$ where i is the natural inclusion of L in $\mathcal{O}_{v_\pi}$. We thus obtain the splitting $\mathcal{O}_{v_\pi}=L\oplus Ker(\alpha)=L\oplus Ker(\beta)=L\oplus M$ with $Ker(\alpha)=Ker(\beta)=M$. Now, using this fact, we can write any element of $\mathcal{O}_{v_\pi}$ uniquely in terms of L and M, hence the corresponding projections α and β are the same.

We have shown that $\pi_1=\pi_{1,v_\pi}$, it remains to check that $\pi_n=\pi_{n,v_\pi}$ for all $n\geq 1$. We prove this by induction on n, the case $n=1$ having been established.

By the induction hypothesis and the compatibility requirement between the π_n, for $\{k_0,k_1,\ldots,k_n\}\subset\mathcal{O}_{v_\pi}$;

$$\pi_{n+1}([k_0:k_1:\ldots:k_n:0])=[\pi(k_0):\pi(k_1):\ldots:\pi(k_n):0]\ (*)$$

where π is the residue map on $\mathcal{O}_{v_\pi}$.

Let $C\subset P^{n+1}(K)$ be the closed subvariety defined using coordinates $[x_0:x_1:\ldots:x_{n+1}]$ by the equations $x_0=x_1=\ldots=x_{n-1}=0$. Then by arguments as above and the fact that C is preserved by π_{n+1}, we can find a Krull valuation v' on K with corresponding residue mapping π' such that;

$$\pi_{n+1}([0:\ldots:0:1:k_{n+1}])=[0:\ldots:0:1:\pi'(k_{n+1})] \text{ if } v'(k_{n+1})\geq 0$$

$$=[0:\ldots:0:0:1] \text{ otherwise } (**)$$

Now let D be the closed subvariety of $P^{n+1}(K)$ defined by the equations $x_1=\ldots=x_n$ and $x_0=x_{n+1}$. Again, we have that π_{n+1} preserves D, hence there exists a Krull valuation v'' on K with corresponding residue mapping π'' such that;

$$\pi_{n+1}([k:1:\ldots:1:k])=[\pi''(k):1\ldots:1:\pi''(k)] \text{ if } v''(k)\geq 0$$

$$=[1:0:\ldots:0:1] \text{ otherwise } (***)$$

Let Sum be the closed subvariety of $P^{n+1}(K)\times P^{n+1}(K)\times P^{n+1}(K)$ defined using coordinates $[x_0:x_1:\ldots:x_{n+1}]$, $[y_0:y_1:\ldots:y_{n+1}]$ and $[z_0:z_1:\ldots:z_{n+1}]$ by the equations $x_0y_1z_1+y_0x_nz_1=z_0x_ny_1$

and $x_{n+1}y_1z_1+y_{n+1}x_nz_1=z_{n+1}x_ny_1$. Then, for $k\in K$, we have that $Sum([0:0:\ldots:1:k],[k:1:\ldots:0:0],[k:1:\ldots:1:k])$, hence by the properties of a specialisation that $Sum(\pi_{n+1}([0:0:\ldots:1:k]),\pi_{n+1}([k:1:\ldots:0:0]),\pi_{n+1}([k:1:\ldots:1:k]))$.

In the generic case when $v_\pi(k),v'(k),v''(k)$ are all non-negative, we obtain $Sum([0:0:\ldots:1:\pi'(k)],[\pi(k):1:\ldots:0:0],[\pi''(k):1:\ldots:1:\pi''(k)])$ which gives the relations $0.1.1+\pi(k).1.1=\pi''(k).1.1$ and $\pi'(k).1.1+0.1.1=\pi''(k).1.1$, so $\pi(k)=\pi'(k)=\pi''(k)$.

A simple calculation shows that $v_\pi(k)<0$ iff $v'(k)<0$ iff $v''(k)<0$, hence $\mathcal{O}_{v_\pi}=\mathcal{O}_{v'}=\mathcal{O}_{v''}$. We have now shown the following further compatibility between π_1 and π_{n+1}. Namely;

If $\gamma:P^1(K)\rightarrow P^{n+1}(K)$ is given by $\gamma:[x_0,x_1]\mapsto[0:0:\ldots:x_0:x_1]$ then $\pi_{n+1}\circ\gamma=\gamma\circ\pi_1$. (†)

Finally, let Sum' be the closed subvariety of $P^{n+1}(K)\times P^{n+1}(K)\times P^{n+1}(K)$ defined in coordinates $[x_0:\ldots:x_{n+1}],[y_0:\ldots:y_{n+1}],[z_0:\ldots:z_{n+1}]$ by the $(n+1)$ equations $x_jy_1z_1+y_jx_nz_1+z_jx_ny_1$ for $j\neq n$. Let $[k_0:\ldots:k_{n+1}]$ be an arbitrary element of $P^{n+1}(K)$. Without loss of generality, we may assume that $\{k_0:\ldots:k_{n+1}\}\subset\mathcal{O}_{v_\pi}$ and that $k_n\in\mathcal{O}^*_{v_\pi}$. Hence, dividing by k_n, the element is of the form $[k_0:\ldots:k_{n-1}:1:k_{n+1}]$ with $\{k_0,\ldots,k_{n-1},k_{n+1}\}\subset\mathcal{O}_{v_\pi}$. We have that $Sum'([0:\ldots:0:1:k_{n+1}],[k_0:\ldots:k_{n-1}:1:0],[k_0:\ldots:k_{n-1}:1:k_{n+1}])$, hence by specialisation and (†), $Sum'([0:\ldots:0:1:\pi(k_{n+1})],[\pi(k_0):\ldots:\pi(k_{n-1}):1:0],[l_0:\ldots:l_n:l_{n+1}])$. where $\{l_0,\ldots,l_{n+1}\}\subset L$. As is easily checked, the case when $l_n=0$ leads to a contradiction, hence we can assume that $l_n=1$ (multiplying by $1/l_n$). Now the equations give that $l_j=\pi(k_j)$ for $j\neq n$. We have therefore shown that $\pi_{n+1}=\pi_{n+1,v_\pi}$ as required.

Theorem 1.5 is now proved.

□

2. A Model Theoretic Language of Specialisations

We now introduce a model theoretic language which will enable us to describe specialisations in the context of algebraic geometry. In this section, we will assume that K and its residue field have the same characteristic. We will use a many sorted structure $\{\bigcup S_n:n\in\mathcal{N}\}$. Each sort will be the domain of $P^n(K)$ for an algebraically closed field

K. We fix an algebraically closed constant field L which we assume to be countable and let K be some non-trivial extension of L, having the same characteristic. In order to describe algebraic geometry, we introduce sets of predicates $\{V_n^m\}$ on the Cartesian powers S_n^m to describe closed algebraic subvarieties of $P^n(K)$ defined over L. In particular, we have constants to denote the individual elements of $P^n(L)$ on each sort S_n. We introduce function symbols $i_n : S_n \rightarrow S_{n+1}$ to describe the imbeddings $P^n(K) \rightarrow P^{n+1}(K)$ defined above. Finally, we will have symbols $\{\pi_n : n \in \mathcal{N}\}$ to describe the specialisation map $\pi = \cup_{n\geq 1}\pi_n$. Strictly speaking, as $P^n(L)$ is not definable, each π_n will be a union over $l \in P^n(L)$ of unary predicates defined as $\{x \in P^n(K) : \pi_n(x) = l\}$. We denote the language $< \{V_n^m\}, i_n, \pi_n >$ by $\mathcal{L}_{spec}$ and the theory of the structure $< P(K), P(L), \pi >$ in this language by T_{spec}. We denote the theory of the structure $< P(K), P(L) >$ in the language $\mathcal{L}_{spec} \setminus \{\pi_n\}$ by T_{alg}. Note that the structure $< K, 0, 1, +, . >$ is interpretable in the structure $< P(K), P(L) >$ in the language $\mathcal{L}_{spec} \setminus \{\pi_n\}$ $(*)$. This follows by noting that the points $[1 : 0], [0 : 1]$ and $[1 : 1]$ are named as elements in the sort S_1 and the operations of $+, .$ define algebraic subvarieties in the sorts S_1^3. The structure $< L, 0, 1, +, . >$ is not interpretable but any model of T_{alg} will contain an isomorphic copy of $P(L)$ as a substructure. It follows that the models of T_{alg} are exactly of the form $< P(K), P(L) >$ for some algebraically closed field K properly extending L (use the fact that the axiomatisation of $Th(< K, 0, 1, +, . >)$ can be interpreted in T_{alg} and the field structure can be related to the predicates $\{V_n^m\}$ using the imbeddings i_n). We now claim the following;

Theorem 2.1. *The theory T_{spec} is axiomatised by $T_{axioms} = T_{alg} \cup \Sigma$ where Σ is the set of sentences given by;*

(i). The mappings $\{\pi_n\}$ preserve the predicates $\{V_n^m\}$.

(ii). The compatibility requirement $\pi_{n+1} \circ i_n = i_{n+1} \circ \pi_n$ holds.

(see definition 1.1). In particular, T_{axioms} is complete. Moreover, T_{axioms} is model complete.

The proof of this theorem will be based on Theorem 1.5 and the following result by Robinson, given in [6];

Theorem 2.2. *Let K be an algebraically closed field with a non trivial Krull valuation v and residue field l. Then T_K is model complete in the language $\mathcal{L}_{val}$ and admits quantifier elimination in the language*

$\mathcal{L}_{rob}$. *Moreover, the completions of* K *are determined by the pair* $(char(l), char(K))$*, that is* $T_K \cup \Sigma$ *is complete where* Σ *is the possibly infinite set of sentences specifying the characteristic of* K *and* l*.*

Here, by the language $\mathcal{L}_{rob}$ we mean the language of algebraically closed fields together with a binary predicate $Div(x, y)$ denoting $v(x) \leq v(y)$. By the language $\mathcal{L}_{val}$, we mean a 2-sorted language for the value group and the field, with the usual language for the field sort and the language of ordered groups on the group sort. T_K is the theory which asserts that K is an algebraically closed field, the value group Γ is linearly ordered and abelian, the valuation is non-trivial. For our purposes, we will require a slightly refined version of this result. Namely, we will fix a set of constants for an algebraically closed field L which we can assume to be countable, add to T_K the atomic diagram of L, relativized to the field sort, the requirement that $v|L$ is trivial and π, the residue mapping, maps L injectively and homomorphically into the residue field. (Note, the condition that L maps onto the residue field is not definable and that the homomorphism requirement ensures that the residue field l and K have equal characteristic, hence the characteristic of K is already determined by the characteristic of L.) We will denote the corresponding theory by $T_{K,L}$ and the expanded languages by $\mathcal{L}_{rob}$ and $\mathcal{L}_{val}$ again. It is no more difficult to prove that $T_{K,L}$ is model complete, Robinson's original proof in [6] requires the solution of certain valuation equations in the model K given that these equations have a solutions in an extension K', it makes no difference if some of the elements from K are named. In order to show that $T_{K,L}$ is complete, it is sufficient to exhibit a prime model of the theory;

Case 1. $Char(K, L) = (p, p)$, with $p \neq 0$. Take $L(\epsilon)^{alg}$ where ϵ is transcendental over L, define the valuation on L to be zero and extend it to $L(\epsilon)$ non-trivially using say $v_{ord,\epsilon}$, the order valuation in ϵ. Take any extension to $L(\epsilon)^{alg}$.

Case 2. $Char(K, L) = (0, 0)$, define a similar valuation on $L(\epsilon)^{alg}$.

We now show the following lemma;

Lemma 2.1. *Amalgamation of Specialisations*

Let $(P(K_1), P(L), \pi_1)$ and $(P(K_2), P(L), \pi_2)$ be models of T_{axioms}, then there exists a further model $(P(K_3), P(L), \pi_3)$ such that;

$$(P(K_1), P(L), \pi_1) \leq (P(K_3), P(L), \pi_3)$$

and

$$(P(K_2), P(L), \pi_2) \leq (P(K_3), P(L), \pi_3)$$

Proof. By Theorem 1.5, we can find Krull valuations v_1 and v_2 on K_1 and K_2 such that $\pi_1 = \pi_{v_1}$ and $\pi_2 = \pi_{v_2}$. Using the refined version of Robinson's completeness result, we can jointly embed (K_1, v_1) and (K_2, v_2) over L into (K_3, v_3) (*). Let L' be the residue field of v_3, then as K_3 is algebraically closed, so is L' and extends the residue field L of v_1 and v_2. By standard results, we can construct a Krull valuation v on L' with residue field L, for example use the construction given in [2]. Using Theorem 1.5 again, we can construct specialisations $(P(K_3), P(L'), \pi_{v_3})$ and $(P(L'), P(L), \pi_v)$, the composition gives a specialisation $(P(K_3), P(L), \pi_3)$. It remains to see that in fact π_3 extends the specialisations π_1 and π_2. This follows from the fact that if $k \in K_1$ or $k \in K_2$ and there exists $l \in L$ such $v_1(k-l) > 0$ or $v_2(k-l) > 0$ then this relation is preserved in the embedding $(*)$. Hence the specialisation π_{v_3} already extends the specialisations π_1 and π_2 of $P(K_1)$ and $P(K_2)$ into $P(L)$. As the specialisation π_v fixes L, this proves the lemma.

□

Lemma 2.3. *Transfer of Formulas*

Let $(P(K), P(L), \pi)$ be a specialisation with corresponding (K, v), then there exists a map;

$$\sigma : P(K) \rightarrow K^{eq}$$

$$\sigma : \mathcal{L}_{spec}\text{-formulae} \rightarrow \mathcal{L}_{val}\text{-formulae}$$

such that for any $\phi(x_1, \ldots, x_n)$ which is a $\mathcal{L}_{spec}$-formula and $(k_1, \ldots, k_n) \subset P(K)$;

$$(P(K), P(L), \pi) \models \phi(k_1, \ldots, k_n) \text{ iff } (K, v) \models \sigma(\phi)(\sigma(k_1), \ldots, \sigma(k_n))$$

. (†)

Moreover, the definition of the map is uniform in K.

Proof. The map σ is defined on the sorts $P^n(K)$ by sending $[k_0, \ldots, k_n]$ to $(k_0, \ldots, k_n)/\sim_n$ where $\sim_n$ is the equivalence relation defined on K^{n+1} from multiplication by K^*. Similarly, σ maps a variable from the sort S_n to the corresponding variable from the sort in K^{eq} defined by $\sim_n$. A closed algebraic subvariety in $\{V_n^m\}$ is defined by a multi-homogeneous equation in the variables $\{(x_{01}, \ldots, x_{n1}), \ldots, (x_{0m}, \ldots, x_{nm})\}$. Let C_n^m be the algebraic variety in $K^{m(n+1)}$ defined by this equation. Then the corresponding formula in K^{eq} is given by;

$$(y_1, \ldots, y_m) \in (\sim_n)^m[\exists x_1 \ldots x_m(C_n^m(x_1, \ldots, x_m) \wedge \bigwedge_{i=1}^m x_i/\sim_n = y_i)]$$

For the inclusion maps i_n, let us identify each i_n with its graph, then clearly we can define σ to map the formula $i(x) = y$ to a corresponding formula relating the sorts $\sim_n$ and $\sim_{n+1}$ in K^{eq}.

Note that if $l \in P^n(L)$ is a constant, then $\sigma(l) = (l_0, \ldots, l_n)/\sim_n$ where each l_i is a constant from the atomic diagram of L.

Finally, let $\pi_n : P^n(K) \to P^n(L)$ be a specialisation. Again, let us assume that we can identify π_n with its graph. We then have that;

$$\pi_n([x_0 : \ldots : x_n]) = [l_0 : \ldots : l_n]$$

iff

$$\exists z \exists z_0 \ldots \exists z_n((\bigwedge_{i=0}^n x_i z = l_i + z_i) \wedge (\bigwedge_{i=0}^n v(z_i) > 0)).$$

It is now clear how to define $\sigma(\pi_n)$ as a union of formulas in the sort defined by $\sim_n$.

This completes the definition of σ, it is clear that the definition is uniform in K and a straightforward induction on the length of a formula from $\mathcal{L}_{spec}$ shows that it has the required property (†).

□

Theorem 2.1 is now a fairly straightforward consequence of the above lemmas. We first show model completeness. Suppose that we have models of T_{axioms};

$(P(K_1), P(L), \pi_1) \leq (P(K_2), P(L), \pi_2)$

By theorem 1.5, we can find Krull valuations v_1 and v_2 such that $(K_1, v_1) \leq (K_2, v_2)$ and $(K_1, v_1), (K_2, v_2) \models T_{K,L}$. By the refined model completeness result after Theorem 2.2, we have $(K_1, v_1) \prec (K_2, v_2)$, hence using Lemma 2.3, we must have that;

$(P(K_1), P(L), \pi_1) \prec (P(K_2), P(L), \pi_2)$

as required. Completeness now follows directly from Lemma 2.1 and model completeness. Alternatively, one can exhibit a prime model of the theory, this is clearly possible by taking the specialisations corresponding to the prime models of $T_{K,L}$ above.

3. A First Order Definition of Intersection Multiplicity and Bezout's Theorem

We now formulate a non-standard definition of intersection multiplicity in the language $\mathcal{L}_{spec}$. We will do this only in the case of projective curves inside $P^2(L)$, the reader may wish to try formulating a corresponding definition in higher dimensions.

Let C_1 and C_2 be projective curves of degree d and degree e in $P^2(K)$ defined over L. The parameter spaces for such curves are affine spaces of dimension $(d+1)(d+2)/2$ and $(e+1)(e+2)/2$ respectively. We can give them a projective realisation by noting that if $(\bar{l})$ is a non-zero vector defining a curve of degree d, then multiplying it by a constant μ defines the same curve. Let $P^{d(d+3)/2}(K)$ and $P^{e(e+3)/2}(K)$ define these spaces which we will denote by P_d and P_e for ease of notation. Let $Curve_d$ and $Curve_e$ be the closed projective subvarieties of $P_d \times P^2(K)$ and $P_e \times P^2(K)$, defined over the prime subfield of L, such that, for $l \in P_d$, the fibre $Curve_d(l)$ defines the corresponding projective curve of degree d in $P^2(K)$. For l in $P^n(L)$, we denote its infinitesimal neighborhood $\mathcal{V}_l$ to be the inverse image under the specialisation π_n.

Now suppose that C_1 and C_2 (which may not be reduced or irreducible), of degrees d and e respectively, are defined by parameters l_1 and l_2 and intersect at an isolated point l in $P^2(L)$. Then we define;

$Mult(C_1, C_2, l) \geq n$

iff

$\exists x_1, x_2 \in \mathcal{V}_{l_1}, \mathcal{V}_{l_2}, \exists_{y_1 \neq \ldots \neq y_n} \in \mathcal{V}_l(\{y_1, \ldots, y_n\} \subset Curve_d(x_1) \cap Curve_e(x_2))$

Then define $Mult(C_1, C_2, l) = n$

iff

$Mult(C_1, C_2, l) \geq n$ and $\neg Mult(C_1, C_2, l) \geq n + 1$.

Clearly, the statement that $Mult(C_1, C_2, l) = n$ naturally defines a sentence in the language $\mathcal{L}_{spec}$. One consequence of the completeness result given above is that the statement "The curves C_1 and C_2 intersect with multiplicity n at l" depends only on the theory T_{axioms} and is independent of the particular structure $(P(K), P(L), \pi)$. In the paper [3], we showed that this non-standard definition of multiplicity is equivalent to the algebraic definition of multiplicity when computed in the structure $(P(K_{univ}), P(L), \pi_{univ})$ (see the next section). It therefore follows that the non-standard definition of multiplicity is equivalent to the algebraic definition even when computed in a prime model of T_{axioms} which I will denote by $(P(K_{prime}), P(L), \pi_{prime})$.

We now turn to the statement of Bezout's theorem. In algebraic language, this says that if projective algebraic curves C_1 and C_2 of degree d and degree e in $P^2(L)$ intersect at finitely many points $\{l_1, \ldots, l_n\}$, then;

$\sum_{i=1}^{n} I(C_1, C_2, l_i) = de$

where $I(C_1, C_2, l_i)$ is the algebraic intersection multiplicity. The non-standard version of this result can be formulated in the language $\mathcal{L}_{spec}$ by the sentence;

$Bezout(C_1, C_2) \equiv \exists_{m_1, \ldots, m_n; m_1 + \ldots + m_n = de} (\bigwedge_{i=1}^{n} Mult(C_1, C_2, l_i) = m_i)$

Again, in the paper [3], we proved the *algebraic* version of Bezout's theorem by non-standard methods in the structure $(P(K_{univ}), P(L), \pi_{univ})$. It follows that the sentences $Bezout(C_1, C_2)$ are all proved by the theory T_{axioms} and therefore hold in the structure $(P(K_{prime}), P(L), \pi_{prime})$ as well. This demonstrates the fact that we can prove an algebraic statement of Bezout's theorem using only infinitesimals from a straightforward extension of L, namely $L(\epsilon)^{alg}$, in particular in a structure such that the infinitesimal neighborhoods $\mathcal{V}_l$ are all recursively enumerable. This seems to provide some answer to a general objection concerning the use of infinitesimals, originating in [1]. It may also provide an effective alternative method to compute intersection multiplicities generally in algebraic geometry.

4. Constructing a Universal Specialisation

In the papers [2] and [3], we used the existence of a specialisation $(P(K_{univ}), P(L), \pi_{univ})$ having the following "universal" property;

If $L \subset L_m$ is an algebraically closed extension of L with transcendence degree m, and $(P(L_m), P(L), \pi_m)$ is a specialisation, then there exists an L-embedding $\alpha_L : L_m \rightarrow K_{univ}$ with the property that $\pi_{univ} \circ \alpha_L = \pi_m$. (*)

Unfortunately, the construction of K_{univ} was flawed. We correct this difficulty here;

Model theoretically, using theorem 2.1, it is easy to show the existence of such a structure. Namely, let $(P(K_{univ}), P(L), \pi_{univ})$ be a 2^ω saturated model of the theory T_{axioms}. Then, if $L \subset L_m$ is an algebraically closed extension of L of transcendence degree m, clearly $\bigcup_{n \geq 1} Card(S^n(Th(\mathcal{M}))) \leq 2^\omega$, where $\mathcal{M} = (P(L_m), P(L), \pi_m)$. This follows as L was assumed to be countable. Hence, by elementary model theory, there exists an L-embedding α_L with the required properties. For the non-model theorist, we give a more algebraic construction, replacing the use of types by an explicit amalgamation of the possible valuations;

Proof. Suppose, inductively, we have already constructed a specialisation $(P(K_n), P(L), \pi_n)$ which has the property $(*)$ for all extensions $L \subset L_m$ with L_m algebraically closed of transcendence degree $m \leq n$. We will construct K_{n+1} having this property for $m \leq n+1$. By theorem 1.5, we can find a Krull valuation v_n on K_n corresponding to the specialisation π_n. Let t be a new transcendental element. The extensions of v_n to $K_n(t)$ are completely classifiable. In fact, we have the following result in [4] (Theorem 3.9), we refer the reader to the paper for the definition of each family of valuations;

The extensions of v_n are of the form;

(i). $v_{n,a,\gamma}$ where $a \in K_n$ and γ is an element of some ordered group extension of $v(K)$.

(ii). $v_{n,A}$ where A is a pseudo Cauchy sequence in (K_n, v_n) of transcendental type.

Let I be a fixed enumeration of these valuations. Inductively, we assume that $Card(K_n) \leq 2^\omega$ in which case the dimension of $v(K_n)$ as a vector space over $\mathcal{Q}$ has dimension at most 2^ω as well. Clearly then the number of non-isomorphic (over K_n) valuations from (ii) is at most 2^ω and the same holds for the valuations obtained from (i) by noting that the number of order types of γ is at most 2^ω (it is easily checked that 2 new elements of the value group, γ_1 and γ_2, having the same order type, define isomorphic valuations in the case of (i)). Hence, we can assume that I is well ordered and apply the method of transfinite induction to construct a series of specialisations $(P(K_{n,i}), P(L), \pi_{n,i})$ as follows;

For $i = 0$, set $(P(K_{n,0}), P(L), \pi_{n,0}) = (P(K_n), P(L), \pi_n)$

Given $i \in I$ with i not a limit ordinal, let v_{i+1} be the next valuation in the enumeration. Let $(K_n\{t\}, \overline{v_{i+1}})$ be the completion of $(K_n(t), v_{i+1})$ and let $\overline{v_{i+1}}$ also denote the unique extension of this valuation to the algebraic closure $K_n\{t\}^{alg}$. This defines a Krull valuation and hence a specialisation $(P(K_n\{t\}^{alg}), P(L'), \pi_{n,i+1})$ where L' is the algebraic closure of the residue field of v_{i+1}, having transcendence degree at most 1 over L. Using arguments as above, we can construct a specialisation $(P(L'), P(L), \pi)$. Composing these specialisations, we obtain a specialisation $(P(K_n\{t\}^{alg}), P(L), \pi_{n,i+1})$. (One can omit this step by enumerating in I only those valuations which preserve the residue field L) Now, using Lemma 2.1 and Theorem 2.1, amalgamate the specialisations $(P(K_n\{t\}^{alg}), P(L), \pi_{n,i+1})$ and $(P(K_{n,i}), P(L), \pi_{n,i})$ to form a specialisation;

$$(P(K_{n,i}), P(L), \pi_{n,i}) \prec (P(K_{n,i+1}), P(L), \pi_{n,i+1}).$$

For i a limit ordinal, we set;

$$(P(K_{n,i}), P(L), \pi_{n,i}) = \bigcup_{j<i} (P(K_{n,j}), P(L), \pi_{n,j})$$

By the usual union of chains arguments we have that;

$(P(K_{n,j}), P(L), \pi_{n,j}) \prec (P(K_{n,i}), P(L), \pi_{n,i})$ for $j < i$.

Repeating this process, we obtain a structure $(P(K_{n+1}), P(L), \pi_{n+1})$ such that;

$(P(K_n), P(L), \pi_n) \prec (P(K_{n+1}), P(L), \pi_{n+1})$.

It remains to check that this structure has the universal property $(*)$ for $m = n+1$. Let L_{n+1} be an algebraically closed extension of L with transcendence degree $n+1$ and specialisation $(P(L_{n+1}), P(L), \pi)$. Let v_π be the corresponding valuation and its restriction to $L \subset L_n \subset L_{n+1}$, a subfield of transcendence degree n. The corresponding specialisation $(P(L_n), P(L), \pi)$ already factors through $(P(K_n), P(L), \pi_n)$ $(\dagger)$ and the valuation v_π appears as v_i in the enumeration I when restricted to $L_n(t)$. By a standard result in valuation theory, see [5], there exists an $L_n(t)$-embedding $\tau : L_n(t)^{alg} \rightarrow L_n\{t\}^{alg}$ such that $v_\pi = \overline{v_i} \circ \tau$ $(\dagger\dagger)$ (see notation above). Combining $(\dagger)$ and $(\dagger\dagger)$, we obtain an embedding $\alpha : (P(L_{n+1}), P(L)) \rightarrow (P(K_{n,i}), P(L))$ such that $\pi = \pi_{n,i} \circ \alpha$. This proves the result. It is now clear that the structure

$(P(K_{univ}), P(L), \pi_{univ}) = \bigcup_{i>0}(P(K_i), P(L), \pi_i)$

has the required universal property, is a model of T_{axioms} and;

$(P(K_i), P(L), \pi_i) \prec (P(K_{univ}), P(L), \pi_{univ})$ for $i > 0$.

$\square$

References

[1] "The Analyst: a Discourse addressed to an Infidel Mathematician", George Berkeley (1734)

[2] Zariski Structures and Algebraic Curves, Tristram de Piro, available on my website, (2007), old version "Zariski Structures and Algebraic Geometry" on AG ArXiv (0402301) (2004)

[3] A Non-Standard Bezout Theorem for Curves, Tristram de Piro, available on my website, (2007), old version "A Non Standard Bezout Theorem" on AG/LO ArXiv (0406176) (2004)

[4] Value groups, residue fields and bad places of function fields, Franz-Viktor Kuhlmann, (2003)

[5] Algebraic Number Theory, Jurgen Neukirch, Volume 322, Springer (1992)

[6] Complete Theories, Abraham Robinson, Studies in Logic and the Foundations of Mathematics, North-Holland, (1956).

Mathematics Department, The University of Camerino, Camerino, Italy

E-mail address: `tristam.depiro@unicam.it`