A TOPOLOGICAL NOTE

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Abstract. We prove a technical result, required in footnote 10 of [2].

Lemma 0.1. *Let $C \subset P^2$ be a compact Riemann surface, in coordinates (x,y), and suppose that, for some $p_{i,j} \in P^2$, $\epsilon > 0$;*

$$pr_x : (C \cap D(p_{i,j}, \epsilon)) \rightarrow D(x_{p_{i,j}}, \epsilon)$$

is a double cover, ramified (in the sense of Zariski structures) at $\{z_{i,j}^1, z_{i,j}^2\} \subset P^2$, with $x_{i,j}^1 = pr_x(z_{i,j}^1) \neq x_{i,j}^2 = pr_x(z_{i,j}^2)$. Then $pr_x^{-1}(D(x_{p_{i,j}}, \epsilon))$ is topologically isomorphic to two annuli $\{A_1, A_2\}$, joined by circles $\theta_i : S^1 \rightarrow A_i$, $1 \leq i \leq 2$.

Proof. Let $l = \{tx_{i,j}^1 + (1-t)x_{i,j}^2 : 0 \leq t \leq 1\}$ be a line segment, $l \subset D(x_{p_{i,j}}, \epsilon)$, and let l^0 denote its interior. Choose $U_{i,j} \subset D(x_{p_{i,j}}, \epsilon)$ simply connected, with $(U_{i,j} \cap l) = l^0$. Let $\gamma = \delta U_{i,j}$, and, wlog, assume that γ is a smooth path, with $\gamma = (\gamma_2^{-1} \circ \gamma_1)$, where $\{\gamma_s : 1 \leq s \leq 2\}$ are smooth paths, $\gamma_s(0) = x_{i,j}^1$, $\gamma_s(1) = x_{i,j}^2$, $(\gamma_1 \cap \gamma_2) = \{x_{i,j}^1, x_{i,j}^2\}$. Then $pr : (C \cap D(p_{i,j}, \epsilon) \cap pr^{-1}(U_{i,j})) \rightarrow U_{i,j}$ is an unramified double cover. We claim that;

$$(C \cap D(p_{i,j}, \epsilon) \cap pr^{-1}(U_{i,j})) = U_{i,j}^1 \sqcup U_{i,j}^2$$

where, for $1 \leq k \leq 2$, $pr : U_{i,j}^k \rightarrow U_{i,j}$, is an (analytical) isomorphism, and $U_{i,j}^k$ are simply connected, $(*)$. In order to see $(*)$, pick $p \in U_{i,j}$, and let $\{p\} \subset U \subset U_{i,j}$ be maximal with the property that $(*)$ holds (with U replacing $U_{i,j}$), and corresponding $\{U^k : 1 \leq k \leq 2\}$. Suppose, for contradiction, that $U \neq U_{i,j}$. Pick $x_0 \in \delta U$, then $pr_x : (C \cap D(p_{i,j}, \epsilon))$ is unramified at $pr^{-1}(x_0) = \{x_0^1, x_0^2\}$. By the inverse function theorem, there exists a disc $D(x_0, \delta) \subset D(x_{p_{i,j}}, \epsilon)$, and, for $1 \leq k \leq 2$, simply connected open sets $U_{x_0}^k$, with $(U_{x_0}^1 \cap U_{x_0}^2) = \emptyset$, $(\dagger)$ and $x_0^k \in U_{x_0}^k$. Without loss of generality, we can assume that $D(x_0, \delta) \cap U$ is connected. Then, as, using the hypotheses of $(*)$, U and $D(x_0, \delta)$ are simply connected, so is $U \cup D(x_0, \delta)$. Then, it follows easily, that, wlog $U^k \cap U_{x_0}^k \neq \emptyset$, for

$1 \leq k \leq 2$, and using $(*)$ and $(\dagger)$, that $(U^1 \cup U^1_{x_0}) \cap (U^2 \cup U^2_{x_0}) = \emptyset$. We obtain easily, that $pr_x : (U^k \cup U^k_{x_0}) \rightarrow U \cup D(x_0, \delta)$, for $1 \leq k \leq 2$, is an (analytical) isomorphism, and, therefore, for $1 \leq k \leq 2$, that $(U^k \cup U^k_{x_0})$ are simply connected. obtaining $(*)$, and contradicting the assumption of maximality. It follows that $(*)$ holds for $U_{i,j}$, as required. We claim that;

$$(C \cap D(p_{i,j}, \epsilon) \cap pr^{-1}(\overline{U_{i,j}})) = \overline{U^1_{i,j}} \sqcup_{z^1_{i,j}, z^2_{i,j}} \overline{U^2_{i,j}}$$

where $\{\overline{U^1_{i,j}}, \overline{U^2_{i,j}}\}$ are closed and simply connected, $(**)$. We have that $\overline{U_{i,j}} = U_{i,j} \cup \gamma$ is simply connected, and, for $1 \leq k \leq 2$, $\overline{U^k_{i,j}}$ are simply connected. We have that $(C \cap D(p_{i,j}, \epsilon) \cap pr^{-1}(\overline{U_{i,j}})) = U^1_{i,j} \sqcup U^2_{i,j} \sqcup (C \cap D(p_{i,j}, \epsilon) \cap pr^{-1}(\gamma))$. Moreover, for $1 \leq s \leq 2$, $1 \leq k \leq 2$ $z^s_{i,j} \in \overline{U^k_{i,j}}$, (using limits), and, if $x_1 \in Im(\gamma) \setminus \{x^1_{i,j}, x^2_{i,j}\}$, then, as $pr_x^{-1}(\gamma) \subset \overline{U^1_{i,j}} \cup \overline{U^2_{i,j}}$, and $|pr_x^{-1}(x_1)| = 2$, we have that $U^1_{i,j} \setminus \{z^1_{i,j}, z^2_{i,j}\} \cap U^1_{i,j} \setminus \{z^1_{i,j}, z^2_{i,j}\} = \emptyset$. Hence, $(**)$ holds, as required.

Now, choose $\epsilon' > 0$, and closed discs $\{\overline{D}(x^l_{i,j}, \epsilon') : 1 \leq l \leq 2\} \subset U_{i,j}$. Choose open sets $\{V_l : 1 \leq l \leq 2\}$, with $z^l_{i,j} \in V_l$ and $pr_x(V_l) \subset D(x^1_{i,j}, \epsilon')$. For a choice of local coordinates $\lambda_l : D(0, \epsilon'') \rightarrow V_l$, $\mu_l : D(0, \epsilon') \rightarrow D(x^l_{i,j}, \epsilon')$, $1 \leq l \leq 2$, we have that the maps $(\mu_l^{-1} \circ pr_x \circ \lambda_l : D(0, \epsilon'') \rightarrow D(x^l_{i,j}, \epsilon') \rightarrow D(0, \epsilon')$ are analytic, and, using Theorem 18.3 of [1], $Mult_0(\mu_l^{-1} \circ pr_x \circ \lambda_l) = 2$, and $(\mu_l^{-1} \circ pr_x \circ \lambda_l) = z^2 u_l(z)$, with u analytic, $u_l(0) \neq 0$. Let $v_l(z) = u_l(z)^{\frac{1}{2}}$ (principal branch), $v_l(0) \neq 0$, then $(\mu_l^{-1} \circ pr_x \circ \lambda_l) = (zv_l(z))^2$, $ord_0(zv_l(z)) = 1$. Hence, after a further analytic change of coordinates, $\nu_l : D(0, \epsilon''') \rightarrow D(0, \epsilon'')$, we have that the maps $\Gamma_l = (\mu_l^{-1} \circ pr_x \circ \lambda_l \circ \nu_l) = z^2$. Let $V^l_{i,j} = \overline{D}(x^l_{i,j}, \epsilon') \cap U_{i,j}$, $\gamma'^{,l,s} = (\overline{D}(x^l_{i,j}, \epsilon') \cap \gamma_s)$ and $W^l_{i,j} = (\mu_l)^{-1}(V^l_{i,j})$, $W^l_{i,j} \subset D(0, \epsilon')$, $\gamma^{l,s} = (\mu_l)^{-1})(\gamma'^{,l,s})$. Let $\Gamma : D(0, \epsilon''') \rightarrow graph(z^2)$, $\Gamma(z) = (z, z^2)$, (x', y'), and $B^l_{i,j} = (pr_x \circ \lambda_l \circ \nu_l \circ \Gamma^{-1})^{-1}(\overline{D}(x^l_{i,j}, \epsilon'))$, (reducing ϵ' if necessary), $B^l_{i,j} \subset graph(z^2)$. We claim that $B^l_{i,j}$ is topologically isomorphic to a closed disc E^l, centred at $(0, 0)$, $(***)$. By $(**)$, we have, taking ϵ' sufficiently small, that $pr_{y'}^{-1}(V^l_{i,j}) = Y^1_{i,j,l} \sqcup_{(0,0)} Y^2_{i,j,l}$, with $Y^k_{i,j,l}$ simply connected, for $1 \leq k \leq 2$. For $1 \leq k \leq 2$, $1 \leq l \leq 2$, $1 \leq s \leq 2$, we define $\{\gamma^{k,l,s}\} \subset Y^k_{i,j,l}$ by;

$$\gamma^{k,l,s} = pr_{y'}^{-1}(\gamma^{l,s}) \cap Y^k_{i,j,l}$$

We invert $pr_{y'}$, by taking the two branches $\{g_{1,l}, g_{2,l}\}$ of $exp(\frac{log(x')}{2})$, and cutting the disc $D(0,\epsilon')$, along contours θ_l from 0 to $\delta D(0,\epsilon')$, (excluding 0) with $\theta_l \subset W^l_{i,j}$, $(\theta_l \cap \gamma^{l,s}) = \emptyset$, see [3]. Then, wlog, $g_{1,l}$ glues together the edges $\{\gamma^{1,l,2}, \gamma^{2,l,1}\}$, and $g_{2,l}$ glues together the edges $\{\gamma^{1,l,1}, \gamma^{2,l,2}\}$, along $g_{1,l}(D(0,\epsilon') \setminus W^l_{i,j})$ and $g_{2,l}(D(0,\epsilon') \setminus W^l_{i,j})$ respectively. (If, say, $g_{1,l}$ glued together the edges $\{\gamma^{1,l,2}, \gamma^{1,l,1}\}$ of $Y^1_{i,j,l}$, then, using the fact that $pr_{y'} : Y^1_{i,j,l} \setminus \{0\} \rightarrow W^l_{i,j} \setminus \{0\}$ is an isomorphism, we could analytically continue $g_{1,l}$ across the contour θ_l.) This gives the result $(***)$. If, we let $\gamma^{',l,3} = (\delta D(x^l_{i,j}, \epsilon') \cap U_{i,j}$, $\gamma^{l,3} = (\mu_l)^{-1})(\gamma^{',l,3})$, and $\gamma^{k,l,3} = pr^{-1}_{y'}(\gamma^{l,3}) \cap Y^k_{i,j,l}$, Then the discs $C^1_{i,j} = ((\lambda_l \circ \nu_l) \circ \Gamma^{-1})(B^1_{i,j})$ and $C^2_{i,j} = ((\lambda_l \circ \nu_l) \circ \Gamma^{-1})(B^2_{i,j})$ are glued together, by $\underline{((\lambda_l \circ \nu_l) \circ \Gamma^{-1})(\gamma^{1,1,3} \cup \gamma^{2,1,3}) \text{ and } ((\lambda_l \circ \nu_l) \circ \Gamma^{-1})(\gamma^{1,2,3} \cup}$ $\gamma^{2,2,3})$, along $pr^{-1}_x(\overline{U_{i,j} \setminus \{D(x^1_{i,j}, \epsilon') \cup D(x^2_{i,j}, \epsilon')\}})$. For $1 \leq l \leq 2$, letting $l^l = (l \cap \overline{D}(x^l_{i,j}, \epsilon')$, and $l^{l,lift} = (pr^{-1}_x(\overline{U_{i,j}}) \cap l^l \cap C^l_{i,j})$, and $l^{',lift} = pr^{-1}_x(\overline{U_{i,j} \setminus \{D(x^1_{i,j}, \epsilon') \cup D(x^2_{i,j}, \epsilon')\}})$ it is clear that the interior $l^{\circ,l,lift} \cap \delta(C^l_{i,j}) = \emptyset$, and the loop $pr^{-1}_x(l \cap C)$, passing through $\{z^1_{i,j}, z^1_{i,j}\}$ consists of the union of the three line segments $\{l^{l,lift}, l^{2,lift}, l^{',lift}\}$. It follows easily from $(***)$, that $C \cap pr^{-1}_x(U_{i,j} \cup D(x^1_{i,j}, \epsilon') \cup D(x^2_{i,j}, \epsilon'))$ consists of two annuli $\{B_t : 1 \leq t \leq 2\}$ joined along $pr^{-1}_x(l \cap C)$. It is easy to see that $C \cap pr^{-1}_x(U_{i,j} \cup D(x^1_{i,j}, \epsilon') \cup D(x^2_{i,j}, \epsilon'))$ is topologically equivalent to $C \cap pr^{-1}_x(D(p_{i,j}, \epsilon))$, as the cover pr_x is unramified restricted to $C \cap (pr^{-1}_x(D(p_{i,j}, \epsilon) \setminus (U_{i,j} \cup D(x^1_{i,j}, \epsilon') \cup D(x^2_{i,j}, \epsilon'))))$ Moreover, the interiors of the annuli are uneffected, that is $A_t = (B_t \cup C_t)$, where C_t is an annulus, with $(\delta(A_t) \cap \delta(B_t) \cap \delta(C_t)) = \emptyset$. Hence, the result follows. □

References

[1] A Nonstandard Approach to the Theory of Algebraic Curves, Tristram de Piro, submitted to the Journal of Logic and Analysis, (2012).

[2] A Theory of Harmonic Variations, Tristram de Piro, available at http://www.magneticstrix.net, (2014).

[3] Introduction to Complex Analysis, OUP, H.A. Priestley, (1990).

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