RAPHAEL AND THE COSMATI GEOMETRY OF CURVES

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In this chapter, we will look more closely at the aesthetic ideas discussed above, and see how they can be used in geometry. The use of spirals and inflexions in the artwork of Raphael, together with the braiding patterns of Cosmati artists, suggests an intelligence behind the depiction of curves in the plane, which, I will argue, is extremely useful in the study of the theory of braids, a field of study in algebraic curves connected with topology and the notion of "fundamental groups". We also examined the understanding of harmony in medieval architecture, with particular reference to the chevet form. I will argue these ideas are also extremely important in the representation of algebraic curves, using the notions of class and genus, a field pioneered by the German mathematician Julius Plucker. Finally, in the medieval conception of focusing light, either in the form of rose windows, or in the earlier devices of Romanesque windows, through the analysis of curves in terms of ocsillating patterns, we will find a connection with the analysis of functions on curves, into simpler harmonics, an area now referred to as Fourier analysis, after the French scientist Joseph Fourier, but possibly invented by his adviser Laplace.

In order to make these ideas more precise, I will give a brief explanation of some of the above mentioned mathematical terms. The term "fundamental group" was first used by Henri Poincare, in his paper Analysis Situs, of 1895. The idea is to understand some of the geometry of algebraic curves, through the possible one dimensional loops which can exist on their two-dimensional surfaces. Of course, there are an infinite number of loops through any given point, but any two loops, which can be continuously bridged by a path across the surface are called equivalent, see figure 1, ([1]) In the diagram, any loop passing through the point p, on the surface of the sphere S, can be contracted back to the initial point p, so any two loops are formally equivalent. In this case, the fundamental group is trivial, that is consists of a single element. In more complicated surfaces, such as the torus T, there exist loops which cannot be so contracted, see figure 2. Here, the loops γ_a and γ_b cannot be

[1]Formally, if S is a topological space and $s_0 \in S$ is a chosen base point, we define a closed loop to be a continuous map $\gamma : [0,1] \rightarrow S$ with $\gamma(0) = \gamma(1) = p$. We define two loops $\{\gamma_0, \gamma_1\}$ to be homotopically equivalent, $\gamma_0 \sim \gamma_1$, if there exists a continuous map;

$H : [0,1] \times [0,1] \rightarrow S$ with $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$, $H(s,0) = H(s,1) = p$. It is easily checked that $\sim$ is an equivalence relation on loops, and we define $[\gamma]$ to be the equivalence class of a loop γ. One can also define a composition of loops $\gamma_2 \circ \gamma_1$, compatibly with the equivalence relation $\sim$, and an identity class $[1]$. We then define;

$\pi_1^{top}(S,p) = \{[\gamma] : \gamma \textit{ a closed loop}\}$

With the composition $\circ$, this defines a group. If S is connected, it is independent of the chosen base point p, and denoted by $\pi_1^{top}(S)$. If C is a projective algebraic curve over $\mathcal{C}$, then it can be considered as a topological space, and we can define $\pi_1^{top}(C)$.

contracted back to the initial point p, and, moreover, cannot be transformed continuously, one to the other. It is easily shown, that any loop is essentially a combination of these two, ([2]).

The geometric idea of the fundamental group connects easily with that of genus. The modern definition, which could be said to originate with the German mathematician Felix Klein, is that an algebraic curve C has geometric genus g, if, topologically, it is equivalent, ([3]) to a sphere S with g attached handles, see figure 3. However, the idea can be traced back to the earlier work of Julius Plucker, where it is mentioned in his "Theorie der Algebraischen Curven", of 1839, ([4]). For nonsingular curves C, there is an interesting relationship between the fundamental group and the genus. Namely, that, for a curve C of genus g, the fundamental group is generated by exactly $2g$ loops, ([5]). Figure 4 makes this clear, the loops γ_a^1 and γ_b^1 attached to the first handle, passing through the point p, are not equivalent, and, a similar argument applies, for each of the g handles.

The class of an algebraic curve C, which, also appears in the above work of Julius Plucker, counts the number of tangent lines, passing through a general point in the plane. This notion is illustrated in figure 5, in which the general point is represented by p, and the tangent lines (in black) of C (in brown), count the class as 8. There is a simple relationship between the class m and the degree of C, for nonsingular curves, known as the elementary Plucker formula, $m = d(d-1)$, in particular, the class of such curves is even. One can also obtain a relationship with the genus, using the degree-genus formula, ([6]).

One can refine the study of loops on a surface, by allowing an algebraic curve to have singularities. We discussed the concept of nodal curves in Chapter 5, and, will, again, in Chapter 11. A related idea is that of an inflexion. Formally, a point $p \in C$ is an inflexion if its tangent line, see the discussion in Chapter 5, has contact at least 3 with the curve, see figure 6 and Chapter 11. As we explain in Chapter 11, any algebraic curve is birational to a plane curve with at most nodes as singularities, and, for which the inflexions are distinct from the nodes. One can then restrict attention to loops or paths on such surfaces which are restricted by its geometry of nodes and inflexions. In figure 7, we have two loops, based

[2]More precisely, the fundamental group is generated by the 2 loops, γ_a and γ_b, subject to the single relation $\gamma_a\gamma_b\gamma_a^{-1}\gamma_b^{-1} = 1$.

[3]Two surfaces, S_1 and S_2, are said to be topologically equivalent, if, one can be continuously transformed into the another, by stretching, without tearing or glueing

[4] Plucker's definition, which, I believe, was later adopted by the Italian mathematician Francesco Severi, can be shown to be equivalent to the modern definition. The proof, assuming the topological degree-genus formula, which asserts that, for a nonsingular curve C, $g = \frac{(d-1)(d-2)}{2}$, where d is the degree of C, see Chapter 5, can be found in [20]. A direct proof of the degree-genus formula, using Severi's definition, can also be found in [20]. A direct proof of the topological genus formula can also be given using Plucker's methods, see [21]. A further important reference is [11].

[5]More specifically, the group is generated by the loops $\{\gamma_a^1, \gamma_b^1, \ldots, \gamma_a^g, \gamma_b^g\}$, satisfying the single relation;

$$((\gamma_a^1)(\gamma_b^1)(\gamma_a^1)^{-1}(\gamma_b^1)^{-1}) \ldots ((\gamma_a^g)(\gamma_b^g)(\gamma_a^g)^{-1}(\gamma_b^g)^{-1}) = 1$$

The proof of this result can be found in [11], see also [15].

[6]This is a simple calculation, from $m = d(d-1)$, $g = \frac{(d-1)(d-2)}{2}$, we obtain $d^2 - d - m = 0$, $d = \frac{1+\sqrt{1+4m}}{2}$, $g = \frac{m}{2} - d + 1$, $g = \frac{m+1+\sqrt{1+4m}}{2}$

at the points p and q. The first is an example of an inflexionary path, with inflexions at i_1 and i_2, these should occur at the actual inflexions of the surface. The second is an example of a nodal path, which passes through the node ν of C, ([7]).

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[7] Formally, by a "nodal path", we mean a closed path $\gamma : (S^1, 0) \rightarrow (C, p)$ with the following further properties;

(a). γ is locally analytic in the sense of real manifolds.

(b). γ is smooth.

(c). γ has at most nodes as singularities, defined by *at least* one of the nodes in C, (†)

In condition (a), we take $\mathcal{R}^4$ to be the object manifold. Then condition (b) means that $\gamma'(t) \neq \bar{0}$, for $t \in S^1$, and condition (c) means that γ is injective, with the exception of pairs $\{t_1, t_2\}$ such that $\gamma(t_1) = \gamma(t_2) = v$ where $v \in C'$ is a node and, in this case, $\{\gamma'(t_1), \gamma'(t_2)\}$ belong to the tangent planes $\{H_{\gamma_1}, H_{\gamma_2}\}$ of the branches γ_1, γ_2 centred at v. We define a "nodal-inflexionary" path to be a path $\gamma : (S^1, 0) \rightarrow (C, p)$, satisfying conditions $(a), (b)$ of (†), with the additional requirement;

$(c)'$ γ has at most nodes as singularities, defined *only* by the nodes of C, the *only* inflexions of γ are defined by the inflexions of C.

Here, by an inflexion $t \in S^1$ of γ, we mean that, for a generic choice of $P \in \mathcal{R}^4$, with conic projection $pr_P : C' \rightarrow \mathcal{R}^2$, t should define an inflexion of the induced curve $pr_P \circ \gamma : S^1 \rightarrow \mathcal{R}^2$, in the sense of [22], (this notion is well defined as inflexions are preserved by homographies of $P^1(\mathcal{C})$). A nodal-inflexionary path clearly reflects the "flow" of the curve C along the finitely many non-ordinary branches, hence seems to be an important object of study.

Let $S = \{v_1^1, v_1^2, \ldots, v_d^1, v_d^2, i_1, \ldots i_m\}$, where $v_j = v_j^1 = v_j^2$, for $1 \leq j \leq d$ denotes a node of C and i_j, for $1 \leq j \leq m$ denotes an inflexion of C. We call a partial ordering on S *good*, if, v_j^1 appears in the order iff v_j^2 appears in the order, and, in this case, $v_j^1 < v_j^2$. Using the natural ordering on S^1, induced by the interval $[0, 1)$, any nodal-inflexionary path determines a good partial ordering on S. One can define a geodesic nodal-inflexionary path γ, by adding the requirement (d) that γ defines a geodesic, and adapt the notion of homotopy to require that intermediate paths are geodesics. Conversely, one can ask;

(a). Given a good ordering on S, can one find a geodesic nodal-inflexionary path realising this ordering?

(b). Given homotopic geodesic nodal-inflexionary paths γ_1, γ_2, are the orderings determined by these paths the same?

(c). Given geodesic nodal-inflexionary paths determining the same ordering, are they homotopic?

(d). Is every path in $\pi_1^{top}(C)$ homotopic to a nodal-inflexionary or geodesic nodal-inflexionary path?

(e). More generally, we can allow S to have repeats, and define the length $l([\gamma]) = min\{l(\gamma_1) : \gamma_1 \sim \gamma\}$, where $l(\gamma_1) = 2d' + m'$, of a geodesic nodal-inflexionary path, where d' is the number of nodes (with repeats) and m' is the number of inflexions (with repeats). How does the length behave under composition, and, is there a bound $0 \leq 2d + m \leq k$, and a finite index subgroup $G \subset \pi_1^{top}(C)$ such that, given a path γ, can one find paths $\{\gamma_1, \gamma_2\}$, with $max\{l([\gamma_1]), l([\gamma_2])\} \leq k$, $[(\gamma \circ \gamma_1^{-1})] = [\gamma_2]$, $[\gamma_2] \in G$?

One can show that any nodal-inflexionary loop in the plane $\mathcal{R}^2$ has an even number of inflexions, see [22], however, suprisingly the number of inflexions of a plane nonsingular algebraic curve can be odd, see below. An interesting exercise is to give a convenient description of nodal loops, without inflexions, in the plane. These are classified as sources, in the sense that one can remove all the nodes from the path, by successively subtracting loops beween adjacent nodes, see [22]. In figure 8, the nodal path γ, centred at p, has two nodes, based at $\{\nu_1, \nu_2\}$. The ordering defined by the path (starting at p), is given by $\{\nu_2^1, \nu_1^1, \nu_1^2, \nu_2^2\}$. Subtracting $\{\nu_1^1, \nu_1^2\}$, we obtain $\{\nu_2^1, \nu_2^2\}$, and, then $\emptyset$. This result is useful in the study of nodal-inflexionary paths on an algebraic curve C, as defined above, and, considered in footnote 7, as the generic projection of a nodal path on C is a nodal path in the plane $\mathcal{R}^{\in}$. The reduction of geometric thinking about algebraic curves to the plane, gives ample scope for the type of visual thinking, about inflexions and spirals, employed by artists such as Raphael and the Cosmati, which we considered in the previous chapter. A successful understanding of such paths, see particularly question (d) in footnote 7 about symmetry groups, might lead to geometric insights into the etale fundamental groups of nodal algebraic curves, and the structure of Galois actions, discussed in Chapter 5, Theorem 0.19, within the context of flash geometry discussed there. It is the author's hope that these considerations might lead to new proofs of Severi's conjecture, ([8]). This type of thinking is also fundamental in the study of braids, see figure 8, ([9]). In figure 9, we have 2 braids, (in pink and red), which are distinct, in the sense that one cannot be continuously transformed into the other. There is an interesting connection with algebraic curves. Given $C \subset P^2$, there exists finitely many points $\{q_1, \ldots, q_r\}$, for which the projection $pr_1 : C \to P^1$ is ramified, that is the line defined by $pr^{-1}(x)$, $x \in P^1$, is tangent to the curve C. Ignoring these points, at which the surface "branches" over the plane, any loop γ centred at $p \in D \subset P^1 \setminus \{pr_1(q_1), \ldots, pr_1(q_n)\}$, lifts uniquely to a path (possibly a loop) commencing at p', with $pr_1(p') = p$, and defines the string of a braid by tracing, over time, the moving shadow of γ' on a disc D', centred at $p_1 = pr_2(p')$, see figure 10. We can then obtain a representation of the fundamental group $\pi_1(D \setminus \{q_1, \ldots, q_r\}, p)$, in the braid group $Br_{n,pr^{-1}(p)}$, where $n = degree(pr_1)$, the cardinality of a typical fibre of the projection pr_1, $(*)$. Now, given a projective algebraic curve C, we can label its nodes and inflexions, together with the nonsingular ramification points of the

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(f). Given positive answers to $(a), (b), (c), (d), (e)$, can one use the induced action of the symmetric group S_{2d+m} on $\pi_1^{top}(C)$ to give an analysis of the automorphism groups of certain Galois extensions, $Gal(C''/C')$, with $C \subset C' \subset C''$, $C' = Fix(G)$ and $Card(Gal(C''/C')) \geq 2d+m$?. Otherwise, can one construct invariants of C on which S_{2d+m} acts? Can one obtain an analysis of the etale fundamental group, possibly in nonzero characteristic, using the specialisation and lifting theorems, see [18]?

[8] A central idea in proving irreducibility of Severi varieties V, is that an automorphism of $[C] \in Sing(V)$ should extend to an analytic map between the sheets of V passing through $[C]$. This might involve some ideas from deformation theory, but also require insights into the realisation of this extension in terms of these Galois actions.

[9] Formally, a braid is a sequence of continuous paths $\{\gamma_1, \ldots, \gamma_n\}$, with $\gamma_i : [0,1] \to [0,1] \times D$, for $1 \leq i \leq n$, where D denotes a complex disc, such that $\{\gamma_i(\delta) : 1 \leq i \leq n\} \subset \{(\delta, p_1), \ldots, (\delta, p_n), \delta \in \{0,1\}\}$, (distinct points), $pr_2 \circ \gamma_i = Id_{[0,1]}$ and, for $t \in (0,1)$, $1 \leq i < j \leq n$, $\gamma_i(t) \neq \gamma_j(t)$. Similarly to the above description of the fundamental group, we can define a notion of isotopy for braids, namely $\overline{\gamma}_0 \sim \overline{\gamma}_1$, if there exists a continuous $H : [0,1] \times [0,1]^n \to [0,1] \times D$, with $H(0, \overline{t}) = \overline{\gamma}_0(\overline{t})$, $H(1, \overline{t}) = \overline{\gamma}_1(\overline{t})$, and $H(s, \overline{t})$ defining a braid for any $s \in [0,1]$. As above, we can define a composition of braids, compatible with $\sim$ and an identity [1]. We let $Br_{n,\overline{p}} = \{[\gamma] : \gamma \ a \ braid\}$, which we call the braid group on n strings, associated to $\overline{p} = \{p_1, \ldots, p_n\}$.

projection $pr_1 : C \to P^1$, ([10]). One can then attempt to address some of the questions in footnote 7, by combining the theory of nodal-inflexionary paths in $\mathcal{R}^2$, see [22], with Moishezon's theory of braid monodromy, see [17], ([11]).

Nowhere is an understanding of how the aesthetics of harmony and balance can be used in the geometry of algebraic, more evident, than in the work of Julius Plucker, to whom we alluded to above. Julius Plcker was born on June 16, 1801 in Elberfeld, Germany, and died on May 22, 1868, Bonn. He was a mathematician of the highest order, making important advances in the theory of algebraic curves, and, in later life, a physicist. Plcker attended the universities in Heidelberg, Bonn, Berlin, and Paris. He obtained his doctorate in 1824, from the University of Marburg. In 1829, after four years as an unsalaried lecturer, he became a professor at the University of Bonn, where he wrote "Analytisch-geometrische Entwicklungen, The Development of Analytic Geometry, in 2 volumes, between 1828 and 1831. Here, he begins to develop the notation of projective geometry, in which lines rather than points become the fundamental elements. The idea, here, is that projective space, $P^2(\mathcal{C})$ considered as the set of lines passing through the origin in $\mathcal{C}^3$, can be written in homogenous coordinates $\{[X : Y : Z] : (X, Y, Z) \neq \bar{0}\}$, with an embedding of affine space $\mathcal{C}^2$, given by $\{[x : y : 1] : (x, y) \in \mathcal{C}^2\}$. The equation of a curve $F(x, y)$ in the affine plane can then be converted into an equation in projective space, by making the substitution $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$, and clearing coefficients. On (p97, Vol.2, Essen (1831)), he writes;

"Die Coordinaten dieses Mittelpunctes, den wir durch (y', x') bezeichnun wollen, sind folgende:

$x' = \frac{B}{A}\ y' = \frac{D}{A}$

and bases a number of calculation on this substitution. Plucker considers mainly problems concerning curves of low degree, with an exceptional interlude on Cramer's paradox, that the number of points of intersection of two higher-order curves can be greater than the number of arbitrary points that are usually needed to define one such curve, "Sobald $n = 3$, oder $n > 3$, und also;

$n^2 \geq (\frac{n(n+3)}{1.2})$

[10]Formally, let $\{\nu_1, \ldots, \nu_s, i_1, \ldots, i_t, q_1, \ldots, q_r\}$ denote these points. Any nodal inflexionary loop $\gamma \in \pi_1(D \setminus \{pr_1(\nu_1), \ldots, pr_1(\nu_s), pr_1(q_1), \ldots, pr_1(q_r)\})$, with inflexions defined by $\{pr_1(i_1), \ldots, pr(i_t)\}$, lifts to a nodal-inflexionary path γ' in C, and, if C is nonsingular, $\gamma' \subset C \setminus \{pr_1^{-1}(q_j) : 1 \leq j \leq r\}$ is a nodal-inflexionary path, with $pr_1(\gamma'(0)) = pr_1(\gamma'(1))$ then $pr_1 \circ \gamma'$ is a nodal inflexionary loop in $\pi_1(D \setminus \{pr_1(q_1), \ldots, pr_1(q_r)\})$. How does the ordering of the inflexions on the path γ effect its winding numbers around the projected ramification points, and, therefore, by $(*)$, its representation in the braid group?

[11]This requires the positioning of the ramification of nonsingular curves in terms of the intersections of lines, proved in [21], coming from Severi's conjecture.

begegnet man einer Art von paradox...", ([12]). In Chapter 3, Section 3, "Das Princip der Reciprocitat", (p259), of the same text, Plucker formulates the principle of duality, ([13]), "Die Polaren aller Puncte einer gegebenen geraden Linie gehen durch den Pol derselben...Die pole aller geraden Linien, welch durch einen gegebenen Punct gehen, liegen auf der Polaren dieses Punctes." Plucker does not give a general definition of class here, for an arbitrary algebraic curve, but uses it to count the number of "parallele Tangenten", for specific cubic curves, see p96. He is clearly experimenting with the idea at this stage, but addresses the problem with greater confidence in his "System der Analytischen Geometrie", Berlin, (1835), written after Plcker became an ordinary professor of mathematics at the University of Halle, in 1834. We now consider this text in greater detail.

Plucker seems to be interested now in the use of the methods developed above, to represent algebraic curves in the plane. He seems to be mainly interested in real algebraic curves, but many of his ideas extend immediately to the complex case. In Chapter 5, we considered, in depth, Newton's classification and analysis (with diagrams) of cubic curves, using asymptotes. Plucker continues this project in Section 3, "Allgemeine geometrische Construction der Curven dritter Ordnung" (Several geometric constructions of curves of the third order) and Section 5, "Aufzahlung der verschiedenen Arten der Curven dritter Ordnung" (List of different kinds of curve of the third order", (p221), obtaining 219 types. Here, he begins by considering types of cubic with the property that the three finite points where the three asymptotes intersect the curve lie on a straight line l, dividing the curves into groups, according to the relative position of this line with the intersection points of the asymptotes. In figure 11, (referring to p227) he considers the case where the line l intersects the triangle formed by the intersections, above and below the extreme vertices. He maintains Newton's terminology of an oval, to refer to the irreducible component (for real algebraic curves), formed within the triangle, and is addressing the question of how these "double intersections" (of the asymptotes) might help us to analyse algebraic curves. The aesthetics of harmony and balance are clearly employed here. In figure 12, a typical representation of an algebraic curve (of higher degree) is given in which three of its asymptotes are bitangents to the curve. Although Plucker doesn't introduce bitangents until later, where he makes the connection with nodal curves, by duality, the method of finding such asymptotes, for a suitable choice of coordinates, is critical, in finding representations, where the class points, are assigned in pairs, to the asymptote intersections, thus forming an oval and a facing hyperbola. This method, for nonsingular curves, is outlined in [21], Lemma 2.7, for, "harmonic" arrangements of lines, which radiate concentrically around a central polygon, see the aesthetic discussions in the previous chapter and figure 14, which is again discussed in relation to genus below. The pairs theorem is given in Lemma 2.8. Here, the use of duality is important in the technical detail of aligning the bitangent points along two

[12] Plucker is clearly aware of Bezout's theorem, which was published in 1779 by tienne Bzout, "Thorie gnrale des quations algbriques", see also [28], he also makes an interesting reference to a numerical notion of genus, observing that, (p244), $n^2 - (\frac{n(n+3)}{2}) = \frac{(n-1)(n-2)}{2}$, see footnote 4, however, this could be a coincidence, we will return to this point later

[13] Namely, any line in $l \in P^2(\mathcal{C})$ can be defined by an equation $aX + bY + cZ = 0$, (up to multiplication by a scalar), hence, defines a corresponding point $p_l \in (P^2)(\mathcal{C})$, $p_l = [a : b : c]$. A point $p \in P^2(\mathcal{C})$ can be written as the intersection of lines $\{cX + dY + eZ : cp_1 + dp_2 + ep_3 = 0\}$, hence determines a line $l_p \subset (P^2)(\mathcal{C})$, given by $p_1X + p_2Y + p_3Z = 0$. Moreover these constructions are inverses

parallel axes, see footnote 8. Plucker clearly understands the general duality map, in which a curve C is mapped to its set of tangents, forming a new curve C^*, ([14]), and the definition of class at this stage, as, on p291, he notes that "Eine gegebene Curve n. Ordnung hat zu ihrer Polar-Curve eine Curve der n(n-1) Ordnung". "A Plane curve of degree n has a dual curve of degree $n(n-1)$, Although there is no formal proof given of this statement, it is interesting to speculate, whether this correct statement, now usually referred to as the "Elementary Plucker Formula", was obtained by generalising results for curves of low degree, or, more likely, was proved but unpublished. A later geometric proof of this result was given by Severi, see Chapter 11, and reproduced in [20];

Theorem 0.1. *Let C be a plane projective algebraic curve of order n and class m, with d nodes. Then;*

$$n + m + 2d = n^2$$

Proof: See [20].

Lemma 0.2. *Let C be a projective algebraic curve, with finitely many flexes, then $Cl(C)$, as defined in Definition 4.1, is the same as $deg(C^*)$ and $deg(C)$ is the same as $Cl(C^{*)}$. In particular, if C has at most nodes as singularities, then;*

$$deg(C^*) = n(n-1) - 2d$$

Proof: See [20].

Plucker obtains, again without proof, the correct formula for the number $3n(n-2)$ of inflexions on a nonsingular complex curve C of degree n, $3n(n-2) = 3(m-n)$, see the discussion below;

"Nach dem 6. Paragraphen, auf den ich mich hier stillschweigund ofter bezogen habe, hat eine Curve n. Ordnung $3n(n-2)$ Wendungstangenten."

and also a formula for the number of bitangents to a real algebraic curve;

"und also auch die Anzahl jener doppeltengenten $\frac{n(n-2)(n^2-9)}{2}$"

giving his famous result of 28 bitangents, for quartic curves. This seems to be correct for generic real curves, the correct formula $\frac{n(n-2)(n^2-6)}{2}$ bitangents for complex nonsingular algebraic curves, is given in [21], footnote 5. It seems that, here, these results were genuinely obtained, from particular constructions of lower degree, as he obtains a deeper understanding of duality in his following "Theorie der Algebraischen Curven", Bonn, published in 1839. At this stage, he was a full professor of mathematics, until 1847, where he succeeded Karl von Mnchow in 1836, and, had married, in 1837, a Miss Altsttten.

[14] see [20] for a formal algebraic definition

There, he obtains, on p211, the following result, concerning the transformation of branches by duality;

"v die Anzahl der Wendungspuncte der gegebenen Curve und also die Anzahl der Spitzen ihrer Polar-Curve;

u die Anzahl der Doppeltangenten der gegebenen und also der Doppelpuncte ihrer Polar-Curve;

y die Anzahl der Spitzen der gegebenen Curve und also der Wendungspuncte ihrer Polar-Curve;

x die Anzahl der Doppelspuncte der gegebenen Curve und also der Doppeltangenten ihrer Polar-Curve.

A rigorous proof of this statement can be found in [20];

Theorem 0.3. *Transformation of Branches by Duality*

Let C be a plane projective algebraic curve, with finitely may flexes, then, if γ is a branch of C with character (α, β), such that $\{\alpha, \alpha+\beta\}$ are coprime to $char(L) = p$, the corresponding branch of C^* has character (β, α).

Proof: See [20].
The result is illustrated in figure 13. Here, we see how an inflexion corresponds to a cusp, in duality, which is formally described by a character, ([15]), and, then, how a bitangent, corresponds to a node, this also requires the definition of the duality map, see footnote 14.

Plucker also obtains immediately afterward the formulas, which are still associated to his name;

"Alsdann besteben die folgenden, durch das Princip der Reciprocitat paarwise mit einander verknupften, Gleichungen:

$m = n(n-1) - 2x - 3y$, (1)

$n = m(m-1) - 2u - 3v$, (2)

$v = 3n(n-2) - 6x - 8y$, (3)

$y = 6m(m-2) - 6u - 8v$, (4)

$u = \frac{n(n-2)(n^2-9)}{2} - (2x+3y)(n(n-1)-6) + 2x(x-1) + \frac{9y(y-1)}{2} + 6xy$ (5)

[15]This is explained in Chapter 11, and was a major preoccupation of the geometer Severi

$x = \frac{m(m-2)(m^2-9)}{2} - (2u+3v)(m(m-1)-6) + 2u(u-1) + \frac{9v(v-1)}{2} + 6uv$ (6)
"

Plucker's observation in the "System der Analytischen Geometrie", for the number of inflexions of a nonsingular curve C, follows from (3), $i = 3(m-d) = 3(d(d-1)-d) = 3d(d-2)$. A rigorous proof of this result (3), follows from the class formula, known as Plucker III' in Severi, and Plucker's earlier elementary Plucker formula above. The class formula is given in [20], along with the proofs relevant to the other claims;

Theorem 0.4. *Let C be a normal plane projective curve, not equal to a line, having at most nodes as singularities, with the convention on summation of branches given above and $\{m, n, \rho, d\}$ as defined in the previous lemmas. Then, we obtain the class formula;*

$3m - 3n = \sum_\gamma (\beta(\gamma) - 1)$ (1)

and the genus formula;

$6\rho + 3n - 6 = \sum_\gamma (\beta(\gamma) - 1)$ (2)

and the node formula;

$3n(n-2) - 6d = \sum_\gamma (\beta(\gamma) - 1)$ (3)

In particular, if C has at most ordinary flexes, and i is the number of these flexes, we obtain the class formula, referred to as Plucker III' in [33]*;*

$3m = 3n + i$ (4)

and the genus formula;

$6\rho = i - 3n + 6$ (5)

and the node formula, referred to as Plucker III in [33]*;*

$6d = 3n(n-2) - i$ (6)

Proof: See [20].

It seems unclear exactly how Plucker obtained these results. Again, Plucker formulates some notion of the genus of an algebraic curve, on p215, "Das gesuchte Maximum wird hiernach:

$z = \frac{(n-1)(n-2)}{2}$

as the maximum number of double points, an irreducible curve can possess, with a footnote to the Swiss mathematician Gabriel Cramer, (1704-1752). This is consistent with the degree-genus formula, for a curve, of degree n, having at most d nodes as singularities, $g = \frac{(n-1)(n-2)}{2} - d$, see also footnote 4, as we always have that $g \geq 0$. There are exactly $\frac{(n-1)(n-2)}{2}$ "alcoves", in finite position, formed by a general set of n lines, see [21], Lemma 1.12. This gives a clear connection between the genus g of a complex algebraic curve C, and the number of ovals of real algebraic curves, see the attached figure 14, in which there are 5 lines $\{l_i : 1 \leq i \leq 5\}$, 6 alcoves $\{a_i : 1 \leq i \leq 6\}$ and $g = 6$. The earlier reference by Plucker to Cramer, and his discussion of ovals, suggests that he might have been impressed by this idea from previous stages of his career, and, here, formulates, a precise geometric definition. The interest of Cramer in the dimensions of linear systems of curves, as in his paradox, and the strong connection with Plucker's formulae, seem to motivate Severi's definition of genus. Moreover, the intuition of figure 12, relating nonsingular curves to harmonic arrangements of lines, is the main idea in proving the equivalence of Severi's and Klein's later definition, again see footnote 4. As before, the author would suggest that much of Plucker's, or possibly even Cramer's, work on this subject was unpublished, but, perhaps, later obtained by Severi, who, was, in fact known to have written his major work, "Vorlesungen uber algebraische Geometrie", (1907), in German. Severi's proof of Plucker's formulae relies heavily on his geometric definition of genus, which is preserved by birational maps, see Chapter 11. It seems possible that many of Plucker's unpublished ideas were later rediscovered by Severi, but, perhaps, this would be to discredit the great achievements of this geometer and the later work of the "Italian School" in algebraic surfaces, ([16])

Plucker was also a professor of physics at Bonn, from 1847 to 1868. He produced a further important geometric work, "System der Geometrie des Raumes in neuer analytischer Behandlungsweise", (Dusseldorf), in 1846, which, refined his earlier results. However, this upset a parallel school of synthetic geometry, ([17]), associated with Jakob Steiner, and he mainly concentrated on research into physics after this date. In 1847 he began research into the behaviour of crystals in a magnetic field, establishing results central to a deeper knowledge of magnetic phenomena. At first alone and later with the German physicist Johann W. Hittorf, Plucker investigated the magnetic deflection of cathode rays, ([18]) Together they also made many important discoveries in spectroscopy, anticipating the German chemist Robert Bunsen and the German physicist Gustav R. Kirchhoff, who later announced that spectral lines were characteristic for each chemical substance. In 1862 Plucker pointed out that the same element may exhibit different spectra at different temperatures. According to his pupil J.W. Hittorf, Plucker was the first to identify the three lines of the hydrogen spectrum, which a few months after his death were recognized in the spectrum of solar radiation, and preceded the celebrated experiments of R. Bunsen and G. Kirchhoff in Heidelberg. The discreteness of spectral lines was an important motivation in the later development of quantum physics. Plucker wrote 59 papers on pure physics, published primarily in the "Annalen der Physic

[16] See [2] for a good modern survey of their results in this area.

[17] In which infinitesimals are defined algebraically as "vanishing powers", without the use of coordinates

[18] The earliest version of the cathode ray tube, now used in televisions, was invented by the German physicist Ferdinand Braun in 1897, possibly influenced by the work of the English physicist J. J. Thomson, who identified the electron on the basis of the deflection of cathode rays in both electric and magnetic fields. Plucker was greatly influenced by the English scientist Michael Faraday, (1791-1867), with whom he corresponded, and who may well have motivated his work on magnetic fields. Plucker later studied the phenomena of electrical discharge in evacuated gases. A good modern text on electrodynamics is [8]

und Chemie" and the "Philosophical Transactions of the Royal Society", ([19])

Following Steiners death in 1863, Plucker returned to the study of mathematics with his final work in geometry, "Neue Geometrie des Raumes gegrndet auf die Betrachtung der geraden Linie als Raumelement" (New Geometry of Space Founded on the Treatment of the Straight Line as Space Element). He died before finishing the second volume, which was edited and brought to completion by his gifted young pupil Felix Klein, who served as Pluckers physical assistant from 1866 to 1868, and whom we referred to above. Although Pluckers accomplishments were unacknowledged in Germany, English scientists appreciated his work more than his compatriots did, and, in 1868, he was awarded the Copley Medal.

Other scientists, who, I will argue, used aesthetic ideas in their work, in particular the analysis of wave and periodic motions, are the contemporary French mathematicians Pierre Simon Laplace and Joseph Fourier. Laplace was born in Beaumont-en-Auge, Normandy, in 1749. In 1765, he began to read theology at the University of Caen, where he remained for 5 years. In 1768, he published his first paper, "Recherches sur le calcul integral aux differences infiniment petites, et aux differences finies", (1766-1769, Volume 4, Melanges de Turin), which resulted in correspondence with another famous French mathematician Lagrange. Here, he advocates the use of a finite difference method to find solutions of ordinary differential equations;

"Plusieurs principes du calcul integral aux differences infiniment petites, ont egalement lieu pour les differences finies, ainsi toutte fonction de x, par ex. qui satisfera pour y^x", dans l'equation A, (differential equation omitted, see text (formulation of counterpart) e qui refermera un nombre n, de constantes arbitraires, en fera l'integrale complette. Ce principe qui est de plus grand usage dans le calcul integral aux differences infiniment petites, n'est pas d'un usage moins etendu, dans le calcul aux differences finies." (p301, Section XIV)

There is the use of integral notion, early in the text, and in Section XV, p302, he gives a definition of the finite difference operator and its iterates;

"Comme on a

$\Delta.y^x = y^{x+1} - y^x$

$\Delta^2.y^x = y^{x+2} - 2y^{x+1} + y^x$

$\Delta^3.y^x = y^{x+3} - 3y^{x+2} + 3y^{x+1} - y^x$

..."

[19] His interestingly named study of wave surfaces (1839) and the reflection of light at quadric surfaces, (1847), combine both physics and geometry, and are counted among his 41 mathematical papers.

In Section XV1, p303, he obtains a series solution to a difference equation, by this difference method;

"Soit propose d'integrer l'equation du premier ordre

$X^x = y^x + H^x y^{x+1}$

$y^x = p^{x-1}(A - \Sigma \centerdot [\frac{X^x}{p^{x-1}}]$"

The above passage and the simultaneous use of sum and integral notation, show that Laplace was, at an early age, comfortable with the idea of switching between step-by-step difference methods and ideas from calculus, in solving ordinary differential equations. This was, of course, due to the influence of Newton's (and Liebniz's) use of infinitesimals, rather than limits, in the definition of integration and differentiation, see Chapter 5, a point of view which was unaltered until the 19th century. As we saw there, Newton construed integrals as sums of infinitesimals, hence the sums obtained, by the difference method, of simple differential equations such as $\frac{dy}{dx} = G(x, y(x))$, $(*)$ can be interpreted immediately as integral solutions $y(x) = \int_0^x G(x', y(x'))dx'$, which solve the original differential equation by the Fundamental Theorem of Calculus. These arguments are, in some sense, unacceptable by modern standards, however, now, with the development of nonstandard analysis, it is possible to show these two viewpoints are possible simultaneously, provided that one replaces equality by "infinitely close". The interested reader can find a good discussion of the finite difference method in [25], where a number of analogues of simple calculus results are proved. The nonstandard definition of the integral, which we briefly encountered in Chapter 5, can be found in [27], together with conditions under which the nonstandard integral is infinitely close to the standard integral, Definition 3.26 and Theorem 3.28. A rigorous demonstration of $(*)$ is in [27], Theorem 4.1,([20])

In 1771, Laplace moved to Paris, having supposedly abandoned theology and become an atheist. Intent on becoming a great mathematician, he carried a letter of introduction to Jean le Rond d'Alembert. After gaining a teaching position at the Ecole Militaire, on his recommendation, he threw himself into original research until 1787. In 1778, he published "Un memoire sur le calcul integral", ([21]), and, in 1785, "Theorie des Attractions des Spheroids et de la Figure des Planetes". In this text, he considers the gravitational potential function and shows that it always satisfies the partial differential equation;

"$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$"

This is still known today as Laplace's equation. Later, there is an attempt to solve the equation using power series, which we considered in Chapter 5. Real valued solutions of

[20] It can be shown that, under quite general conditions, solutions to ordinary differential equations, obtained by the difference method, will be infinitely close to the standard solution. However, there are some fairly exceptional cases. We will consider the corresponding problem for partial differential equations below.

[21] Here, he demonstrates that $\int_{\mathcal{R}} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$, predating the development of Cauchy's later complex analysis.

Laplace's equation are known as harmonic functions. In two dimensions, with variables (x, y), a function V is harmonic iff it is the real part of a complex analytic function, ([22]). The Laplacian operator occurs in a number of important partial differential equations, involving more than one spatial variable, ([23]), and can even be generalised to an operator on an arbitrary algebraic curve. Understanding the eigenvalues of this operator, that is solutions to the partial differential equation $\nabla^2 W = \lambda W$, for λ infinitely close to real values, seems, in conjunction with methods, later developed by Laplace and Fourier, to be the clue to using the finite difference method to solve partial differential equations, under quite general conditions, on surfaces, we discuss this question in greater depth below.

Laplace continued his analytical research into the solar system with his "Mechanique Cleste" published in five volumes. The first two volumes, published in 1799, contain methods for calculating the motions of the planets, determining their figures, and resolving tidal problems. In the third volume, published in 1802, (Complete Works, Tome 3), Laplace applies these methods to understand geodesic lines on the surface of the earth. On p118, he defines geodesics in terms of meridians, and rotated overlapping triangles;

"Telle est donc la propriete caracteristique de la courbe tracee par les operations geodesiques. Son premier cote, dont la direction peut etre supposee quelconque, est tangent a la surface de la Terre; son second cote est le prolongement de cette tangente, plie suivant une verticale; son troisieme cote est le prolongement du second cote, plie suivant une verticale, et ainsi de suite"

and observes that such lines are the shortest distance between two points on a surface;

"Ainsi les lignes tracees par les mesures geodesiques ont la propriete d'etre les plus courtes que l'on puisse mener sur la surface de spheroide, entre deux de leurs points quelconques;"

---

[22] The subject of complex analysis was developed later, by the French mathematician Augustin-Louis Cauchy, (1789-1857), who is thought to have essentially proved the famous Cauchy's Integral Theorem in 1814, publishing it entirely in "Memoire sur les integrales definies prises entre des limites imaginaires" (Memorandum on definite integrals taken between imaginary limits), which was submitted to the Acadmie des Sciences on February 28, 1825. Cauchy defined residues in "Sur un nouveau genre de calcul analogue au calcul infinitsimal" (On a new type of calculus analogous to the infinitesimal calculus), Exercices de Mathematique, vol. 1, p. 11 (1826), and proved the residue theorem in Mmoire sur les rapports qui existent entre le calcul des Residus et le calcul des Limites, et sur les avantages qu'offrent ces deux calculs dans la resolution des equations algebriques ou transcendantes (Memorandum on the connections that exist between the residue calculus and the limit calculus, and on the advantages that these two calculi offer in solving algebraic and transcendental equations), presented to the Academy of Sciences of Turin, November 27, 1831. Cauchy is sometimes associated with the introduction of limits, but, this is unclear. It seems, from these publications, that he was ambivalent, and adapted to the new perspective in his later work. His "Cours d'Analyse", (1821), uses $\epsilon - \delta$ notation and infinitesimals.

[23]Examples include the heat equation, the wave equation and Schrodinger's equation, see [16] and [6] for discussions of these equations in the plane. The concept of potential also occurs in close connection with Maxwell's equations and Navier-Stokes equations.

He finally deduces, using an infinitesimal argument, that the second derivative of the geodesic path is perpendicular to the surface, ([24]).;

"Ainsi les lignes tracees par les mesures geodesiques ont la propriete d'etre les plus courtes que l'on puisse mener sur la surface de spheroide, entre deux de leurs points quelconques;"

"on aura donc, par la nature de la verticale et en supposant que $u = 0$ soit l'equation de la surface de la Terre,

$0 = \frac{\partial u}{\partial x} d^2 y - \frac{\partial u}{\partial y} d^2 x$

$0 = \frac{\partial u}{\partial x} d^2 z - \frac{\partial u}{\partial z} d^2 x$

$0 = \frac{\partial u}{\partial y} d^2 z - \frac{\partial u}{\partial z} d^2 y$
"

An important analysis occurs on p131, with the attached figure 15. There is a clear understanding of periodic motion here, with the geodesical paths traced out by circular orbits around a sphere, and inclusion of the sine and cosine functions. Laplace states;

"Mais dans l'hypothesese de la Terre spherique on a;

$\frac{d\phi}{ds} = \frac{sin.\lambda}{cos.\psi}$ $\frac{dd\phi}{ds^2} = \frac{2.sin.\lambda.cos.\lambda}{cos.\psi}.tang.\psi$

$\frac{d\psi}{ds} = cos.\lambda$ $\frac{dd\psi}{ds^2} = -sin.^2\lambda.tang.\psi$"

obtaining expressions for the variation of angle with arc length. In the fourth volume, published in 1805, we obtain further applications (p515) in the study of the satellites of Saturn, Laplace develops his theory of gravitational potential to calculate the eccentricity of their orbits, which he assumes to be elliptical. There is clearly some development of the field, now known as Lagrangean mechanics, a good discussion of this topic can be found in [13]. The fifth volume, published in 1825, contains figure 16, giving a more geometric analysis, ([25])

In 1807, Joseph Fourier published his treatise "Memoire sur la propogation de la chaleur dans les corps solides", which exerted a great influence on Laplace. In Fourier's " Memoire sur la Theorie Analitique de la Chaleur", (1822), p594, which is, presumably, taken from the memoire of 1807, Fourier defines the heat equation (Section 3,p586);

[24]This is equivalent to the property that $\bigtriangledown_{\gamma'}(\gamma') = 0$, for the induced connection, obtained by an orthogonal projection of the covariant derivative $\bigtriangledown_{\overline{\gamma'}}(\overline{\gamma'})$ onto the surface S. It seems likely that Gauss was aware of this interpretation by 1825, when he published "Disquisitiones generales circa superficies curvas". Good modern references on differential geometry are [4] and, for curves and surfaces, [19].

[25]No doubt Laplace was influenced by Newton's earlier theory of planetary motion, in which he gave a derivation of Kepler's laws, in his Philosophiae Naturalis Principia Mathematica of 1687, see Chapter V.

"Or l'equation differentielle de mouvement lineaire de la chaleur est $\frac{dv}{dt} = \frac{k}{C.D}\frac{d^2v}{dx^2}$, et si l'on ecrit $\frac{kt}{C.D}$ au lieu de t, on a $\frac{dv}{dt} = \frac{d^2v}{dx^2}$"

and employs a series solution involving sine functions;

"Nous employons en premier lieu l'expression suivante:

$v = \sum e^{-i^2t} sin(ix)\alpha_i$, $(**)$, ([26]).

en designant par α_i une fonction inconnue du temps t qui contient aussi l'indice i, ([27]);"

Laplace, perhaps motivated by his work we have considered on planetary motion, recognised that Fourier's method of series for solving the heat equation could only apply to a limited region of space as the solutions were periodic. Such a view seems reasonable, as periodic solutions on an infinite line would be a physically unrealistic model of diffusion, and a periodic solution reflects the geometry of a bounded interval. In figure 17, we see how the sine function, restricted to a bounded interval $[-\pi, \pi]$ can be wrapped around a circle, to provide a new continuously differentiable function. This property turns out to be not only physically realistic, ([28]), but mathematically essential, in the sense that the Fourier series of such functions converge uniformly. In this case, a complete analysis of a function in terms of its harmonics can be obtained, see figure 18, ([29]). A proof using infinitesimals is given in [26], ([30]);

Theorem 0.5. *For $g \in C^\infty(\mathcal{S})$, there is a non standard proof of the uniform convergence of its Fourier series.*

Proof: The idea is to obtain the result first using infinitesimals $\frac{1}{\eta}$, the summation being up to η . This turns out to be relatively easy, by the finite difference method, ([31]), without

[26]This is basically correct, the general periodic solution, with values in $\mathcal{C}$, on $\mathcal{R}_{\geq \prime} \times [-\pi, \pi]$ is given by, $v(x,t) = \sum_{n=-\infty}^{\infty} A_n e^{-n^2t} e^{-inx}$, hence the real solutions are given by $\sum_{n=-\infty}^{\infty} a_n e^{-n^2t} sin(nx) + b_n e^{-n^2t} cos(nx)$

[27]Fourier makes a miscalculation, here, by requiring that the terms α_i depend on time t

[28]It turns out that solutions to the closely related wave equation with fixed endpoints have this property, see [29]

[29]The harmonics, α_i in Fourier's notation, refer to the coefficients of the sine/cosine terms in the series, the $sin(ix)$ terms. A formal definition of these can be found in [26], Definition 0.1 and Remarks 0.3. The definition of Fourier coefficients in terms of infinitesimals, infinitely close to the standard Fourier coefficients, is given in Definition 0.8. It seems likely that both ideas were again used simultaneously in the early development of the subject.

[30]The standard proof of this result can be found in [34], based on Dirichlet's memoir of 1829, "Sur la convergence des series trigonometriques qui servent a representer une fonction arbitraire entre des limites donnees". According to [35], Fourier had originally claimed that any periodic function could be written as a series of sines and cosines, this being true only if one construes equality in the measure theoretic sense of "almost everywhere"

[31]This could be classed as a result in the theory of fast or discrete Fourier transforms, see [14], there is some evidence that an analogue of the result could be found in Gauss's "Theoria Interpolationis Methodo

imposing any particular restrictions on g;

$g_\eta(x) = \frac{1}{2}\sum_{m\in\overline{\mathcal{Z}}_\eta} \hat{g}_\eta(m) exp_\eta(\pi i x m)$ $(*)$

The continuously differentiable property is required to show that the tail terms of this series converge like $\frac{1}{m^2}$, hence rapidly enough to ensure that the total of the summands with infinitely large indices is infinitesimal. It follows that the infinitesimal series is infinitely close to the standard series.

It is not clear whether a proof along these lines was known at this time, though it does seem more intuitive than Dirichlet's later result of 1829,([32]) However, as we will now see, Laplace, in his "Memoire sur divers points dAnalyse", (Journal de lEcole Polytechnique, Tome VIII., 229-265, Oeuvres Completes, XIV, 178-214, (1809)) approached the question of finding solutions that diffused indefinitely in space, using his earlier method of finite differences, and obtained the correct result. On p189, he defines the heat equation using both the method of finite differences and as a standard partial differential equation;

"L'equation aux differences finies,

$\Delta^2 y_{x,x'} = \Delta' y_{x,x'}$

se change dans une equation aux differences infiniment petites, en y substituant $\frac{\partial}{\partial x}$ et $\frac{\partial}{\partial x'}$, au lieu des caracteristiques Δ et Δ', (Memoires de l'Academie des Sciences, 1779), et en y changeant $y'_{x,x'}$ en y, on a

$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial x'}$

noting, on p188, the step by step method of solution;

"L'equation precedente aux differences partielles donne

$y_{x,x'+1} = y_{x,x'} + \Delta^2 y_{x,x'}$.

and obtains, on p190, the solution;

$y = \frac{1}{\sqrt{\pi}} \int e^{-z^2} \phi(x + 2z\sqrt{x'}) dz$, $(*)$, ([33])

Nova Tractata", which was published posthumously in Volume 3 of Gauss's collected works, appearing from 1863 to 1871, but predated Fourier's work of 1807.

[32]It is claimed in [1], that Fourier began to tackle the problem, in the style that Dirichlet later used, within a later work of 1822, see below, but the evidence involves limits which can't be found in the original text, though they may just have been introduced at this time, see 22.

[33]Here, $\phi(w)$ denotes the boundary condition $y(0, w)$. The solution is usually now written in the form $\frac{1}{\sqrt{4\pi x'}} \int_{\mathcal{R}} e^{\frac{-(x-w)^2}{4x'}} g(w) dw$. The two expressions are easily seen to be the same by making the substitution $w = x + 2z\sqrt{x'}$

In order to understand how Laplace arrives at this answer, we need to consider his contemporary paper "Memoire sur les Approximations des Formules qui sont Fonctions de Tres Grands Nombres et sur Leur Application aux Probabilites" (Memoire de l'Academie des Sciences, 1809). On p336, he notes;

"Considerons le cas de $i=\frac{1}{2}$; on aura pour l'integrale de equation (u)

$\phi(r,n)=\int\frac{1}{\sqrt{x}}e^{\frac{-x^2}{6}}(acosrx+bsinrx)dx$, ([34])

a et b etant deux constantes arbitraires, et l'integrale etant prise depuis x nul jusqu'a x infini"

The expression on the right is essentially the transform which he is still known for, in this case applied to the function $\frac{1}{\sqrt{x}}e^{\frac{-x^2}{6}}$, ([35]). On p332, Laplace gives the expression;

$\phi'(r,n)=\sqrt{\frac{3}{2\pi}}e^{\frac{-3r^2}{2}}[1-\frac{30}{n}(1-6r^2+3r^3)+\ldots]$

There is a similarity here, with the calculation, for a transform named after Fourier, ([36]), namely that $\mathcal{F}(K)=e^{-tr^2}$, where $K(t,x)=\frac{1}{\sqrt{4\pi t}}e^{\frac{-x^2}{4t}}$, the function appearing in Laplace's earlier solution $(*)$. This calculation, using Fourier transforms, is an essential step in recovering Laplace's solution from the reduction of the original partial differential equation to a set of ordinary differential equations indexed by the space step x, ([37]). Laplace clearly knew how to solve the simpler 1 variable equations from (†), using the difference method, which he presented in 1768, indeed, on p328 of his approximation memoir, he gives the step by step solution;

"on a donc cette equation aux differences partielles finies et infiniment petites

$(p)\phi(r',n-1)-\phi(r'',n-1)=\frac{3}{\sqrt{n}}\phi'(r,n)$, ([38])

The final step, to obtain Laplace's solution, requires the inversion of the Fourier transform, ([39]). Laplace is carrying out a parallel calculation with the Laplace transform, which is more difficult to assess, as a simple inversion theorem is not known, ([40]). Laplace's concern with

[34]The reader should not confuse the function of 2 variables, $\phi(r,n)$, here, with the function of 1 variable ϕ given above.

[35]The precise definition is $\mathcal{L}(f)(r)=\int_0^\infty f(x)e^{-ixr}dx$

[36] We define the Fourier transform by $\mathcal{F}(f)(r)=\int_{\mathcal{R}}f(x)e^{-ixr}dx$; there is a clear connection with the Laplace transform, in that the domain of integration is changed from the real line $\mathcal{R}$ to the half line $(0,\infty)$, although in modern treatments r is usually treated as a complex variable for the Laplace transform.

[37]Namely, if f solves the heat equation, $f_t-f_{xx}=0$, $f(0,w)=g(w)$, then $d\frac{\mathcal{F}(f)}{dt}+x^2\mathcal{F}(f)=0$, (†) $\mathcal{F}(f)(0,x)=\mathcal{F}(g)(x)$, for each $x\in\mathcal{R}$.

[38]The standard solution of (†) is given by $A(x)e^{-tx^2}$, where $A(x)=\mathcal{F}(g)(x)$

[39]The method of solving the heat equation using the difference method, and its relation with the inversion theorem for Fourier transforms can be found in [23]

[40]The interested reader can find the author's result for the inversion of Laplace transforms in [24]

the heat equation arose mainly from considerations in probability, we will return to this point below. Inspired by his memoir, Joseph Fourier wrote the "Thorie du mouvement de la chaleur dans les corps solides", a prize essay, deposited with the Institut on September 28, 1811, and published in Memoires lAcadmie Royale des Sciences de lInstitut de France, years 1819, 1820, 185-556. Unfortunately, this work is difficult to trace (it appears in Fourier's complete works, Tome1), however, the contents are reproduced in " Memoire sur la Theorie Analitique de la Chaleur", (1822). In Ch(IX) Section 1,342-371, (p428-429), Fourier addresses the solution of the heat equation on an infinite line;

"et l'on aura

$\frac{du}{dt} = k\frac{d^2u}{dt^2}$"

obtaining the solution;

"Il en resulte que la valeur de u pourra etre exprimee ainsi:
$u = \int dqQ.cos.qxe^{-kq^2t}$
"

Here, Fourier writes the solution as the (inverse) Fourier transform applied to the function e^{-kq^2t}. The reader should observe the similarity between this expression and the one he obtained in (∗∗), replacing the sum by an integral, and the coefficients, by the (transform) of the boundary condition. This analogy is an important feature of their approach using infinitesimals, indeed, the inversion theorem for Fourier transforms is, from a nonstandard perspective, exactly analogous to the convergence of Fourier series. The following result is taken from [27], the reader can compare this with the previous theorem;

Theorem 0.6. *For $g \in S(\mathcal{R})$, the Fourier Inversion Theorem holds and admits a non standard proof.*

Proof: Again, one obtains the result using infinitesimals $\frac{1}{\eta}$, the nonstandard integral converting to a sum up to η^2.
$g_\eta(x) = \frac{1}{2}\int_{\overline{\mathcal{R}}_\eta} \hat{g}_\eta(r)exp_\eta(\pi ixr)d\lambda_\eta(r)$

The condition that $g \in S(\mathcal{R})$, meaning rapid decay at infinity, analogous to the previous continuous differentiability condition, is required to show that the transform $\hat{g}_\eta(r)$ also decays like $\frac{1}{r^2}$, hence rapidly enough to ensure that the tail of the integral above infinite values is infinitesimal. It follows that the integral of the nonstandard transform is infinitely close to the integral of the standard transformation.

The result is illustrated in figure 19. A single application of the transformation alters the dispersion and amplitude of the function, and a second application alters just the amplitude, the initial function is essential preserved by two applications of the transformation, ([41]).

[41]Precisely, for the definition in footnote 36, $(\mathcal{F})^2(g)(x) = 2\pi g(-x)$, the double transformation is a constant multiple of the mirror image of the original function.

Consideration of the text shows that Fourier arrived at the result by analogy with his previous paper. It seems likely that Laplace and Fourier saw that their two different integral solutions to the heat equation, implied an inversion property for a particular type of function, ([42]), which might imply a general inversion theorem for (Fourier) transforms. On p561, Chapter IX, Fourier states the Inversion Theorem;

"Il faut maintenant, dans le second membre de l'equation;

$fx = \frac{1}{2\pi}\int d\alpha f\alpha \int dpcos.(px - p\alpha)$", ([43])

However, the proof is unclear. It is interesting to speculate whether Laplace and Fourier had unpublished work on this result, using infinitesimals, before Dirichlet's paper. As we noted above, Fourier believed that any function could be decomposed this way, and observes on p558;

"La fonction fx acquiert en quelque sorte, par cette transformation, toutes les proprietes des quantites trigonometriques; les differentiacions, les integrations et la sommation des suites s'appliquent ainsi a des fonctions generales de la meme maniere qu'aux fonctions trigonometriques exponentielles."

We have seen, in the nonstandard proof, that, at the level of infinitesimals, Fourier was correct, and the inversion theorem holds for any function. However, restrictions on the functions are required when recapturing the classical result, ([44]).

Fourier's considerations are guided by his geometric intuition of the propagation of heat. In figure 20, we see his visualisation of how an initial symmetric temperature profile, on part of an infinite line, propagates to an asymmetric function, in order to equalise the zero temperature on the rest of the line;

"Nous considerons d'abord le premier cas, qui est celui ou a chaleur se propage librement dana la ligne infinie dont une partie ab a recu des temperatures initiales quelconques; tous les autres points ayant la temperature initiale o. Si l'on eleve en chaque point de la barre l'ordoonee d'une courbe plane qui represente la temperature actuelle de ce point, on voit qu'apres une certaine valeur du temps t, l' etat du solide est exprime par la figure de'la courbe. Nous designerons par $v = Fx$ l'equation donnee qui correspond a l'etat initial, et nous supposons d'abord pour rendre le calcul plus simple que la figure initiale de Ia courbe, est composee de deux parties symetriques, en sorte que l' on a la condition $Fx = F(-x)$."

This type of visual thinking, in terms of symmetric and asymmetric functions, guides his considerations on series. Fourier observes on p254, see also figure 21, that any function can

[42] $\mathcal{F}^{-1}(\frac{e^{\frac{-x^2}{4x'}}}{\sqrt{4\pi x'}}) = e^{-x'x^2}, \mathcal{F}(e^{-x'x^2}) = \frac{e^{\frac{-x^2}{4x'}}}{\sqrt{4\pi x'}}$

[43] Again, this is nearly correct, if f is real-valued, $f(x) = \frac{1}{2\pi}\int_{\mathcal{R}} d\alpha f(\alpha)\int_{\mathcal{R}} dpcos.(px - p\alpha) + \frac{1}{2\pi}\int_{\mathcal{R}} d\alpha f(\alpha)\int_{\mathcal{R}} dpsin.(px - p\alpha)$, Fourier makes a similar mistake in his definition of Fourier series.

[44] Classical proofs of the inversion theorems and convergence of Fourier Series can be found in [12], with further developments in [10]

be decomposed into a symmetric and asymmetric component,

"La fonction ϕx, developpee en cosinus d'arcs multiples, est representee par une ligne formee de deux arcs egaux places symetriquement de part et d'autre de l'axe des y, dans l'intervalle de $-\pi$ a $+\pi$ (voy. fig. 11); cette condition est exprimee ainsi $\phi x = \phi(-x)$. La ligne qui represente la fonction $\psi(x)$ est au contraire formee dans la meme intervalle de deux ares opposes, ce qu'exprime l'equation

$$\psi x = -\psi(-x)$$

Une fonction quelconque Fx, representee par une ligne tracee arbitrairement dans l'intervalle de $-\pi$ a $+\pi$, peut toujours etre partagee en deux fonctions telles que ϕx et ψx."

later developing a seperate treatment of these in terms of cosines and sines,([45]).

As well as considering the heat equation on an infinite line, Fourier reformulates it on the solid sphere, (Ch V,283-289), the solid cylinder (Ch VI,290-305), the rectangular prism (Ch VII,306-320), and the solid cube (Ch VIII, 321-341). Fourier uses the Laplacian notation we discussed above, to describe the equation in this case;

$$\frac{\partial v}{\partial t} = \frac{K}{C.D}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

and obtains some of the solutions using separation of variables, and reduction to the one dimensional case. The problem of solving the heat equation on surfaces or algebraic curves has always been relevant, and general solutions to this problem are still unknown, ([46]). The general idea for solving partial differential equations (depending on space and time) on surfaces (algebraic curves), and, I think, Fourier and Laplace understood this intuitively, is to determine the eigenvalues for the corresponding Laplacian, ([47]). The exponential terms e^{inx}, with $n \in \mathcal{Z}$, or e^{itx}, with $t \in \mathcal{R}$, are eigenfunctions of the 1-dimensional Laplacian operator $\frac{d^2}{dx^2}$ on a bounded interval or an infinite line, and most of the properties of and proofs around Fourier series or transforms rely on this fact. Fourier himself observes this point on p557;

"On aura donc, en designant par i un nombre entier quelconque

$$\frac{d^{2i}}{dx^{2i}} fx = \pm \int d\alpha f\alpha \int dp p^{2i} cos.(px - p\alpha)$$

[45] see also [29] and [30], in which the uniform convergence of cosine/sine series to such functions is discussed, and an asymmetric reflection method is used to solve the wave equation.

[46] One of the major obstacles to scientists working in quantum physics was finding solutions to the related Schrodinger's equation on surfaces such as the sphere, cube or torus. This would be necessary in order to determine the correct orbits for charged particle paths, by relating the discrete frequencies arising from the generalised Fourier series, we explain this further below, and those arising from classical calculations.

[47] There is a general result, known as the Hodge Theorem, which says that the eigenvalues of the Laplacian are discrete (with finite multiplicity) and any smooth function can be decomposed into a series, with the eigenfunctions, corresponding to these eigenvalues, replacing the classical periodic sine and cosine terms. However, the proof of uniform convergence is unknown.

On ecrit le signe superieure lorsque i est pair, et le signe inferieur lorsque i est impair. On aura en suivant cette meme regle relative au choix du signe

$\frac{d^{(2i+1)}}{dx^{(2i+1)}} fx = \mp \int d\alpha f\alpha \int dpp^{2i+1} sin.(px - p\alpha)$"

With the eigenfunctions in place, one can then develop the series or transform method for solving the equation as we have discussed above, exactly for the 1 dimensional case. The problems of uniform convergence and inversion can be handled analogously using the nonstandard method,([48]).

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[48] The idea being to investigate the discrete set of standard eigenvalues $\{\lambda_i : i \in I\} \subset \mathcal{R}$, and smooth solutions for Laplace's equation $\Delta f = \lambda f$ on a projective algebraic curve C. Using the method of [5], (nonstandard solutions to ordinary differential equations), one can find an (internal) set of distinct nonstandard eigenvalues $\{\mu_i : i \in {}^*I\} \subset {}^*\mathcal{C}$, such that;

(a). Given $i \in I$, there exists $i' \in {}^*I$, with ${}^\circ\mu_{i'} = \lambda_i$.

(b). $Card(A_i) = mult(\lambda_i)$, where $A_i = \{i' \in I : {}^\circ\mu_{i'} = \lambda_i\}$

As the operator ${}^*\Delta$ is almost symmetric, on an internal space V, which includes the lifts $\overline{f}$ of smooth functions $f \in C^\infty(C)$, the corresponding eigenfunctions $\{f_i : i \in {}^*I\}$ are almost orthogonal and form a basis for V. Using the Gramm-Schmidt orthogonalisation procedure, and the method of [26], to control the decay rate of the nonstandard Fourier coefficients, one can show that, for $f \in C^\infty(C)$;

$\overline{f} \simeq \sum_{i \in {}^*I} a_i f_i,\ a_i = \int_C f(x,y)\overline{f_i(x,y)} d\mu(x,y)$

and, taking standard parts, that the Fourier series $f = \sum_{i \in I''} b_i g_i$, where $\{g_i : i \in I''\}$ are the corresponding standard eigenfunctions, taking into account multiplicity, is uniformly convergent.

Using this method, we can define the nonstandard Fourier transform $\hat{\overline{f}} : Sp({}^*\Delta) \rightarrow {}^*\mathcal{C}$, by $\hat{\overline{f}}(\mu_i) = a_i$, to obtain a nonstandard inversion formula, for arbitrary functions;

$\overline{f}(x,y) = \int_{Sp({}^*\Delta)} \hat{\overline{f}}(\mu)\overline{g(\mu,x,y)} d\lambda(\mu)$

where $g(\mu,x,y)$ parametrises the nonstandard eigenfunctions, and $d\lambda(\mu)$ is a counting measure on $Sp({}^*\Delta)$. One can generalise this formula in the case when C is not compact, using the method of [25]. Taking standard parts, we aim to obtain the inversion formula for smooth functions on algebraic curves;

$f(x,y) = \int_{Sp(\Delta)} \hat{f}(\lambda) h(\lambda,x,y) d\rho(\lambda)$

where $h(\lambda,x,y)$ parametrises the standard eigenfunctions, including multiplicity. We can use this technique, combined with the method in [23], to solve partial differential equations on algebraic curves, such as the heat and wave equation, or Schrodinger's equation. Generalising this method, one might approach outstanding questions such as the Atiyah-Singer Index Theorem for noncompact algebraic curves, or provide an alternative approach to the existing theory for projective curves, see [32]. This might involve insights into the geometrical representations of genus g curves we discussed above. The decay rate of Fourier coefficients on an algebraic curve C satisfying certain periodic relations (possibly obtained from the fundamental group) is also addressed by the Ramanujan-Petersson and Weil conjectures, although in simple cases, an ordinary analysis seems to suffice, see [31]. Interest in the Weil conjectures can be traced back to Gauss's "Disquisitiones Arithmeticae", 1798, the modern proof can be found in [7]

Although not a fervent Christian, in his later years, like Newton, Laplace remained curious about the question of God. According to [35], he frequently discussed Christianity with the Swiss astronomer Jean-Frdric-Thodore Maurice, and told Maurice that "Christianity is quite a beautiful thing" and praised its civilizing influence. According to Hahn, "Nowhere in his writings, either public or private, does Laplace deny God's existence." Indeed, he is known to have made the following touching remark to his son, in 1809, "Je prie Dieu qu'il veille sur tes jours. Aie-Le toujours prsent ta pense, ainsi que ton pre et ta mre." (I pray that God watches over your days. Let Him be always present to your mind, as also your father and your mother). Like Laplace, Fourier had an early acquaintance with theology, training for the priesthood and entering the Benedictine abbey of St Benoit-sur-Loire, in 1787, which is one of the surviving examples of the chevet form in France. He became a teacher at the Benedictine college, cole Royale Militaire of Auxerre, where he had studied, in 1790. Plucker's religious views are unknown, although the 5-petalled rose window in the cathedral of Cologne, near to where he was born and worked, may have been a source of inspiration in his work.

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