PIERO DELLA FRANCESCA AND THE TRUE GEOMETRY OF LIGHT

TRISTRAM DE PIRO

Piero della Francesca was born in the Umbrian town of Sansepolcro, the exact year being unknown, but, it is speculated, between 1420 and 1422. There is no doubt that Piero della Francesca was an extremely accomplished mathematican, not only in geometry, but also in algebra. He wrote "Trattato d'Abaco", which deals with radicals and algebra resolving "equations above the second degree that cannot be reduced" and "De quinque corporibus regolaribus", a treatise on Platonic solids. P. Arrighi described Piero as "the greatest mathematician of his era". Vasari also explains how; "In his youth Piero applied himself to mathematics, and although when he became fifteen it was decided that he should be a painter he nevertheless always kept up his earlier studies".

As we mentioned in the previous chapter, he is known to have worked in Florence, between 1439 and 1442, where he came across the geometric constructions of Alberti, in particularly, one point perspective, from his Della pittura [On Painting] (1435). Possibly the first work involving the geometry of projection can be found in Euclid's "Ottica" (Optics), (c 300 BC). Theorem 11 states that, "Tra i piani che stanno sopra l'occhio i (piu) lontani appaiano piu in basso." (Amongst the planes which fall on the eye, the furthest away appear to be the lowest). Euclid's simple proof of this proposition is illustrated in Figure 1; the points x, y and z illustrating the diminishing height of a object as it recedes from the eye, situated at O. Although Euclid does not explicitly make this connection, it was realised, possibly by Alberti, that, as the object recedes to infinity, it converges to the point ∞, the vanishing point, marked on the baseline of Figure 1. Alberti's one-point construction can be found in Figure 2, Euclid's planar figure being represented by the planes Owz and Oxu, which intersect with P, at the vanishing point y, along the normal line between the plane P and the eye, situated at O; the lines zw and xu being perpendicular to the plane P. It seems reasonable to say that Albertian perspective was the basis for the modern subject of linear algebra. The projection, described in Figure 2, has the property that it maps a line l, either to a line l' or to a point q, in the plane P. For a suitable choice of coordinates, the projection pr described in Figure 2, has the property $pr(\alpha x' + y') = \alpha pr(x') + pr(y')$, for a scalar α, which, is the modern definition of a linear map. This is easily visualised by taking the eye O to be at infinity, y to be the origin of the coordinate system in the plane P, with the rays, defining parallel perpendiculars to the plane P. In fact, using the well known "rank-nullity" theorem, see [5] or [4], any linear map can be decomposed into a projection and a linear transformation of the plane to itself.

Towards the end of his life, Piero wrote "De Prospectiva Pingendi", on perspective for painters. Perhaps dissatisfied with the limitations of the linearity of Albertian perspective, it is innovative for being the first scientific work to deal with the laws of perspective applied to curves; indeed it is written as a series of purely geometrical constructions of curves onto a plane. "De Prospectiva Pingendi" was completed before 1482, 10 years before his death in 1492. There has been some speculation that Piero was blind when it was written. The following quotation is from [2];

"According to Vasari, the aged Piero was blind, and in the mid sixteenth century a man still lived who claimed that, as a boy, he had led Piero about Borgo San Sensepolcro by the hand. This story has been doubted, but it must contain more than a grain of truth even though in 1490, two years before his death, Piero still wrote a beautiful hand."

Irrespective of the truth of this claim, there can be no doubt that "De Prospectiva Pingendi" was written at least 5 years before this period, as one can find a record from 5 July 1487, which records that Piero appears before the notary Ser Lionardo di ser Mario Fedeli, elderly yet "sound of mind and body", see [2]. Piero della Francesca was a major influence on the geometric style of later Renaissance artists. In particular, the representation of "smooth" curves in 3-dimensions as curves in a 2-dimensional plane, having at most "nodes as singularities", is clearly reflected in the general drawing style of that period.

In figures 3 and 4, which correspond to figures 61 and 62, of "De Prospectiva Pingendi", we see the geometric idea behind Piero's method. An object, in this case, a pillar, is projected onto the plane of the canvas, corresponding to the vertical line in figure 61. This is done, by choosing an (x, y) grid of points on the object, and drawing lines connecting these points to the eye. The rendering of the object in the plane, is obtained by finding the intersections of these lines with the vertical. Piero, first, finds the intersections of the mesh points in the y-direction, and, then, repeats the construction in the x-direction. Observe that the pillar is being drawn above and to the right. In figure 62, the final result, of connecting the projected points, is shown, and, the calculation is repeated for various distances of the canvas. The interested reader can find a detailed description of this construction, in pp51-55 of the text. Note that, unlike Alberti's construction, the method easily incorporates curved figures.

The geometrical idea discovered by Piero is now referred to as the method of Conic Projections, and was highly influential on the Italian development of algebraic geometry, in the 19th century, particularly the work of Severi, Castelnuovo and Enriques. Severi was born in Arezzo, Italy, on 13 April 1879, and was the major figure of the Italian school. He wrote an article on [3] in [10], perhaps becoming interested in his work, due to the proximity of Sansepolcro to his place of birth. We can describe the construction in a more mathematical language, using the framework of algebraic curves, which we introduced in Chapter 4. If $C \subset P^3$ is an algebraic curve, ([1]), and $P \notin C$, we define $Cone_P(C) = \bigcup_{x \in C} l_{xP}$. Then, we have the following theorem;

Lemma 0.1. *Let $C' \subset P^3$ be a nonsingular projective curve, which is not contained in a plane, then, if $O \in P^3$, and H is a 2-dimensional hyperplane, with $O \notin H$, we have that $(Cone_O(C') \cap H)$ defines a projective algebraic curve C.*

[1] P^3 is the standard terminology for 3-dimensional projective space.

The proof is basically a simple consequence of the fact that $Cone_P(C')$ has dimension 2, H has dimension 2, $Cone_P(C') \neq H$, and the fact that two such surfaces intersect in a curve, the reader should look at [8] for further details. Of course, the proof is visually obvious, see figure 5, but, Piero's treatise is interesting because it provides some of the first insights into how visual and logical thinking are related.

In order to explain one of the most important mathematical results, motivated by this type of optical thinking, we require some terminology. We say that two curves C and C' are birational, if there exist open subsets $U \subset C$, $U' \subset C'$, and an isomorphism between U and U'. We introduced Newton's notion of an infinitesimal in Chapter 4. This notion was intuitively extended by Severi, in his discussion of algebraic curves, see [11], by defining an infinitesimal neighborhood $\mathcal{V}_p(C)$, of a nonsingular curve C, to be a disc of infinitesimal radius, surrounding a point p. ([2].) In [11], Severi combined these two ideas, to give an intuitive description of the local geometry of a, possibly singular, curve. Namely, given a curve C, he first showed that there exists a nonsingular curve C', birational to C, ([3]. This lack of rigour was one of the main criticisms of the Italian geometers approach.), see [11] or [8]. Secondly, given a point $p \in C$, located at a singularity, he defined the branches $\gamma_p^i(C)$, for $1 \leq i \leq n$, centred at p, to be the infinitesimal neighborhoods of the points $\{p_1, \ldots, p_n\}$, corresponding to p, on C'. That this is a good definition, is visually obvious, see figure 6, a rigorous proof can be found in [8]. Severi then defined a node p to be the origin of two branches $\{\gamma_p^1(C), \gamma_p^2(C)\}$, which intersect transversely, that is, their tangent lines are distinct, see figure 7, ([4]).

The main achievement of this visual method of proof, was to show the following theorem, the main idea being due to Severi and Castelnuovo;

Theorem 0.2. *Let $C' \subset P^w$ be a nonsingular projective curve, then C' is birational to a plane projective curve $C \subset P^2$, having at most nodes as singularities.*

Again, the geometric idea is clear, however, a clear formulation of the idea of a branch is required to find a rigorous proof, even to formulate the notion of a node correctly. The proof

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[2]A more rigorous modern formulation can be found in [12]. The proof that this is equivalent to Severi's definition can be found in [6]

[3]More precisely, he showed that C' has no multiple points, that is points p with contact at least 2, (see footnote 4) for any hyperplane H passing through p. This turns out to be equivalent to the algebraic definition of nonsingularity, see Lemmas 3.2 and 4.14 of [6], Lemma 4.14 of [7]

[4]The notion of a tangent line at a branch γ of a plane curve C requires the development of Severi's notion of "contact", (contatto), between a branch γ and a line l, which I refer to as $I_{italian}(p, \gamma, C, l)$, in [8]. Namely, a line l and a branch γ have contact of order m, if, varying the line generically by an infinitesimal amount, defines m distinct intersections along the branch, see figure 8, (compare figure 2 of Chapter 4 for nonsingular points). The tangent line l_γ is defined uniquely by the requirement that the contact has order at least two. That is a good definition depends heavily on the idea of birationality, namely the intersection points are preserved by the isomorphism between open sets, and we can isolate the intersections along the branch γ, by counting them near to the corresponding nonsingular point, see figure 8 below and figure 2 of Chapter 4. That this corresponds to Newton's idea of a branch for nonsingular points, using power series, see Chapter 4, relies heavily on the method developed in [7]. This notion is also required to make the notion of tangent line for a branch precise, as, for nonsingular points, it is simple to define the tangent line algebraically, and a power series representation easily shows that it is unique. to A full description of the possible singularities of a curve C was given by Cayley, in the 19th century.

proceeds by first showing the following lemma, attributed to Castelnuovo. A rigorous proof can be found in [8].

Lemma 0.3. *Let $C' \subset P^3$ and suppose that $\{A, B\}$ are independent generic points of C'. Then if C' is not contained in a plane, the line l_{AB} does not otherwise meet the curve.*

The idea might seem clear from a visual point of view, by considering the shadow of, say, a smooth loop of thread on a surface, and revolving the thread until the shadow has no triple intersections. However, to make this idea geometrically precise seems difficult, as the loop could, in principle, be very complicated. The essence of the proof can be found in figure 9, if there existed a third point $P \in C'$, on the line l_{AB}, then projecting from P onto a hyperplane H, the tangent line l_A would lie in the plane spanned by the tangent line l_B and the line l_{AB}, $(*)$, as the projected point $pr_P(A)$ and $pr_P(B)$ coincide, and, as this point is nonsingular on $pr_P(C')$, so do the projected tangent lines $pr_P(l_A)$ and $pr_P(l_B)$. As A is generic with respect to B, this property $(*)$ holds for all but finitely many points x on the curve, (i.e replace A with x) and entails a strange symmetry. Now, projecting from B, see figure 9, all the tangent lines of the projected curve $pr_P(C')$ pass through $Q = pr_B(l_B)$. It is easily shown that the only curve with this property is a line l, which implies that the original curve C' is contained in a plane $pr^{-1}(l)$.

With this lemma shown, one can then show that the union of trisecant lines on a smooth curve $C' \subset P^3$, that is lines passing through three distinct points, on a smooth curve C' is two dimensional, (†). It is easily shown that every point in P^3 lies on a bisecant line of C', unless C' is contained in a plane, so the dimension of the bisecants is larger than that of the trisecants. In order to prove the theorem, one defines the following sets;

(i). The union of tangent lines to C'.

(ii).The union of bitangent chords to C', that is chords l_{ab}, for which the tangent lines l_a and l_b, at the two ends are parallel.

(iii). The union of osculatory chords of C', that is chords l_{ab}, for which the plane H of maximal contact with C' at a, passes through b.

(iv). The singular cone, which is the union of two dimensional cones $Cone_a(C')$, for the finitely many points which are "inflexions".

(v). Points lying on infinitely many bisecant lines of C'.

Similarly to (†), one can show that all these sets have dimension at most two. Hence, one can pick a point $O \in P^3$, avoiding all of them, as well as the union of trisecant lines. The idea is then to project the curve C' from this point O onto a hyperplane H. As O avoids (v), the projection is birational. As O avoids the trisecant lines, and C' is smooth, there are only finitely many points $\{p_1, \ldots, p_n\}$ on C', which project onto points of C, which are the origin of two branches. As O avoids (iv), these points are not inflexions. As O avoids (i), the projections of these branches do not "cusp". As O avoids (iii), and using (iv), the projections are not inflexions. As O avoids (ii), the tangent lines are distinct, so, by Severi's definition, $\{pr_P(p_1), \ldots, pr_P(p_n)\}$ are nodes. The proof is summarized in figure 10.

The purpose of this chapter has been to demonstrate how the aesthetic ideas of projection, simple fragmentation, in the form of a cross, and, the circle, combine. The notion of a branch can be considered as a microscopic circle surrounding a point on a curve, which could be the origin of a node (the cruciform shape). The idea of projection using conics captures the idea of a macroscopic circle of light emanating from a point and projected onto a surface by shadows. This aesthetic intuition of the interplay between circles and lines, as we have seen, facilitates a deeper understanding of the geometry of curves, which a purely rational, non-visual, analysis could not provide, ([5]).

References

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[5] The reader can find out more about the geometry of curves and surfaces embedded in three dimensional space, in [9] and [1]