A NONSTANDARD APPROACH TO SOLVING N'TH-ORDER, LINEAR, INHOMOGENEOUS ODE'S WITH SMOOTH FUNCTION COEFFICIENTS

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Abstract. We show the Euler method for solving n'th-order, linear, inhomogeneous ODE's with smooth function coefficients converges.

Lemma 0.1. *Let $[a,b]\subset\mathcal{R}$ be a bounded interval, and suppose that, for all $\alpha>0$, $G\in C^{\infty}((a-\alpha,b+\alpha)\times\mathcal{R}^n,\mathcal{R}^n)$. Let $\overline{x}_0\in(\mathcal{R})^n$ be given, $\eta\in{}^*\mathcal{N}$ infinite, and suppose that there exists $K>0$, $L>0$, such that for all $t\in[a,b]$ and $\overline{x}\in\mathcal{R}^n$;*

$||G(t,\overline{x})||_{\mathcal{R}^n}\leq K||\overline{x}||_{\mathcal{R}^n}+L$

Let $\overline{X}:{}^*[a,b]\rightarrow({}^*\mathcal{R})^n$ be defined inductively by;

$\overline{X}(a)=\overline{x}_0$

$\overline{X}(a+\frac{i+1}{\eta})-\overline{X}(a+\frac{i}{\eta})=\frac{1}{\eta}({}^*G)(a+\frac{i}{\eta},\overline{X}(a+\frac{i}{\eta}))\ 0\leq i\leq[\eta(b-a)]-1$

$\overline{X}(a+\tau)=\overline{X}(a+\frac{[\eta\tau]}{\eta})\ \tau\in{}^*[0,b-a)$

$\overline{X}(b)=\overline{X}(b-\frac{1}{\eta})\ (*)$

Then $\overline{X}$ is S-continuous, and letting $\overline{x}={}^{\circ}\overline{X}:[a,b]\rightarrow(\mathcal{R})^n$, we have that;

$\overline{x}(t)-\overline{x}_0=\int_a^t G(s,\overline{x}(s))ds$

In particular, $\overline{x}\in C^{\infty}([a,b])$, ([1]), and solves the differential equation;

[1]In the sense that $\overline{x}|_{(a,b)}\in C^{\infty}(a,b)$, and for all $n\in\mathcal{N}$, there exist $\overline{g_n}\in C[a,b]$, with $\overline{g_n}|_{(a,b)}=(\overline{x}|_{(a,b)})^{(n)}$

$\overline{x}'(t) = G(t, \overline{x}(t))$, *for* $t \in (a,b)$, *with initial condition* $\overline{x}(a) = \overline{x}_0$.

Proof. Assume first that $||G(t,\overline{x})||_{\mathcal{R}^n} \leq c$, for all $(t,\overline{x}) \in [a,b] \times \mathcal{R}^n$. We have that, for $\{\tau, \tau'\} \subset {}^*[a,b]$, $||\overline{X}(\tau') - \overline{X}(\tau)||_{({}^*\mathcal{R})^n} \leq c|\tau' - \tau|$. This follows, by internal induction, see Lemma 2.12(ii) of [2]. We have that $||\overline{X}(\tau) - \overline{X}(\tau)||_{({}^*\mathcal{R})^n} = 0$, and, if $||\overline{X}(\tau + \frac{i}{\eta}) - \overline{X}(\tau)||_{({}^*\mathcal{R})^n} \leq \frac{ci}{\eta}$;

$||\overline{X}(\tau + \frac{i+1}{\eta}) - \overline{X}(\tau)||_{({}^*\mathcal{R})^n}$

$\leq ||\overline{X}(\tau + \frac{i+1}{\eta}) - \overline{X}(\tau + \frac{i}{\eta})||_{({}^*\mathcal{R})^n}$

$+||\overline{X}(\tau + \frac{i}{\eta}) - \overline{X}(\tau)||_{({}^*\mathcal{R})^n}$

$\leq ||\frac{1}{\eta}({}^*G)(a + \frac{i}{\eta}, \overline{X}(a + \frac{i}{\eta}))||_{({}^*\mathcal{R})^n} + \frac{ci}{\eta} = \frac{c}{\eta} + \frac{ci}{\eta} = \frac{c(i+1)}{\eta}$

Then $\overline{X}$ is S-continuous, in the sense that, if $\{\tau'', \tau'''\} \subset {}^*[a,b]$, with $\tau'' \simeq \tau'''$, then $||\overline{X}(\tau'') - \overline{X}(\tau''')||_{({}^*\mathcal{R})^n} \simeq 0$. It follows that $\overline{x} = {}^\circ\overline{X} \in C([a,b])$. Now let $F : {}^*[a,b] \rightarrow ({}^*\mathcal{R})^n$ be defined by;

$F_1(\tau) = {}^*G(\tau, \overline{X}(\tau))$

$F(a + \tau) = F_1(a + \frac{[\eta\tau]}{\eta})$, $\tau \in {}^*[0, b-a)$

$F(b) = F(b - \frac{1}{\eta})$

Then, for all $\tau \in {}^*[0, b-a]$, we have, using the fact that $\overline{X}$ and ${}^*G$ are S-continuous;

$F(a + \tau) = F_1(a + \frac{[\eta\tau]}{\eta})$

$= {}^*G(a + \frac{[\eta\tau]}{\eta}, \overline{X}(a + \frac{[\eta\tau]}{\eta}))$

$\simeq G({}^\circ(a + \frac{[\eta\tau]}{\eta}), {}^\circ(\overline{X}(a + \frac{[\eta\tau]}{\eta})))$

$= G({}^\circ(a + \frac{[\eta\tau]}{\eta}), (\overline{x}({}^\circ(a + \frac{[\eta\tau]}{\eta}))))$

so that $F(\tau) \simeq G({}^\circ\tau, \overline{x}({}^\circ\tau))$, for $\tau \in {}^*[a,b]$. $(**)$

Moreover, F is measurable with respect to the $*$-algebra $\mathcal{C}_\eta$ and, as F is bounded, F is S-integrable. It follows that, using the definition $(*)$, Remarks 3.10, Theorems 3.20 of [2], Theorem 3.14 of [1] and $(**)$, that if $t \in [a,b]$;

$$\overline{x}(t) = {}^\circ\overline{X}(t) = {}^\circ\overline{X}(a + \tfrac{[(t-a)\eta]}{\eta})$$

$$= {}^\circ(\tfrac{1}{\eta}{}^*\textstyle\sum_{i=0}^{[(t-a)\eta]-1}{}^*G(a+\tfrac{i}{\eta}, \overline{X}(a+\tfrac{i}{\eta}))) + \overline{X}(a)$$

$$\overline{x}(t) - \overline{x}_0$$

$$= {}^\circ(\textstyle\int_a^t F(\tau)d\lambda_\eta(\tau))$$

$$= \textstyle\int_a^t {}^\circ F(\tau)dL(\lambda_\eta)(\tau)$$

$$= \textstyle\int_a^t G({}^\circ\tau, \overline{x}({}^\circ\tau))d\mu$$

$$= \textstyle\int_a^t G(s, \overline{x}(s))d\mu(s)$$

where $d\mu$ denotes Lebesgue measure, λ_η is the counting measure on ${}^*[a,b]$, with respect to the $*$-algebra $\mathcal{C}_\eta$, and $L(\lambda_\eta)$ is the corresponding Loeb measure. By the Fundamental theorem of Calculus, and the condition on G, we obtain the final claim.

For the general case, we claim that that there exists a finite real constant $C > 0$, such that $||\overline{X}(\tau)||_{({}^*\mathcal{R})^n} \leq C$, for all $\tau \in {}^*[a,b]$. We have that, for $0 \leq i \leq [\eta(b-a)]$, $||\overline{X}(a+\frac{i}{\eta})||_{({}^*\mathcal{R})^n} \leq (1+\frac{K}{\eta})^i||\overline{x}_0||_{({}^*\mathcal{R})^n} + L\frac{i}{\eta}$, $(***)$. By internal induction; we have that $(***)$ holds for $i = 0$, suppose true for $0 \leq i_0 < [\eta(b-a)]$. Then, using transfer of the boundedness assumption on G to $({}^*G)$;

$$||\overline{X}(a+\tfrac{i_0+1}{\eta})||_{({}^*\mathcal{R})^n}$$

$$= ||\overline{X}(a+\tfrac{i_0}{\eta}) + \tfrac{1}{\eta}({}^*G)(a+\tfrac{i_0}{\eta}, \overline{X}(a+\tfrac{i_0}{\eta}))||_{({}^*\mathcal{R})^n}$$

$$\leq (1+\tfrac{K}{\eta})^{i_0}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(\tfrac{i_0}{\eta}) + \tfrac{1}{\eta}[K((1+\tfrac{K}{\eta})^{i_0}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(\tfrac{i_0}{\eta})) + L]$$

$$= (1+\tfrac{K}{\eta})^{i_0+1}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(\tfrac{i_0}{\eta^2}) + \tfrac{L}{\eta} \leq L(\tfrac{i_0+1}{\eta})$$

giving $(***)$. Hence, for all $\tau \in {}^*[a,b]$, $||\overline{X}(\tau)||_{({}^*\mathcal{R})^n} \leq (1+\frac{K}{\eta})^{\eta([b-a+1])}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(\frac{[\eta(b-a)]}{\eta})$. We have that $(1+\frac{K}{\eta})^\eta \simeq e^K$, as $lim_{n\rightarrow\infty}(1+\frac{x}{n})^n = e^x$, for $x \in \mathcal{R}$. Hence;

$||\overline{X}(\tau)||_{({}^*\mathcal{R})^n} \leq e^{[K+1][b-a+1]}||\overline{x}_0||_{({}^*\mathcal{R})^n} + L(b-a) = C$

Now, clearly, by compactness, there exists a finite real constant $F > 0$, such that;

$||G(t,\overline{x})||_{({}^*\mathcal{R})^n} \leq F$, for $(t,\overline{x}) \in {}^*[a,b] \times ({}^*B)(\overline{0}, C)$

Then, we can use the proof in the previous part, to give the result, as $\overline{X}(\tau) \in ({}^*B)(\overline{0}, C)$, for $\tau \in {}^*[a,b]$.

□

Lemma 0.2. *Let $[a,b] \subset \mathcal{R}$ be a bounded interval, $n \in \mathcal{N}$, and, for all $\alpha > 0$, $\{c_0, \ldots, c_{n-1}, d_0\} \subset C^\infty(a-\alpha, b+\alpha)$. Let $\overline{x}_0 \in \mathcal{R}^n$ be given, then there exists $x \in C^\infty([a,b])$ such that;*

$\frac{d^n x}{dt^n} + c_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots c_1(t)\frac{dx}{dt} + c_0(t)x + d_0 = 0$, $t \in (a,b)$

and $x(a) = \overline{x}_0^1, \ldots, x^{n-1}(a) = \overline{x}_0^n$ (†)

Moreover, for $s \in [a,b]$, $0 \leq j \leq n-1$, $\frac{d^j x}{dt^j}(s) = x_{j+1}(s) = {}^\circ X_{j+1}(s)$, where, for $1 \leq k \leq n$, $X_k(s) = (pr_k \circ \overline{X})(s)$, $x_k(s) = (pr_k \circ \overline{x})(s)$ and $\overline{X} : {}^*[a,b] \rightarrow {}^*\mathcal{R}^n$ is defined inductively, as in Lemma 0.1, with G given by;

$G(s, x_1, \ldots, x_n) = (x_2, \ldots, x_n, -c_{n-1}(s)x_n \ldots - c_1(s)x_2 - c_0 x_1(s) - d_0)$

Proof. Let $M = max_{0 \leq i \leq n-1}(||c_i||_{C[a,b]}, ||d_0||_{C[a,b]})$. Then;

$||G(t, x_1, \ldots, x_n)||_{\mathcal{R}^n}$

$\leq \sum_{2 \leq i \leq n} |x_i| + M((\sum_{1 \leq i \leq n} |x_i|) + 1)$

$\leq (M+1)\sum_{1 \leq i \leq n} |x_i| + M$
$\leq (M+1)||\overline{x}||_{\mathcal{R}^n} + M$

Hence, G satisfies the conditions of Lemma 0.1. Therefore, if $\overline{x} = {}^\circ\overline{X}$, then $\overline{x}$ solves the differential equation;

$\overline{x}'(t) = G(t, x_1(t), \ldots, x_n(t))$

$= (x_2(t), \ldots, x_n(t), -c_{n-1}(t)x_n \ldots - c_1(t)x_2(t) - c_0(t))$, $t \in (a,b)$ $(**)$

Setting $x(t) = x_1(t)$, we obtain that $x'(t) = (x_1)'(t) = x_2(t), \ldots, (x_j)'(t) = x_{j+1}(t)$, for $1 \leq j \leq n-1$, hence, $\frac{d^j x}{dt^j}(s) = x_{j+1}(t)$, for $1 \leq j \leq n-1$ and;

$$(x_n)'(t) = -c_{n-1}(s)x_n(t) \ldots - c_1(t)x_2(t) - c_0(t)x_1(t) - d_0(t)$$

$$\frac{d^n x}{dt^n} = (\frac{d^{n-1}x}{dt^{n-1}})'(s) = -c_{n-1}(s)\frac{d^{n-1}x}{dt^{n-1}}(s) \ldots - c_1(t)\frac{dx}{dt} - c_0(t)x(t) - d_0(t)$$
$(*)$

as required. □

Remarks 0.3. *Given an ODE of the form* $(\dagger)$, *with solution* $\overline{x}$, *and* $\epsilon \in \mathcal{R}_{>0}$;

$$S_\epsilon = \{\eta : (\forall \overline{y} \in {}^*[a,b])(||^*(\overline{x})(\overline{y}) - \overline{X}_\eta(\overline{y})||_{({}^*\mathcal{R})^n} < \epsilon)\}$$

is an internal set, which includes all infinite positive integers $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$. *By overflow, see* [2], *there exists a positive integer* $B(\epsilon) \in \mathcal{N}^{>0}$, *with* $S_\epsilon \supset \mathcal{N}^{\geq B(\epsilon)}$. *By transfer, it follows that* $max_{t\in[a,b]}||\overline{x}(t) - \overline{Y}_n(t)||_{({}^*\mathcal{R})^n} < \epsilon$, *for* $n \geq B(\epsilon)$, *where* $\overline{Y}_n$ *is given by the* $2 + [n(b-a)]$-*step discrete algorithm, given on* $[a,b]$ *by* G *and* $(*)$ *of Lemma 0.1. This guarantees the convergence of computer programs based on the Euler method, for solving ODE's of the form (dag), such as ode45.*

References

[1] A Non-Standard Representation for Brownian Motion and Ito Integration, R. Anderson, Israel Journal of Mathematics, Volume 25, (1976).

[2] Applications of Nonstandard Analysis to Probability Theory, T. de Piro (2013)

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