AN APPLICATION OF FOURIER ANALYSIS TO RIEMANN SUMS

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Abstract. We develop a method for calculating Riemann sums using Fourier analysis.

1. Poisson Summation Formula

Definition 1.1. *If $f \in L^1(\mathcal{R})$, we define;*

$(f)^\wedge(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx$

$(f)_-(y) = f(-y)$

$(f)^\vee(y) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixy}dx$

and, if $g \in L^1([0,1])$, $m \in \mathcal{Z}$, we define;

$(g)^\wedge(m) = \int_0^1 g(x)e^{-2\pi ixm}dx$

Remarks 1.2. *If $f \in \mathcal{S}(\mathcal{R})$, we have that;*

$f(x) = \int_{-\infty}^{\infty} (f)^\wedge(y)e^{2\pi ixy}dy$, $(x \in \mathcal{R})$

and, if $g \in C^\infty([0,1])$, ([1]), the series;

$\sum_{m \in \mathcal{Z}} (g)^\wedge(m)e^{2\pi ixm}$

converges uniformly to g on $[0,1]$. See [4],[2] *and* [3].

Also observe that $(f)^\vee = (f_-)^\wedge$ and $(f)^\wedge = (f_-)^\vee$.

Theorem 1.3. *Let $f \in \mathcal{S}(\mathcal{R})$, and let;*

[1] By which we mean that $g|_{(0,1)} \in C^\infty(0,1)$, and there exist $\{g_k \in C[0,1] : k \in \mathcal{Z}_{\geq 0}\}$, such that $g_k|_{(0,1)} = g^{(k)}$, and $g_k(0) = g_k(1)$.

$g(y)=\sum_{m\in\mathcal{Z}} f(y+m)$, $(y\in[0,1])$

Then $g\in C^{\infty}([0,1])$ *and the series*

$\sum_{m\in\mathcal{Z}}(f)^{\wedge}(m)e^{2\pi iym}$

converges uniformly to g *on* $[0,1]$.

In particular;

$\sum_{m\in\mathcal{Z}} f(m)=\sum_{m\in\mathcal{Z}}(f)^{\wedge}(m)$

Proof. Observe that, as $f\in\mathcal{S}(\mathcal{R})$, for $y_0\in[0,1]$, $r\in\mathcal{Z}_{\geq 0}$, $n\geq 2$;

$\sum_{m\in\mathcal{Z}}|\frac{d^r f}{dy^r}|_{y_0+m}|$

$\leq\sum_{m\in\mathcal{Z}}\frac{C_{r,n}}{(1+|y_0+m|^n)}$

$\leq\sum_{m\in\mathcal{Z}}\frac{C_{r,n}}{(1+|m|^n)}$

$\leq C_{r,n}+2C_{r,n}\sum_{m\geq 1}\frac{1}{m^n}$

$\leq C_{r,n}+2C_{r,n}(1+[\frac{y^{-n+1}}{-n+1}]_1^{\infty}$

$=C_{r,n}(1+2(1+\frac{1}{n-1})\leq 5C_{r,n}$ $(*)$

where $C_{r,n}=sup_{w\in\mathcal{R}}(|w|^n\frac{d^r f}{dx^r}|_w)$

Suppose, inductively, that $\frac{d^r g}{dy^r}|_{y_0}=\sum_{m\in\mathcal{Z}}\frac{d^r f}{dy^r}|_{y_0+m}$, for $y_0\in[0,1]$, ([2]), then, using $(*)$, we have, for $r\geq 1$, that;

$\frac{d^{r+1}g}{dy^{r+1}}|=\frac{d}{dx}(\sum_{m\in\mathcal{Z}}\frac{d^r f_m}{dy^r})=\sum_{m\in\mathcal{Z}}\frac{d^{r+1}f_m}{dy^{r+1}}$

where $f_m(x)=f(x+m)$, for $m\in\mathcal{Z}$. Moreover, for $r\geq 0$;

$\frac{d^r g}{dy^r}|_0=\sum_{m\in\mathcal{Z}}\frac{d^r f_m}{dy^r}|_0=\sum_{m\in\mathcal{Z}}\frac{d^r f_m}{dy^r}|_1=\frac{d^r g}{dy^r}|_1$

It follows that $g\in C^{\infty}[0,1]$. Moreover, we have that, for $n\in\mathcal{Z}$;

[2]Given $\frac{d^r g}{dy^r}$, we interpret $\frac{d^{r+1}g}{dy^{r+1}}|_0=lim_{h\rightarrow 0,+}\frac{1}{h}(\frac{d^r g}{dy^r}|_h-\frac{d^r g}{dy^r}|_0)$

$(g)^\wedge(n) = \int_0^1 g(y)e^{-2\pi iyn}dx$

$= \int_0^1 (\sum_{m\in\mathcal{Z}} f(y+m))e^{-2\pi iyn}dx$

$= \int_0^1 (\sum_{m\in\mathcal{Z}} f(y+m))e^{-2\pi i(y+m)n}dx$

$= \int_{-\infty}^{\infty} f(y)e^{-2\pi iyn}dx = (f)^\wedge(n)$

Using Remark 1.2, the series;

$\sum_{m\in\mathcal{Z}} \hat{f}(m)e^{2\pi iym}$

converges uniformly to g on $[0,1]$ as required.

□

Lemma 1.4. *If $h \in C^2(\mathcal{R})$, and there exists $C \in \mathcal{R}$, with;*

$sup_{x\in\mathcal{R}}(|x|^2|h(x)|, |x|^2|h'(x)|, |x|^2|h''(x)|) \leq C$

then the Inversion theorem holds for h. That is $(h)^\wedge \in L^1(\mathcal{R})$ and;

$h(x) = \int_{\mathcal{R}}(h)^\wedge(y)e^{2\pi ixy}dy \ (x \in \mathcal{R})$

Proof. The result follows from inspection of the proof in [2], see Remark 0.4.

□

Lemma 1.5. *If h satisfies the conditions of Lemma 1.4, and $f = (h)^\vee$, then $(f)^\wedge = h$.*

Proof. As h satisfies the conditions of Lemma 1.4, so does h_-, and, therefore, the inversion theorem holds for h_-. Then;

$((h_-)^\wedge)^\vee = (((h_-)^\wedge)^\wedge)_- = (h_-)$

therefore;

$(((h_-)^\wedge)^\wedge) = h$. As $f = h^\vee = (h_-)^\wedge$, we have that;

$(f)^\wedge = (h_-)^\wedge)^\wedge = h$

□

Lemma 1.6. *Let f be given by Lemma 1.5. Then, if there exists $D \in \mathcal{R}$, with;*

$$sup_{x\in\mathcal{R}}(|x|^4|h(x)|, |x|^4|h'(x)|, |x|^4|h''(x)|) \leq D$$

we have that $f \in C^2(\mathcal{R})$, and, moreover, there exists a constant $F \in \mathcal{R}$, such that;

$$sup_{y\in\mathcal{R}}(|y|^2|f(y)|, |y|^2|f'(y)|, |y|^2|f''(y)|) \leq F.$$

Proof. Letting $E = ||h|_{[-1,1]}||_{C[-1,1]}$, we have that, for $y \in \mathcal{R}$, $|x| \geq 1$;

$$|h(x)e^{2\pi ixy}| = |h(x)| \leq \frac{D}{|x|^4} \leq \frac{D}{|x|^2}$$

$$|2\pi ixh(x)e^{2\pi ixy}| = 2\pi|x||h(x)| \leq \frac{2\pi D}{|x|^3} \leq \frac{2\pi D}{|x|^2}$$

$$|-4\pi^2x^2h(x)e^{2\pi ixy}| = 4\pi^2|x|^2|h(x)| \leq \frac{4\pi^2 D}{|x|^2} \leq \frac{4\pi^2 D}{|x|^2}$$

and, for $y \in \mathcal{R}$, $|x| \leq 1$;

$$|x|^2|h(x)e^{2\pi ixy}| \leq |h(x)| \leq E$$

$$|x|^2|2\pi ixh(x)e^{2\pi ixy}| \leq 2\pi|h(x)| \leq 2\pi E$$

$$|x|^2|-4\pi^2x^2h(x)e^{2\pi ixy}| \leq 4\pi^2|h(x)| \leq 4\pi^2 E$$

Hence;

$$sup_{x\in\mathcal{R}}\{|x|^2|h(x)e^{2\pi ixy}|, |x|^2|2\pi ixh(x)e^{2\pi ixy}|, |x|^2|-4\pi^2x^2h(x)e^{2\pi ixy}|\}$$

$$\leq 4\pi^2 max(D, E)$$

and $\{h(x)e^{2\pi ixy}, 2\pi ixh(x)e^{2\pi ixy}, -4\pi^2x^2h(x)e^{2\pi ixy}\} \subset C(\mathcal{R})$. It follows that, for $y_0 \in \mathcal{R}$, we can differentiate under the integral sign, to obtain that $\{f(y_0), f'(y_0), f''(y_0)\}$ are all defined. By the DCT, using the fact that $-4\pi^2x^2h(x) \in L^1(\mathcal{R})$, we obtain that $f'' \in C(\mathcal{R})$, hence, $f \in C^2(\mathcal{R})$. Differentiating by parts, using the fact that;

$$\{h, h', h'', xh, xh', xh'', x^2h, x^2h', x^2h''\} \subset (L^1(\mathcal{R}) \cap C_0(\mathcal{R}))$$

by the hypotheses of Lemma 1.4 and this Lemma, we have that;

$(h'')^{\vee} = -4\pi y^2 (h)^{\vee} = -4\pi y^2 f$

$(4\pi i h' + 2\pi i x h'')^{\vee} = ((2\pi i x h)'')^{\vee} = -4\pi y^2 (2\pi i x h)^{\vee} = -4\pi y^2 f'$

$(8\pi^2 h + 16\pi^2 x h' + 4\pi^2 x^2 h'')^{\vee} = ((4\pi^2 x^2 h)'')^{\vee}$

$= -4\pi y^2 (4\pi^2 x^2 h)^{\vee} = -4\pi y^2 f''$, $(*)$

We have, by $(*)$, for $|y| \geq 1$, that;

$|f(y)| \leq \frac{|(h'')^{\vee}(y)|}{4\pi y^2} \leq \frac{||h''||_{L^1(\mathcal{R})}}{4\pi y^2} \leq \frac{\frac{2D}{3}+2E''}{4\pi y^2}$

$|f'(y)| \leq \frac{|(4\pi i h' + 2\pi i x h'')^{\vee}(y)|}{4\pi y^2}$

$\leq \frac{||(4\pi i h' + 2\pi i x h'')||_{L^1(\mathcal{R})}}{4\pi y^2}$

$\leq \frac{2||h'||_{L^1(\mathcal{R})} + ||x h''||_{L^1(\mathcal{R})}}{2y^2}$

$\leq \frac{2(\frac{2D}{3}+2E') + D + 2E''}{2y^2}$

$|f''(y)| \leq \frac{|(8\pi^2 h + 16\pi^2 x h' + 4\pi^2 x^2 h'')^{\vee}(y)|}{4\pi y^2}$

$\leq \frac{||(8\pi^2 h + 16\pi^2 x h' + 4\pi^2 x^2 h'')||_{L^1(\mathcal{R})}}{4\pi y^2}$

$\leq \frac{2\pi ||h||_{L^1(\mathcal{R})} + 4\pi ||x h'||_{L^1(\mathcal{R})} + \pi ||x^2 h''||_{L^1(\mathcal{R})}}{y^2}$

$\leq \frac{2\pi(2\frac{D}{3}+2E) + 4\pi(D+2E') + \pi(2D+2E'')}{y^2}$

where $E' = ||h'|_{[-1,1]}||_{C[-1,1]}$ and $E'' = ||h''|_{[-1,1]}||_{C[-1,1]}$

For $|y| \leq 1$, we have that;

$|f(y)| \leq ||h||_{L^1(\mathcal{R})} \leq \frac{2D}{3} + E$

$|f'(y)| \leq ||(2\pi i x h)||_{L^1(\mathcal{R})} \leq 2\pi D + 2E$

$|f''(y)| \leq ||-4\pi^2 x^2 h||_{L^1(\mathcal{R})} \leq 8\pi^2 D + 2E$

Hence, we can take $F = max(8\pi^2 D + 2E, \frac{22\pi D}{3} + 4\pi E + 8\pi E' + 2\pi E'')$

□

Definition 1.7. *Let f be given by satisfying the conditions of Lemmas 1.5 and 1.6, we let;*

$g(y)=\sum_{m\in\mathcal{Z}} f(y+m)$, $(y\in[0,1])$

Lemma 1.8. *Let g be given by Definition 1.7, then $g\in C^2[0,1]$.*

Proof. Using Lemma 1.6 and Weierstrass' M-test, we have that the series;

$\sum_{m\in\mathcal{Z}} f(y+m)$, $\sum_{m\in\mathcal{Z}} f'(y+m)$, $\sum_{m\in\mathcal{Z}} f''(y+m)$

are uniformly convergent on $[0,1]$. It follows, that $g\in C^2(0,1)$, and clearly;

$g'_+(0)=\sum_{m\in\mathcal{Z}} f'(m)=\sum_{m\in\mathcal{Z}} f'(m+1)=g'_-(1)$

hence, $g\in C^2[0,1]$.

□

Lemma 1.9. *Let $f\in L^1(\mathcal{R})$, such that;*

$g(y)=\sum_{m\in\mathcal{Z}} f(y+m)$

is defined, for $y\in[0,1]$. Then, if $g\in C^2[0,1]$, we have that the series $\sum_{m\in\mathcal{Z}}(f)^\wedge(m)e^{2\pi iym}$ converges uniformly to g on $[0,1]$. In particular;

$\sum_{m\in\mathcal{Z}} f(m)=\sum_{m\in\mathcal{Z}}(f)^\wedge(m)$

Proof. Following through the calculation in Theorem 1.3, we have that $g\in L^1([0,1])$, and $(g)^\wedge(m)=(f)^\wedge(m)$, for $m\in\mathcal{Z}$. Using the result of [3] or [4], we obtain the second part, the final claim is clear.

□

Lemma 1.10. *Let f be given by satisfying the conditions of Lemmas 1.5 and 1.6, with respect to h, then;*

$\sum_{m\in\mathcal{Z}} f(m)=\sum_{m\in\mathcal{Z}} h(m)$

Proof. Using Lemmas 1.5 and 1.6, we have that $g\in C^2[0,1]$, where g is defined by 1.8, and $(f)^\wedge(m)=h(m)$, for $m\in\mathcal{Z}$. By Lemmas 1.8

and 1.9, we have that;

$\sum_{m\in\mathcal{Z}} f(m) = \sum_{m\in\mathcal{Z}} (f)^\wedge(m)$

Hence;

$\sum_{m\in\mathcal{Z}} f(m) = \sum_{m\in\mathcal{Z}} h(m)$

as required.

□

Lemma 1.11. *If $s \in \mathcal{Z}_{\geq 2}$, s even, then;*

$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{(-1)^{\frac{s+2}{2}}(2\pi)^s B_s}{2(s!)}$

Proof. The proof of this result can be found in [4]. □

Definition 1.12. *If $s \in \mathcal{C}$, with $Re(s) \geq 4$, and $r \in \mathcal{Z}_{\geq 1}$, we define;*

$h_{s,r}(x) = \frac{1}{x^s}$, $(x \geq r)$

$h_{s,r}(x) = \frac{(-1)^s}{x^s} = \frac{e^{-i\pi s}}{x^s}$, $(x \leq -r)$

Remarks 1.13. *$h_{s,r}$ is symmetric, that is $h_{s,r}(x) = h_{s,r}(-x)$, for $|x| \geq r$.*

Lemma 1.14. *There exists a polynomial $p_{s,r}$ of degree $2r+3$, with the properties;*

(i). $p_{s,r}$ is symmetric, that is $p_{s,r}(x) = p_{s,r}(-x)$, for $x \in \mathcal{R}$.

(ii). $p_{s,r}(n) = \frac{1}{n^s}$, for $1 \leq n \leq r$.

(iii). $p_{s,r}^{(k)}(r) = h_{s,r}^{(k),+}(r)$, $(0 \leq k \leq 2)$

(iv). $p_{s,r}^{(k)}(-r) = h_{s,r}^{(k),-}(-r)$, $(0 \leq k \leq 2)$

Proof. We let, for $1 \leq j \leq 1+r$, $1 \leq k \leq r$;

$$\overline{A}_r = \begin{pmatrix} 1 & 1 & \dots & 1 & \dots & 1 \\ 1 & 2^2 & \dots & 2^{2j} & \dots & 2^{2(r+1)} \\ \dots \\ 1 & k^2 & \dots & k^{2j} & \dots & k^{2(r+1)} \\ \dots \\ 1 & r^2 & \dots & r^{2j} & \dots & r^{2(r+1)} \\ 0 & 2r & \dots & 2jr^{2j-1} & \dots & 2(r+1)r^{2r+1} \\ 0 & 2 & \dots & 2j(2j-1)r^{2j-2} & \dots & 2(r+1)(2r+1)r^{2r} \end{pmatrix}$$

$$\overline{b}_{s,r} = \begin{pmatrix} 1^{-s} \\ 2^{-s} \\ \dots \\ k^{-s} \\ \dots \\ r^{-s} \\ -sr^{-(1+s)} \\ s(s+1)r^{-(2+s)} \end{pmatrix}$$

We have that $det(\overline{A}_r) \neq 0$, hence, we can solve the equation $\overline{A}_r(\overline{a}_{s,r}) = \overline{b}_{s,r}$. Let $p_{s,r}(x) = \sum_{j=0}^{r+1} (\overline{a}_{s,r})_{(j+1)} x^{2j}$. We have, by construction, that $p_{s,r}(-x) = p_{s,r}(x)$, and $p_{s,r}^{(k)}(r) = h_{s,r}^{(k),+}(r)$. As both $p_{s,r}$ and $h_{s,r}$ are symmetric, we also have that, $p_{s,r}^{(k)}(r) = h_{s,r}^{(k),-}(-r)$, as required.

□

Definition 1.15. *We define;*

$g_{s,r}(x) = h_{s,r}(x)$, *(if* $|x| \geq r$*)*

$g_{s,r}(x) = p_{s,r}(x)$, *(if* $|x| \leq r$*)*

Lemma 1.16. *We have that* $g_{s,r} \in C^2(\mathcal{R})$*,* $g_{s,r}$ *is symmetric, and, moreover, the hypotheses of Lemmas 1.4 and 1.6 hold for* $g_{s,r}$*.*

Proof. The fact that $g_{s,r} \in C^2(\mathcal{R})$ follows immediately from Conditions (iii) and (iv) of Lemma 1.14. The symmetry condition is a consequence of Condition (i). If $x \geq r$, we have that;

$|g_{s,r}(x)| \leq |x^{-Re(s)}||x^{-Im(s)}| \leq |x|^{-4}$

Hence, as $g_{s,r}$ is symmetric, $|g_{s,r}(x)| \leq |x|^{-4}$, for $|x| \geq r$.

If $|x| \leq r$;

$|g_{s,r}(x)| = |p_{s,r}(x)| \leq r^{2r+2} \sum_{j=0}^{r+1} |(\overline{a}_{s,r})_{(j+1)}| \leq r^{2r+2}\sqrt{r+2}||\overline{a}_{s,r}||$

It follows that $sup_{x\in\mathcal{R}}(|x|^4|g_{s,r}(x)|) \leq max(1, r^{2r+6}\sqrt{r+2}||\overline{a}_{s,r}||)$. Similarly, as $g'_{s,r}(x) = \frac{-s}{x^{s+1}}$, $g''_s(x) = \frac{s(s+1)}{x^{s+2}}$, $|x| > r$, then, if $|x| > r$, we have that;

$|g'_{s,r}(x)| \leq |s||x|^{-5}$

$|g''_{s,r}(x)| \leq |s||s-1||x|^{-6}$

and, if $|x| \leq r$;

$|g'_{s,r}(x)| = |p'_{s,r}(x)| \leq r^{2r+2}(\sum_{j=1}^{r+1} |2j(\overline{a}_{s,r})_{(j+1)}|) \leq 2(r+1)r^{2r+2}\sqrt{r+2}||\overline{a}_{s,r}||$

$|g''_s(x)| = |p''_s(x)| \leq r^{2r+2}(\sum_{j=1}^{r+1} |2j(2j-1)(\overline{a}_{s,r})_{(j+1)}|) \leq (2r+2)(2r+1)r^{2r+2}\sqrt{r+2}||\overline{a}_s||$

so that;

$sup_{x\in\mathcal{R}}(|x|^5|g'_{s,r}(x)|) \leq max(|s|, 2(r+1)r^{2r+7}\sqrt{r+2}||\overline{a}_s||)$

$sup_{x\in\mathcal{R}}(|x|^6|g''_{s,r}(x)|) \leq max(|s||s-1|, (2r+2)(2r+1)r^{2r+8}\sqrt{r+2}||\overline{a}_s||)$ $(*)$

It follows that Lemmas 1.4 and 1.6 holds for $g_{s,r}$, with $C = D = max(|s||s-1|, (2r+2)(2r+1)r^{2r+8}\sqrt{r+2}||\overline{a}_s||$.

□

Definition 1.17. *We let* $f_{s,r}(y) = \int_{\mathcal{R}} g_{s,r}(x)e^{2\pi ixy}dx$.

$R_{s,r,1} = \sum_{n\in\mathcal{Z}_{\neq 0}}(\int_r^\infty \frac{e^{2\pi inx}}{x^s}dx)$

$R_{s,r,2} = \sum_{n\in\mathcal{Z}_{\neq 0}}(\int_r^\infty \frac{e^{-2\pi inx}}{x^s}dx)$

$R_{s,r} = R_{s,r,1} + R_{s,r,2}$

$P_{s,r,1} = \sum_{n\in\mathcal{Z}_{\neq 0}}(\int_0^r p_{s,r}(x)e^{2\pi inx}dx)$

$P_{s,r,2}=\sum_{n\in\mathcal{Z}_{\neq 0}}(\int_0^r p_{s,r}(x)e^{-2\pi inx}dx)$

$P_{s,r}=P_{s,r,1}+P_{s,r,2}$

Lemma 1.18. *We have that $f_{s,r}$ is symmetric, and $f_{s,r}$ satisfies the conclusions of Lemmas 1.5 and 1.6. Moreover;*

$f_{s,r}(0)+P_{s,r}+R_{s,r}=p_{s,r}(0)+2\sum_{n=1}^{\infty}\frac{1}{n^s}$

Proof. The second part follows immediately from Definition 1.17, and Lemmas 1.16, 1.5 and 1.6. It follows that $f_{s,r}\in L^1(\mathcal{R})$, and;

$f_{s,r}(-y)=\int_{\mathcal{R}}g_{s,r}(x)e^{-2\pi ixy}dx$

$=\int_{\mathcal{R}}g_{s,r}(-x)e^{2\pi ixy}dx$

$=\int_{\mathcal{R}}g_{s,r}(x)e^{2\pi ixy}dx$

$=f_{s,r}(y)$

Hence, $f_{s,r}$ is symmetric. By Lemma 1.10 , we have that;

$\sum_{n\in\mathcal{Z}}f_{s,r}(n)=\sum_{n\in\mathcal{Z}}g_{s,r}(n)$

As both $f_{s,r}$ and $g_{s,r}$ are symmetric, using Definition 1.15 and property (ii) of Lemma 1.14, we obtain;

$f_{s,r}(0)+P_{s,r}+R_{s,r}$

$=f_{s,r}(0)+\sum_{n\in\mathcal{Z}_{\neq 0}}f_{s,r}(n)$

$=g_{s,r}(0)+2(\sum_{n=1}^{\infty}g_{s,r}(n))$

$=p_{s,r}(0)+2(\sum_{n=1}^{\infty}\frac{1}{n^s})$

□

Lemma 1.19. *We have that;*

$|R_{s,r}|\leq\frac{2|s|^2}{3(Re(s)+1)r^{Re(s)+1}}$

$P_{s,r} = p_{s,r}(0) + p_{s,r}(r) + 2\sum_{l=1}^{r-1} p_{s,r}(l) - 2\int_0^r p_{s,r}(x)dx$

Proof. We have that;

$R_{s,r,1} = \sum_{n\in\mathcal{Z}_{\neq 0}} \frac{-1}{2\pi i n r^s} + \sum_{n\in\mathcal{Z}_{\neq 0}} \frac{s}{2\pi i n}\int_r^\infty \frac{e^{2\pi i n x}}{x^{s+1}}dx$

$= \sum_{n\in\mathcal{Z}_{\neq 0}} \frac{-s}{r^{s+1}(2\pi i n)^2} + \sum_{n\in\mathcal{Z}_{\neq 0}} \frac{s(s+1)}{(2\pi i n)^2}\int_r^\infty \frac{e^{2\pi i n x}}{x^{s+2}}dx$

$= \frac{2s}{4r^{s+1}\pi^2}\sum_{n=1}^\infty \frac{1}{n^2} + D_{s,r,1}$

$= \frac{s}{2r^{s+1}\pi^2}\frac{\pi^2}{6} + D_{s,r,1}$

$= \frac{s}{12r^{s+1}} + D_{s,r,1}$

where;

$|D_{s,r,1}| \leq \frac{|s(s+1)|C_{s,r,1}}{4\pi^2}\sum_{n\in\mathcal{Z}_{\neq 0}} \frac{1}{n^2} = \frac{|s(s+1)|C_{s,r,1}}{12}$

and $C_{s,r,1} \leq \int_r^\infty \frac{1}{|x^{s+2}|}dx$

$= \int_r^\infty \frac{dx}{x^{Re(s)+2}}$

$= \frac{1}{(Re(s)+1)r^{Re(s)+1}}$

It follows that;

$|R_{s,r,1}| \leq \frac{|s|}{12r^{Re(s)+1}} + \frac{|s(s+1)|}{12(Re(s)+1)r^{Re(s)+1}}$

$\leq \frac{|s|(Re(s)+1)}{12(Re(s)+1)r^{Re(s)+1}} + \frac{|s(s+1)|}{12(Re(s)+1)r^{Re(s)+1}}$

$= \frac{|s|(Re(s)+1)+|s(s+1)|}{12(Re(s)+1)r^{Re(s)+1}}$

$\leq \frac{2|s(s+1)|}{12(Re(s)+1)r^{Re(s)+1}}$

$\leq \frac{|s|^2}{3(Re(s)+1)r^{Re(s)+1}}$

Similarly, $|R_{s,r,2}| \leq \frac{|s|^2}{3(Re(s)+1)r^{Re(s)+1}}$, so that $|R_{s,r}| \leq \frac{2|s|^2}{3(Re(s)+1)r^{Re(s)+1}}$.

We have that;

$P_{s,r,1}=\sum_{n\in\mathcal{Z}_{\neq 0}}(\int_0^r p_{s,r}(x)e^{2\pi inx}dx$

$=\sum_{n\in\mathcal{Z}_{\neq 0}}(\sum_{l=0}^{r-1}\int_l^{l+1}p_{s,r}(x)e^{2\pi inx}dx)$

$=\sum_{l=0}^{r-1}(\sum_{n\in\mathcal{Z}_{\neq 0}}(\int_l^{l+1}p_{s,r}(x)e^{2\pi inx}dx))$

$=\sum_{l=0}^{r-1}(\sum_{n\in\mathcal{Z}_{\neq 0}}(\int_0^1 p_{s,r}(x+l)e^{2\pi in(x+l)}dx))$

$=\sum_{l=0}^{r-1}(\sum_{n\in\mathcal{Z}_{\neq 0}}(\int_0^1 p_{s,r}(x+l)e^{2\pi inx}dx))$

$=\sum_{l=0}^{r-1}(\sum_{n\in\mathcal{Z}_{\neq 0}}(p^l_{s,r})^{\vee}(n))$

$=\sum_{l=0}^{r-1}(\sum_{n\in\mathcal{Z}}(p^l_{s,r})^{\vee}(n)-\int_0^1 p^l_{s,r}dx)$, ([3])

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[3]If $f\in(C[0,1]\cap C^2(0,1))$, and there exist $\{a_{+,j},a_{-,j}:0\leq j\leq 2\}\subset\mathcal{C}$, with $lim_{x\rightarrow 0^+}f^{(j)}(x)=a_{+,j}$ and $lim_{x\rightarrow 1^-}f^{(j)}(x)=a_{-,j}$, (†), for $0\leq j\leq 2$, then a classical result in the theory of Fourier series, says that;

$\lim_{N\rightarrow\infty}\sum_{n=-N}^{N}(f)^{(\wedge)}(n)e^{2\pi inx}=f(x)\ (x\in(0,1))$

$\lim_{N\rightarrow\infty}\sum_{n=-N}^{N}(f)^{(\wedge)}(n)=\frac{a_{+,0}+a_{-,0}}{2}$

We give a simple proof of this result. First observe that there exists a polynomial $p\in\mathcal{C}[x]$, with $deg(p)=5$, such that $p^{(j)}(0)=0$ and $p^{(j)}(1)=a_{-,j}-a_{+,j}$, for $0\leq j\leq 2$. This follows from the fact that we can find $\overline{c}\subset\mathcal{C}^3$, such that $\overline{M}\centerdot\overline{c}=\overline{a}$, where $\overline{a}(j)=a_{-,j-1}-a_{+,j-1}$, for $1\leq j\leq 3$, and;

$$\overline{M}=\begin{pmatrix}1&1&1\\3&4&5\\6&12&20\end{pmatrix}$$

as $det(\overline{M})\neq 0$, and, setting $p(x)=\sum_{k=0}^{2}c_kx^{3+k}$. We have that $p+f\in C^2(S^1)$, in which case the result follows from [3]. Hence, it is sufficient to verify the result for the powers $\{x^k:0\leq k\leq 5\}$. We have that, for $k\geq 1$, $n\in\mathcal{Z}_{\neq 0}$;

$\int_0^1 x^ke^{-2\pi inx}dx$

$[\frac{x^ke^{-2\pi inx}}{-2\pi in}]_0^1+\frac{k}{2\pi in}\int_0^1 x^{k-1}e^{-2\pi inx}dx$

$\frac{-1}{2\pi in}+\frac{k}{2\pi in}\int_0^1 x^{k-1}e^{-2\pi inx}dx$

$\int_0^1 x^ke^{-2\pi inx}dx$

$=-(\sum_{l=1}^{k}\frac{k!}{(k-l+1)!(2\pi in)^l})+\int_0^1 e^{-2\pi inx}dx$

$=-(\sum_{l=1}^{k}\frac{k!}{(k-l+1)!(2\pi in)^l})$

$= \sum_{l=0}^{r-1}(\frac{p_{s,r}^l(1)+p_{s,r}^l(0)}{2} - \int_0^1 p_{s,r}^l dx)$

$= \sum_{l=0}^{r-1}(\frac{p_{s,r}(l+1)+p_{s,r}(l)}{2} - \int_0^1 p_{s,r}(x+l)dx)$

$= \sum_{l=0}^{r-1}(\frac{p_{s,r}(l+1)+p_{s,r}(l)}{2} - \int_l^{l+1} p_{s,r}(x)dx)$

$= \frac{p_{s,r}(0)+p_{s,r}(r)}{2} + \sum_{l=1}^{r-1} p_{s,r}(l) - \int_0^r p_{s,r}(x)dx$

Similarly;

$P_{s,r,2} = \frac{p_{s,r}(0)+p_{s,r}(r)}{2} + \sum_{l=1}^{r-1} p_{s,r}(l) - \int_0^r p_{s,r}(x)dx$

so that;

$P_{s,r} = P_{s,r,1} + P_{s,r,2}$

$lim_{N\rightarrow\infty} \sum_{n=-N}^{N} (x^k)^\wedge(n)$

$= \frac{1}{k+1} - 2\sum_{l=1}^{k}\sum_{n=1}^{\infty} \frac{k!}{(k-l+1)!(2\pi in)^l}$

Case $k = 1$, we obtain $S_k = \frac{1}{2}$

$k = 2$, $S_k = \frac{1}{3} + \frac{2.2}{4\pi^2}(\sum_{n=1}^{\infty} \frac{1}{n^2})$

$k = 3$, $S_k = \frac{1}{4} + \frac{2.3}{4\pi^2}(\sum_{n=1}^{\infty} \frac{1}{n^2})$

$k = 4$, $S_k = \frac{1}{5} + \frac{2.4}{4\pi^2}(\sum_{n=1}^{\infty} \frac{1}{n^2}) - \frac{2.24}{16\pi^4}(\sum_{n=1}^{\infty} \frac{1}{n^4})$

$k = 5$, $S_k = \frac{1}{6} + \frac{2.5}{4\pi^2}(\sum_{n=1}^{\infty} \frac{1}{n^2}) - \frac{2.120}{16\pi^4}(\sum_{n=1}^{\infty} \frac{1}{n^4})$

Using Lemma 1.11, we have that;

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{-\pi[cot(\pi z)z]^{(2)}|_0}{2.2!} = \frac{-\pi.-4\pi}{6.2.2!} = \frac{\pi^2}{6}$

$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{-\pi[cot(\pi z)z]^{(4)}|_0}{2.4!} = \frac{-\pi.-48\pi^3}{90.2.4!} = \frac{\pi^4}{90}$

$S_2 = \frac{1}{3} + \frac{2.2}{4\pi^2}(\frac{\pi^2}{6}) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

$S_3 = \frac{1}{4} + \frac{2.3}{4\pi^2}\frac{\pi^2}{6} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$S_4 = \frac{1}{5} + \frac{2.4}{4\pi^2}\frac{\pi^2}{6} - \frac{2.24}{16\pi^4}\frac{\pi^4}{90} = \frac{1}{2}$

$S_5 = \frac{1}{6} + \frac{2.5}{4\pi^2}\frac{\pi^2}{6} - \frac{2.60}{16\pi^4}\frac{\pi^4}{90} = \frac{1}{2}$

$= p_{s,r}(0) + p_{s,r}(r) + 2\sum_{l=1}^{r-1} p_{s,r}(l) - 2\int_0^r p_{s,r}(x)dx$

$\square$

Lemma 1.20. *If $Re(s) \geq 4$, $r \geq 1$, we have that;*

$\sum_{n=r}^{\infty} \frac{1}{n^s}$

$= \int_r^{\infty} \frac{dx}{x^s} + \frac{p_{s,r}(r)}{2} + \frac{R_{s,r}}{2}$

$= \frac{1}{(s-1)r^{s-1}} + \frac{R_{s,r}}{2} + \frac{r^s}{2}$

If $r \geq 2$;

$\sum_{n=1}^{r-1} \frac{1}{n^s}$

$= \sum_{j=0}^{r+1} (\overline{a}_{s,r})_{j+1} (\frac{B_{2j+1}(r)}{2j+1})$

Proof. The first claim is just a simple rearrangement of the claim in Lemma 1.18, using Lemma 1.19. We have that;

$\int_r^{\infty} \frac{dx}{x^s} = \frac{1}{(s-1)r^{s-1}}$

and $\frac{p_{s,r}(r)}{2} = \frac{r^s}{2}$, by property (ii) in Lemma 1.14.
Moreover;

$\sum_{n=1}^{r-1} \frac{1}{n^s}$

$= \sum_{l=1}^{r-1} p_{s,r}(l)$

$= \sum_{l=0}^{r-1} \sum_{j=0}^{r+1} (\overline{a}_{s,r})_{j+1} l^{2j}$

$= \sum_{j=0}^{r+1} (\overline{a}_{s,r})_{j+1} (\sum_{l=0}^{r-1} l^{2j})$

$= \sum_{j=0}^{r+1} (\overline{a}_{s,r})_{j+1} (\frac{B_{2j+1}(r) - B_{2j+1}(0)}{2j+1})$

$= \sum_{j=0}^{r+1} (\overline{a}_{s,r})_{j+1} (\frac{B_{2j+1}(r)}{2j+1})$

$\square$

Remarks 1.21. *Using Lemma 1.19, we have that $lim_{r\rightarrow\infty}|R_{s,r}| = 0$, hence, Lemma 1.20 reduces the calculation of $\sum_{n=1}^{\infty} \frac{1}{n^s}$ to a calculation* *involving Bernoulli polynomials. Moreover, letting* $A_{s,r} = \sum_{n=r}^{\infty} \frac{1}{n^s}$*, we have that;*

$\int_r^\infty \frac{dx}{|x^s|} \leq |A_{s,r}| \leq \int_{r-1}^\infty \frac{dx}{|x^s|}$

$\int_r^\infty \frac{dx}{x^{Re(s)}} \leq |A_{s,r}| \leq \int_{r-1}^\infty \frac{dx}{|x^s|}$

$\frac{1}{(Re(s)-1)r^{Re(s)-1}} \leq |A_{s,r}| \leq \frac{1}{3(r-1)^3}$

Observing that;

$\frac{|s|^2}{3(Re(s)+1)r^{Re(s)+1}} \leq \frac{1}{(Re(s)-1)r^{Re(s)-1}}$

if $r \geq |s|\sqrt{\frac{(Re(s)-1)}{3(Re(s)+1)}}$*, we have that the estimate* $\sum_{j=0}^{r+1}(\overline{a}_{s,r})_{j+1}(\frac{B_{2j+1}(r)}{2j+1})+\frac{1}{(s-1)r^{s-1}}+\frac{1}{2r^s}$ *improves upon* $\sum_{j=0}^{r+1}(\overline{a}_{s,r})_{j+1}(\frac{B_{2j+1}(r)}{2j+1})$*, for sufficiently large values of* r*. The coefficients* $(\overline{a}_{s,r})_j$*,* $1 \leq j \leq r+2$ *can be computed using simple linear algebra. The computation of absolutely convergent Riemann sums, and their differences, occurs in the evaluation of* $\zeta(s)$*, for* $0 < Re(s) < 1$*, it is well known that* $\zeta(s) \neq 0$*, for* $Re(s) \geq 1$*. It is hoped that the above method might lead to some progress in the direction of solving the famous Riemann hypothesis, that,* $\zeta(s) = 0$ *iff* $Re(s) = \frac{1}{2}$ *or* $s = -2w$*, for* $w \in \mathcal{Z}_{\geq 1}$*, see* [1].

References

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[2] A Simple Proof of the Inversion Theorem using Nonstandard Analysis, available at http://www.magneticstrix.net, T. de Piro, (2013).

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