RESULTS ON THE NONSTANDARD LAPLACIAN

TRISTRAM DE PIRO

Abstract.

We adopt the following notation;

Definition 0.1. *For $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$, we let;*

$\overline{\mathcal{H}_\eta} = {}^*\bigcup_{0 \leq i \leq 2\eta-1}[-\pi + \pi\frac{i}{\eta}, -\pi + \pi\frac{i+1}{\eta})$

so that $\overline{\mathcal{H}_\eta} = {}^*[-\pi, \pi)$.

We let $\{\mathfrak{C}_\eta\}$ denote the associated $*$-finite algebras generated by the intervals $[-\pi+\pi\frac{i}{\eta}, -\pi+\pi\frac{i+1}{\eta})$, for $0 \leq i \leq 2\eta-1$, and $\{\lambda_\eta\}$ the associated counting measures, defined by $\lambda_\eta([-\pi+\pi\frac{i}{\eta}, -\pi+\pi\frac{i+1}{\eta})) = \frac{\pi}{\eta}$. We let $(\overline{\mathcal{H}_\eta}, L(\mathfrak{C}_\eta), L(\lambda_\eta))$ denote the associated Loeb spaces, see Definition 0.5 of []. We let $([-\pi, \pi], \mathfrak{B}, \mu)$ denote the interval $[-\pi, \pi]$, with the completion $\mathfrak{B}$ of the Borel field, and μ the restriction of Lebesgue measure. We let $({}^*\mathcal{R}, {}^*\mathfrak{D})$ denote the hyperreals, with the transfer of the Borel field $\mathfrak{D}$ on $\mathcal{R}$. A function $f : (\overline{\mathcal{H}_\eta}, \mathfrak{C}_\eta) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{D})$ is measurable, if $f^{-1} : {}^*\mathfrak{D} \rightarrow \mathfrak{C}_\eta$. Observe that this is equivalent to the definition given in [?]. We will abbreviate this notation to $f : \overline{\mathcal{H}_\eta} \rightarrow {}^*\mathcal{R}$ is measurable, $(*)$. The same applies to $({}^*\mathcal{C}, {}^*\mathfrak{D})$, the hyper complex numbers, with the transfer of the Borel field $\mathfrak{D}$, generated by the complex topology. Observe that $f : \overline{\mathcal{H}_\eta} \rightarrow {}^*\mathcal{C}$ is measurable, in this sense, iff $Re(f)$ and $Im(f)$ are measurable in the sense of $(*)$.

We let;

$V(\overline{\mathcal{H}_\eta}) = \{f : \overline{\mathcal{H}_\eta} \rightarrow {}^*\mathcal{C},\ f \text{ measurable } d(\lambda_\eta)\}$

Let $f : \overline{\mathcal{H}_\eta} \rightarrow {}^*\mathcal{C}$ be measurable. As in [?], we define the discrete derivative f' to be the unique measurable function satisfying;

$f'(-\pi + \pi\frac{i}{\eta}) = \frac{\eta}{\pi}(f(-\pi + \pi\frac{i+1}{\eta}) - f(-\pi + \pi\frac{i}{\eta}))$;

for $i \in {}^*\mathcal{N}_{0\leq i\leq 2\eta-2}$.

$f'(\pi-\frac{\pi}{\eta})=\frac{\eta}{\pi}(f(-\pi)-f(\pi-\frac{\pi}{\eta}))$

If $f:\overline{\mathcal{H}_\eta}\rightarrow {}^*\mathcal{C}$ *is measurable, then we define the shift (right);*

$f^{sh}(-\pi+\pi\frac{j}{\eta})=f(-\pi+\pi\frac{j+1}{\eta})$ *for* $0\leq j\leq 2\eta-2$

$f^{sh}(\pi-\frac{\pi}{\eta})=f(-\pi)$

$f^{rsh}(-\pi+\pi\frac{j}{\eta})=f(-\pi+\pi\frac{j-1}{\eta})$ *for* $1\leq j\leq 2\eta-1$

$f^{rsh}(-\pi)=f(\pi-\frac{\pi}{\eta})$

We define the nonstandard Laplacian $\Delta_\eta : V(\overline{\mathcal{H}_\eta})\rightarrow V(\overline{\mathcal{H}_\eta})$ *by;*

$\Delta_\eta(f)=f''$

We let $C^\infty([-\pi,\pi])=\{f\in C[-\pi,\pi]:f|_{(-\pi,\pi)}\in C^\infty(-\pi,\pi),\exists g_k\in C([-\pi,\pi]),g_k(-\pi)=g_k(\pi),g_k|_{(-\pi,\pi)}=f^{(k)},k\in\mathcal{Z}_{\geq 0}\}$.

We let $\Delta : C^\infty([-\pi,\pi])\rightarrow C^\infty([-\pi,\pi])$ *be defined by* $\Delta(f)=f^{(2)}$*, where* $f^{(2)}(-\pi)=g_2(-\pi)=f^{(2)}(\pi)=g_2(\pi)$.

Lemma 0.2. *If* $\lambda\in\mathcal{C}$*,* $f\in C^\infty([-\pi,\pi])$*,* $f\neq 0$*, then* $\Delta(f)=\lambda f$*, iff* $\lambda=-n^2$*, for some* $n\in\mathcal{Z}_{\geq 0}$.

Proof. Let $\tau=-\lambda$, and suppose that $\Delta(f)=-\tau f$. Using Peano's Theorem on $(-\pi,\pi)$, we have that;

$f|_{(-\pi,\pi)}(x)=Ae^{i\sqrt{\tau}(x+\pi)}+Be^{-i\sqrt{\tau}(x+\pi)}$

$=Ccos(\sqrt{\tau}(x+\pi))+Dsin(\sqrt{\tau}(x+\pi))$

where $\{A,B,C,D\}\subset\mathcal{C}$. Using the definition of $C^\infty([-\pi,\pi])$ in Definition 0.1, we obtain;

$f(-\pi)=C=Ccos(2\sqrt{\tau}\pi)+Dsin(2\sqrt{\tau}\pi)=f(\pi)$

If $sin(2\sqrt{\tau}\pi)\neq 0$, $(\sharp)$, we obtain;

$D=\frac{C(1-cos(2\sqrt{\tau}\pi))}{sin(2\sqrt{\tau}\pi)}$. Then;

$f|_{(-\pi,\pi)}(x) = C[cos(\sqrt{\tau}(x+\pi)) + \frac{(1-cos(2\sqrt{\tau}\pi))}{sin(2\sqrt{\tau}\pi)} sin(\sqrt{\tau}(x+\pi))]$

$f^{(1)}|_{(-\pi,\pi)}(x) = C[-\sqrt{\tau} sin(\sqrt{\tau}(x+\pi)) + \sqrt{\tau}\frac{(1-cos(2\sqrt{\tau}\pi))}{sin(2\sqrt{\tau}\pi)} cos(\sqrt{\tau}(x+\pi))]$ $(*)$

Using Definition **??** again, we obtain;

$g_1(-\pi) = \sqrt{\tau}\frac{(1-cos(2\sqrt{\tau}\pi))}{sin(2\sqrt{\tau}\pi)} = \sqrt{\tau} cos(2\sqrt{\tau}\pi)\frac{(1-cos(2\sqrt{\tau}\pi))}{sin(2\sqrt{\tau}\pi)} - \sqrt{\tau} sin(2\sqrt{\tau}\pi) = g_1(-\pi)$

$(1-cos(2\sqrt{\tau}\pi))^2 = -sin^2(2\sqrt{\tau}\pi)$

$cos(2\sqrt{\tau}\pi) = 1$

hence, $\sqrt{\tau} \in \mathcal{Z}$, and $sin(2\sqrt{\tau}\pi) = 0$, contradicting ($\sharp$). It follows that $sin(2\sqrt{\tau}\pi) = 0$, $\sqrt{\tau} \in \mathcal{Z}$, $\lambda = -\tau = -n^2$, for some $n \in \mathcal{Z}_{\geq 0}$. It is easily checked that, if $n \in \mathcal{Z}_{\geq 0}$, $e^{inx} \in C^\infty([-\pi,\pi])$, and $\Delta(f) = -n^2 f$. □

Definition 0.3. *We let* $V^\Delta = \{-n^2 : n \in \mathcal{Z}_{\geq 0}\}$.

Lemma 0.4. *If* $r \in \mathcal{Z}_{\geq 0}$, *then* $(\Delta - \lambda I)^r(f) = 0$ *iff* $r = 1$ *and* $\lambda \in V^\Delta$.

Proof. Using [**?**] and Lemma 0.2, if $f \in C^\infty([-\pi,\pi])$, then;

$f = \sum_{m\in\mathcal{Z}} f^\wedge(m)e^{imx}$, $(*)$

where $f^\wedge(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ixm}dx$, and the convergence is uniform. Using $(*)$, and interchanging the summation and differentiation operations, we have that;

$(\Delta - \lambda I)^{(r)}(f) = \sum_{m\in\mathcal{Z}}(-1)^r(m^2+\lambda)^r f^\wedge(m)e^{imx}$

It follows that, if $(\Delta - \lambda I)^{(r)}(f) = 0$, and $f^\wedge(m) \neq 0$, for some $m \in \mathcal{Z}_{\geq 0}$, then, $< (\Delta - \lambda I)^{(r)}(f), e^{imx} >= (-1)^r(m^2+\lambda)^r f^\wedge(m) = 0$ implies $(m^2+\lambda)^r = 0$, and, therefore, $\lambda = -m^2$, $(*)$, holds. It follows, as $f \neq 0$, that $(*)$ holds and $f(x) = Ae^{imx} + Be^{-imx}$, for some $\{A, B\} \subset \mathcal{C}$. In particularly, $r = 1$.

□

Lemma 0.5. *If* $\mu \in {}^*\mathcal{C}$, $h \in V(\overline{\mathcal{H}_\eta})$, *and* $\Delta_\eta(h) = \mu h$, $(*)$, *then, for* $0 \leq j \leq 2\eta - 1$;

$h(-\pi+\pi\frac{j}{\eta})$

$=(\frac{1}{2}[(1+\frac{\sqrt{\mu}}{\eta})^j+(1-\frac{\sqrt{\mu}}{\eta})^j]-\frac{\eta}{2\pi\sqrt{\mu}}[(1+\frac{\sqrt{\mu}}{\eta})^j-(1-\frac{\sqrt{\mu}}{\eta})^j])h(-\pi)$

$+\frac{\eta}{2\pi\sqrt{\mu}}[(1+\frac{\sqrt{\mu}}{\eta})^j-(1-\frac{\sqrt{\mu}}{\eta})^j]h(-\pi+\frac{\pi}{\eta})$

and;

$h(-\pi)=(\frac{\mu\pi^2}{\eta^2}-1)h(-\pi+\pi\frac{2\eta-2}{\eta})+2h(-\pi+\pi\frac{2\eta-1}{\eta})$

$h(-\pi+\frac{\pi}{\eta})=(\frac{2\mu\pi^2}{\eta^2}-2)h(-\pi+\pi\frac{2\eta-2}{\eta})+(\frac{\mu\pi^2}{\eta^2}+3)h(-\pi+\pi\frac{2\eta-1}{\eta})$

Proof. Suppose $\{\mu,e,h\}\subset {}^*\mathcal{C}$, and let $H:\overline{\mathcal{H}_\eta}\times{}^*\mathcal{C}\times{}^*\mathcal{C}\rightarrow{}^*\mathcal{C}\times{}^*\mathcal{C}$ be defined by;

$H(\tau,x',y')=(y',\mu x')$

The solution to $(*)$, with initial condition $h(-\pi)=e, h'(-\pi)=f$, is then given by, using Definition **??** and the method in **??**;

$h(-\pi)=e, h'(-\pi)=f$

$h(-\pi+\pi(\frac{(j+1)}{\eta}))=h(-\pi+\pi(\frac{j}{\eta}))+\frac{\pi}{\eta}h'(-\pi+\pi(\frac{j}{\eta}))$, $0\leq j\leq 2\eta-2$

$h'(-\pi+\pi(\frac{(j+1)}{\eta}))=h'(-\pi+\pi(\frac{j}{\eta}))+\frac{\pi}{\eta}\mu h(-\pi+\pi(\frac{j}{\eta}))$, $0\leq j\leq 2\eta-2$

$\overline{w}_{j+1}=\overline{A}\overline{w}_j$, $0\leq j\leq 2\eta-2$

$\overline{w}_j=\overline{A}^j\overline{w}_0$, $0\leq j\leq 2\eta-1$

where;

$$\overline{w}_j=\begin{pmatrix}h(-\pi+\pi(\frac{j}{\eta}))\\ h'(-\pi+\pi(\frac{j}{\eta}))\end{pmatrix}$$

and;

$$\overline{A}=\begin{pmatrix}1 & \frac{\pi}{\eta}\\ \frac{\pi\mu}{\eta} & 1\end{pmatrix}$$

The eigenvalues of $\overline{A}$ are given by $\{\frac{\pi\sqrt{\mu}}{\eta}+1,\frac{-\pi\sqrt{\mu}}{\eta}+1\}$, with eigenvectors $\{\overline{v}_1,\overline{v}_2\}$, where;

$\overline{v}_1 = \begin{pmatrix} 1 \\ \sqrt{\mu} \end{pmatrix}$

and;

$\overline{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{\mu} \end{pmatrix}$

so that $\overline{B}^{-1}\overline{AB} = diag(\frac{\pi\sqrt{\mu}}{\eta}+1, \frac{-\pi\sqrt{\mu}}{\eta}+1)$, where;

$\overline{B} = \begin{pmatrix} 1 & 1 \\ \sqrt{\mu} & -\sqrt{\mu} \end{pmatrix}$

$\overline{B}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{\mu}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{\mu}} \end{pmatrix}$

$\overline{w}_j = \overline{B}diag(\frac{\pi\sqrt{\mu}}{\eta}+1, \frac{-\pi\sqrt{\mu}}{\eta}+1)^j\overline{B}^{-1}(\overline{w}_0)$

$= \overline{B}diag((\frac{\pi\sqrt{\mu}}{\eta}+1)^j, (\frac{\pi\sqrt{\mu}}{\eta}-1)^j)\overline{B}^{-1}(\overline{w}_0)$

Hence;

$h(-\pi+\pi(\frac{j}{\eta})) = \frac{[(1+\frac{\pi\sqrt{\mu}}{\eta})^j+(1-\frac{\pi\sqrt{\mu}}{\eta})^j]}{2}e + \frac{[(1+\frac{\pi\sqrt{\mu}}{\eta})^j-(1-\frac{\pi\sqrt{\mu}}{\eta})^j]}{2\sqrt{\mu}}f$

$h'(-\pi+\pi(\frac{j}{\eta})) = \frac{\sqrt{\mu}[(1+\frac{\pi\sqrt{\mu}}{\eta})^j-(1-\frac{\pi\sqrt{\mu}}{\eta})^j]}{2}e + \frac{[(1+\frac{\pi\sqrt{\mu}}{\eta})^j+(1-\frac{\pi\sqrt{\mu}}{\eta})^j]}{2}f$

for $0 \leq j \leq 2\eta-1$, so that there exist $*$-polynomials in $x^{\frac{1}{2}}$, $\{P^i_{j,\eta}(x), Q^i_{j,\eta}(x)\} \subset {}^*\mathcal{C}[x]$, with $1 \leq i \leq 2$;

with $h(-\pi+\pi(\frac{j}{\eta})) = P^1_{j,\eta}(\mu)e + Q^1_{j,\eta}(\mu)f$

$h'(-\pi+\pi(\frac{j}{\eta})) = P^2_{j,\eta}(\mu)e + Q^2_{j,\eta}(\mu)f$, $(**)$

where;

$P^1_{j,\eta}(x) = \frac{1}{2}[(1+\frac{\pi\sqrt{x}}{\eta})^j + (1-\frac{\pi\sqrt{x}}{\eta})^j]$

$Q^1_{j,\eta}(x) = \frac{1}{2\sqrt{x}}[(1+\frac{\pi\sqrt{x}}{\eta})^j - (1-\frac{\pi\sqrt{x}}{\eta})^j]$

$P^2_{j,\eta}(x) = \frac{\sqrt{x}}{2}[(1+\frac{\pi\sqrt{x}}{\eta})^j - (1-\frac{\pi\sqrt{x}}{\eta})^j]$

$Q^2_{j,\eta}(x)=\frac{1}{2}[(1+\frac{\pi\sqrt{x}}{\eta})^j+(1-\frac{\pi\sqrt{x}}{\eta})^j]$

We have that $f=\frac{\eta}{\pi}(h(-\pi+\frac{\pi}{\eta})-h(-\pi))$, $e=h(-\pi)$. Hence, using $(**)$;

$h(-\pi+\pi(\frac{j}{\eta}))=(P^1_{j,\eta}(\mu)-\frac{\eta Q^1_{j,\eta}(\mu)}{\pi})h(-\pi)+\frac{\eta Q^1_{j,\eta}(\mu)}{\pi}h(-\pi+\frac{\pi}{\eta})$

giving the first result. We have, from Definition 0.1;

$h(-\pi)=(\frac{\mu\pi^2}{\eta^2}-1)h(-\pi+\pi\frac{2\eta-2}{\eta})+2h(-\pi+\pi\frac{2\eta-1}{\eta})$ (i)

$h(-\pi+\frac{\pi}{\eta})=(\frac{\mu\pi^2}{\eta^2}-1)h(-\pi+\pi\frac{2\eta-1}{\eta})+2h(-\pi)$ $(\dagger)$

Substituting the expression for $h(-\pi)$ from (i) into $(\dagger)$, we obtain;

$h(-\pi+\frac{\pi}{\eta})=(\frac{2\mu\pi^2}{\eta^2}-2)h(-\pi+\pi\frac{2\eta-2}{\eta})+(\frac{\mu\pi^2}{\eta^2}+3)h(-\pi+\pi\frac{2\eta-1}{\eta})$ (ii)

as required.

□

Lemma 0.6. *If $\mu\in{}^*\mathcal{C}$, $h\in V(\overline{\mathcal{H}_\eta})$, and $\Delta_\eta(h)=\mu h$, $(*)$, then;*

$h(-\pi+\pi(\frac{2\eta-2}{\eta}))=P^1_{2\eta-2,\eta}(\mu)h(-\pi)+Q^1_{2\eta-2,\eta}(\mu)h'(-\pi)$ $(i)'$

$h(-\pi+\pi(\frac{2\eta-1}{\eta}))=P^1_{2\eta-1,\eta}(\mu)h(-\pi)+Q^1_{2\eta-1,\eta}(\mu)h'(-\pi)$ $(ii)'$

$h(-\pi)=(\frac{\mu\pi^2}{\eta^2}-1)h(-\pi+\pi(\frac{2\eta-2}{\eta}))+2h(-\pi+\pi(\frac{2\eta-1}{\eta}))$ $(iii)'$

$h'(-\pi)=(\frac{\mu\pi}{\eta}-\frac{\eta}{\pi})h(-\pi+\pi(\frac{2\eta-2}{\eta}))+(\frac{\mu\pi}{\eta}+\frac{\eta}{\pi})h(-\pi+\pi(\frac{2\eta-1}{\eta}))$ $(iv)'$

Proof. $(i)',(ii)'$ follow from $(**)$ in Lemma 0.9. $(iii)'$ is (i) in Lemma 0.9. Finally, using $h'(-\pi)=\frac{\eta(h(-\pi+\frac{\pi}{\eta})-h(-\pi))}{\pi}$, and (ii) in Lemma 0.9, we obtain $(iv)'$.

□

Lemma 0.7. *We have that μ is an eigenvalue for $\Delta_\eta:V(\overline{\mathcal{H}_\eta})\rightarrow\overline{\mathcal{H}_\eta}$ iff;*

$det(\overline{D})=0$

where;

$$\overline{D}=\begin{pmatrix} 1 & 0 & 1-\frac{\mu\pi^2}{\eta^2} & -2 \\ 0 & 1 & \frac{\eta}{\pi}-\frac{\mu\pi}{\eta} & -(\frac{\mu\pi}{\eta}+\frac{\eta}{\pi}) \\ -P^1_{2\eta-2,\eta}(\mu) & -Q^1_{2\eta-2,\eta}(\mu) & 1 & 0 \\ -P^1_{2\eta-1,\eta}(\mu) & -Q^1_{2\eta-1,\eta}(\mu) & 0 & 1 \end{pmatrix}$$

Proof. This follows easily from Lemma 0.6. □

Lemma 0.8. *We have that, for μ finite;*

$P^1_{2\eta-2,\eta}(\mu) \simeq P^1_{2\eta-1,\eta}(\mu) \simeq \frac{e^{2\pi\sqrt{\mu}}+e^{-2\pi\sqrt{\mu}}}{2}$

$Q^1_{2\eta-2,\eta}(\mu) \simeq Q^1_{2\eta-1,\eta}(\mu) \simeq \frac{e^{2\pi\sqrt{\mu}}-e^{-2\pi\sqrt{\mu}}}{2\sqrt{\mu}}$

In particularly, there exists $C_\mu \in \mathcal{R}$;

with $max(P^1_{2\eta-2,\eta}(\mu), Q^1_{2\eta-2,\eta}(\mu), P^1_{2\eta-1,\eta}(\mu), Q^1_{2\eta-1,\eta}(\mu)) \leq C_\mu$, $(\dagger\dagger)$

Moreover;

$\frac{dP^1_j}{d\mu} = \frac{\pi j}{2\eta}Q^1_{j-1}$, $\frac{dQ^1_j}{d\mu} = \frac{\pi j P^1_{j-1}}{2\eta\mu} - \frac{Q^1_j}{2\sqrt{\mu}}$ *for* $0 \leq j \leq 2\eta-1$

$\frac{dP^1_j}{d\mu} \simeq \frac{\pi Q^1_{j-1}}{2}$, $\frac{dQ^1_j}{d\mu} \simeq \frac{\pi P^1_{j-1}}{2\mu} - \frac{Q^1_j}{2\sqrt{\mu}}$, *for* $j \in \{2\eta-2, 2\eta-1\}$

Proof. For $y \in \mathcal{C}$, $(1+\frac{y}{\eta})^\eta \simeq e^y$, hence, as $(1+\frac{y}{\eta})^\eta$ is S-continuous, $(1+\frac{y}{\eta})^\eta \simeq e^y$, for y finite, $(*)$. It follows that;

$(1+\frac{\pi\sqrt{x}}{\eta})^{2\eta-1} = [(1+\frac{\pi\sqrt{x}}{\eta})^\eta]^2(1+\frac{\pi\sqrt{x}}{\eta})^{-1}$

$\simeq e^{2\pi\sqrt{x}}$

as $(1+\frac{\pi\sqrt{x}}{\eta})^{-1} \simeq 1$. The remaining cases follows similarly, and then the claim $(\dagger\dagger)$ is clear.

The final results are simple calculations, left to the reader.

□

Lemma 0.9. *If $\mu \simeq -n^2$, with $n \in \mathcal{Z}$, then $det(\overline{D}) \simeq 0$. If $\mu \simeq -n^2$, with $n \in \mathcal{Z}_{\neq 0}$, then $\frac{d}{d\mu}(det(\overline{D})) \notin \mathcal{V}_0$*

Proof. Expanding $det(\overline{D})$ along the second row, and using the fact that μ is finite, $\frac{\mu\pi}{\eta} \simeq \frac{\mu\pi}{\eta^2} \simeq 0$, $(\dagger\dagger)$ in Lemma 0.8, and $P^1_{2\eta-2} \simeq P^1_{2\eta-1}$, we obtain that;

$det(\overline{D}) \simeq -(1+P^1_{2\eta-2}-2P^1_{2\eta-1})-\frac{\eta}{\pi}(-Q^1_{2\eta-2}-2E)-\frac{\eta}{\pi}(Q^1_{2\eta-1}+E)$

$\simeq (P^1_{2\eta-1}-1)+\frac{\eta}{\pi}(Q^1_{2\eta-2}-Q^1_{2\eta-1}+E)$

where $E=P^1_{2\eta-2}Q^1_{2\eta-1}-P^1_{2\eta-1}Q^1_{2\eta-2}=\frac{\pi}{\eta}(1-\frac{\pi^2\mu}{\eta^2})^{2\eta-2}$. We have that $\sqrt{\mu}\simeq in$, with $n\in\mathcal{Z}_{\geq 0}$. Hence;

$P^1_{2\eta-1,\eta}(\mu)\simeq cos(2\pi n)=1 \qquad Q^1_{2\eta-1,\eta}(\mu)\simeq \frac{sin(2\pi n)}{n}=0$

We have;

$\frac{\eta}{\pi}((1+\frac{\pi\sqrt{\mu}}{\eta})^{2\eta-1}-(1+\frac{\pi\sqrt{\mu}}{\eta})^{2\eta-2})$

$=(1+\frac{\pi\sqrt{\mu}}{\eta})^{2\eta-2}(\frac{\eta}{\pi}((1+\frac{\pi\sqrt{\mu}}{\eta})-1))$

$=(1+\frac{\pi\sqrt{\mu}}{\eta})^{2\eta-2}\sqrt{\mu}$

Similarly;

$\frac{\eta}{\pi}((1-\frac{\pi\sqrt{\mu}}{\eta})^{2\eta-1}-(1-\frac{\pi\sqrt{\mu}}{\eta})^{2\eta-2})$

$\simeq (1-\frac{\pi\sqrt{\mu}}{\eta})^{2\eta-2}\sqrt{\mu}$

Hence;

$\frac{\eta}{\pi}(P^1_{2\eta-2}-P^1_{2\eta-1})\simeq -\sqrt{\mu}P1_{2\eta-2}\simeq -\sqrt{\mu}\frac{cos(2\pi n)}{\sqrt{\mu}}=-cos(2\pi n)=-1$

$\frac{\eta}{\pi}(Q^1_{2\eta-2}-Q^1_{2\eta-1})\simeq -\sqrt{\mu}Q^1_{2\eta-2}\simeq \frac{-\sqrt{\mu}isin(2\pi n)}{\sqrt{\mu}}=-isin(2\pi n)=0$

We have that;

$\frac{\eta}{\pi}E=\frac{\eta}{\pi}(P^1_{2\eta-2}Q^1_{2\eta-1}-P^1_{2\eta-1}Q^1_{2\eta-2})$

$=\frac{\eta}{\pi}(((P^1_{2\eta-2}-P^1_{2\eta-1})Q^1_{2\eta-1})+((Q^1_{2\eta-2}-Q^1_{2\eta-1})P^1_{2\eta-1}))$

$\simeq -cos(2\pi n)\frac{sin(2\pi n}{n}-isin(2\pi n)cos(2\pi n)$

$=(-\frac{1}{n}-i)sin(2\pi n)cos(2\pi n)=0$

Hence, $det(\overline{D}) \simeq 0$ as required.

For the second part, using the same reasoning as above, we have that;

$det(\frac{d\overline{D}}{d\mu}) \simeq -(\frac{dP^1_{2\eta-2}}{d\mu} - 2\frac{dP^1_{2\eta-1}}{d\mu}) - \frac{\eta}{\pi}(-\frac{dQ^1_{2\eta-2}}{d\mu} - 2\frac{dE}{d\mu}) - \frac{\eta}{\pi}(\frac{dQ^1_{2\eta-1}}{d\mu} + \frac{dE}{d\mu})$

$\simeq \frac{-\pi}{2}(Q^1_{2\eta-3} - 2Q^1_{2\eta-2}) - \frac{\eta}{\pi}(-(\pi\frac{P^1_{2\eta-3}}{2\mu} - \frac{Q^1_{2\eta-2}}{2\sqrt{\mu}}) - 2\frac{dE}{d\mu})$

$-\frac{\eta}{\pi}((\pi\frac{P^1_{2\eta-2}}{2\mu} - \frac{Q^1_{2\eta-1}}{2\sqrt{\mu}}) + \frac{dE}{d\mu})$

$\simeq \frac{-\pi}{2}(Q^1_{2\eta-3} - 2Q^1_{2\eta-2}) - \frac{\eta}{\pi}(-(\pi\frac{P^1_{2\eta-3}}{2\mu} - \frac{Q^1_{2\eta-2}}{2\sqrt{\mu}}) + 2\frac{\pi^2(2\eta-2)E}{\eta^2})$

$-\frac{\eta}{\pi}((\pi\frac{P^1_{2\eta-2}}{2\mu} - \frac{Q^1_{2\eta-1}}{2\sqrt{\mu}}) - \frac{\pi^2(2\eta-2)E}{\eta^2})$

$\simeq \frac{\eta}{2\mu}(P^1_{2\eta-3} - P^1_{2\eta-2}) - \frac{\eta}{2\pi\sqrt{\mu}}(Q^1_{2\eta-2} - Q^1_{2\eta-1}) - E\pi$

$\simeq -\frac{\pi}{2\mu} \neq 0$

□

Lemma 0.10. *For every $n \in \mathcal{Z}_{\geq 0}$, there exists a unique $\mu \simeq -n^2$, an eigenvalue for Δ_η.*

Proof. By Lemma 0.9, we have that, if $n \in \mathcal{Z}_{\geq 0}$, $R_{4\eta}(in) = \epsilon \simeq 0$, ([1]). Let $R_{4\eta,\epsilon} = R_{4\eta} - \epsilon$, and let $p_{4\eta,\epsilon}(x, y) = R_{4\eta,\epsilon}(y) - x$, with coefficients $\{d_{ij} : 0 \leq i \leq 1, 0 \leq j \leq 4\eta\}$ so that $p_{4\eta,\epsilon}(0, in)$ holds. We have

[1] A simple computation shows that;

$det(\overline{D}) = 1 + (\frac{-1}{2}(1 - \frac{\mu\pi^2}{\eta^2}) + \frac{1}{2\sqrt{\mu}}(\frac{\eta}{\pi} - \frac{\mu\pi}{\eta}))(1 + \frac{\pi\sqrt{\mu}}{\eta})^{2\eta-2}$

$+(\frac{-1}{2}(1 - \frac{\mu\pi^2}{\eta^2}) - \frac{1}{2\sqrt{\mu}}(\frac{\eta}{\pi} - \frac{\mu\pi}{\eta}))(1 - \frac{\pi\sqrt{\mu}}{\eta})^{2\eta-2}$

$+(1 - \frac{1}{2\sqrt{\mu}}(\frac{\eta}{\pi} - \frac{\mu\pi}{\eta}))(1 + \frac{\pi\sqrt{\mu}}{\eta})^{2\eta-1}$

$+(1 + \frac{1}{2\sqrt{\mu}}(\frac{\eta}{\pi} - \frac{\mu\pi}{\eta}))(1 - \frac{\pi\sqrt{\mu}}{\eta})^{2\eta-1}$

$+(\frac{-2\mu\pi^2}{\eta^2})(1 - \frac{\pi^2\mu}{\eta^2})^{2\eta-1}$

$= 1 + \frac{\eta}{2\pi\sqrt{\mu}} + {}^*\sum_{i=1}^{2\eta-2}\frac{\pi^{i-1}\mu^{\frac{i-1}{2}}C_i^{2\eta-2}}{2\eta^{i-1}} - {}^*\sum_{i=1}^{2\eta-2}\frac{\pi^{i+1}\mu^{\frac{i+1}{2}}C_i^{2\eta-2}}{2\eta^{i+1}}$

$-\frac{\eta}{2\pi\sqrt{\mu}} + {}^*\sum_{i=1}^{2\eta-2}\frac{\pi^{i-1}\mu^{\frac{i-1}{2}}C_i^{2\eta-2}(-1)^{i+1}}{2\eta^{i-1}} - {}^*\sum_{i=0}^{2\eta-2}\frac{\pi^{i+1}\mu^{\frac{i+1}{2}}C_i^{2\eta-2}(-1)^{i+1}}{2\eta^{i+1}}$

$-\frac{\eta}{2\pi\sqrt{\mu}} - {}^*\sum_{i=1}^{2\eta-1}\frac{\pi^{i-1}\mu^{\frac{i-1}{2}}C_i^{2\eta-1}}{2\eta^{i-1}} + {}^*\sum_{i=1}^{2\eta-1}\frac{\pi^{i+1}\mu^{\frac{i+1}{2}}C_i^{2\eta-1}}{2\eta^{i+1}}$

$+\frac{\eta}{2\pi\sqrt{\mu}} -^* \sum_{i=1}^{2\eta-1} \frac{\pi^{i-1}\mu^{\frac{i-1}{2}} C_i^{2\eta-1}(-1)^{i+1}}{2\eta^{i-1}} +^* \sum_{i=1}^{2\eta-1} \frac{\pi^{i+1}\mu^{\frac{i+1}{2}} C_i^{2\eta-1}(-1)^{i+1}}{2\eta^{i+1}}$

$-^* \sum_{i=1}^{2\eta-2} \frac{\pi^i\mu^{\frac{i}{2}} C_i^{2\eta-2}}{2\eta^i}$

$+^* \sum_{i=1}^{2\eta-2} \frac{\pi^i\mu^{\frac{i}{2}} C_i^{2\eta-2}(-1)^{i+1}}{2\eta^i}$

$+^* \sum_{i=1}^{2\eta-2} \frac{\pi^{i+2}\mu^{\frac{i+2}{2}} C_i^{2\eta-2}}{\eta^{i+2}}$

$+^* \sum_{i=1}^{2\eta-2} \frac{\pi^{i+2}\mu^{\frac{i+2}{2}} C_i^{2\eta-2}(-1)^i}{\eta^{i+2}}$

$+^* \sum_{i=1}^{2\eta-1} \frac{\pi^i\mu^{\frac{i}{2}} C_i^{2\eta-1}}{\eta^i}$

$+^* \sum_{i=1}^{2\eta-1} \frac{\pi^i\mu^{\frac{i}{2}} C_i^{2\eta-1}(-1)^i}{\eta^i}$

$+^* \sum_{i=0}^{2\eta-1} \frac{2\pi^{2i+2}\mu^{i+1} C_i^{2\eta-1}(-1)^{i+1}}{2\eta^{2i+2}} \quad (*)$

.... $det(\overline{D}) = 1 +^* \sum_{i=0}^{2\eta-3} \frac{\pi^i\mu^{\frac{i}{2}} C_{i+1}^{2\eta-2}}{2\eta^i} -^* \sum_{i=2}^{2\eta-1} \frac{\pi^i\mu^{\frac{i}{2}} C_{i-1}^{2\eta-2}}{2\eta^i}$

$+^* \sum_{i=0}^{2\eta-3} \frac{\pi^i\mu^{\frac{i}{2}} C_{i+1}^{2\eta-2}(-1)^i}{2\eta^i} -^* \sum_{i=2}^{2\eta-1} \frac{\pi^i\mu^{\frac{i}{2}} C_{i-1}^{2\eta-2}(-1)^i}{2\eta^i}$

$-^* \sum_{i=0}^{2\eta-2} \frac{\pi^i\mu^{\frac{i}{2}} C_{i+1}^{2\eta-1}}{2\eta^i} +^* \sum_{i=2}^{2\eta} \frac{\pi^i\mu^{\frac{i}{2}} C_{i-1}^{2\eta-1}}{2\eta^i}$

$-^* \sum_{i=0}^{2\eta-2} \frac{\pi^i\mu^{\frac{i}{2}} C_{i+1}^{2\eta-1}(-1)^i}{2\eta^i} +^* \sum_{i=2}^{2\eta} \frac{\pi^i\mu^{\frac{i}{2}} C_{i-1}^{2\eta-1}(-1)^i}{2\eta^i}$

$-^* \sum_{i=1}^{2\eta-2} \frac{\pi^i\mu^{\frac{i}{2}} C_i^{2\eta-2}}{2\eta^i}$

$+^* \sum_{i=1}^{2\eta-2} \frac{\pi^i\mu^{\frac{i}{2}} C_i^{2\eta-2}(-1)^{i+1}}{2\eta^i}$

$+^* \sum_{i=3}^{2\eta} \frac{\pi^i\mu^{\frac{i}{2}} C_{i-2}^{2\eta-2}}{\eta^i}$

$+^* \sum_{i=3}^{2\eta} \frac{\pi^i\mu^{\frac{i}{2}} C_{i-2}^{2\eta-2}(-1)^i}{\eta^i}$

$+^* \sum_{i=1}^{2\eta-1} \frac{\pi^i\mu^{\frac{i}{2}} C_i^{2\eta-1}}{\eta^i}$

$+^* \sum_{i=1}^{2\eta-1} \frac{\pi^i\mu^{\frac{i}{2}} C_i^{2\eta-1}(-1)^i}{\eta^i}$

$+^* \sum_{i=2, i\ even}^{4\eta} \frac{2\pi^i\mu^{\frac{i}{2}} C_{\frac{i}{2}-1}^{2\eta-1}(-1)^{\frac{i}{2}}}{2\eta^i}$

....

$det(\overline{D}) = 1 + (C_1^{2\eta-2} - C_1^{2\eta-1})$

$+^* \sum_{i=3, i\ odd}^{2\eta-3} \frac{\pi^i\mu^{\frac{i}{2}}}{2\eta^i} [C_{i+1}^{2\eta-2}(1+(-1)^i) - C_{i+1}^{2\eta-1}(1+(-1)^i) - C_{i-1}^{2\eta-2}(1+(-1)^i)$

that $\frac{\partial p_{4\eta,\epsilon}}{\partial x}|_{(0,in)} = 1 \neq 0$, hence, by Newton's Theorem, there exists a $*$-power series $\gamma(x) = {}^*\sum_{j\in {}^*\mathcal{Z}_{\geq 0}} ejx^j$, such that $p_{4\eta,\epsilon}(x,\gamma(x)) = 0$, with $\gamma(0) = in$. We have that $|d_{ij}| \leq C+1$, (show bound is uniform in η) for $0 \leq i \leq 1, 0 \leq j \leq 4\eta$, need uniform bound on increase in coefficients of γ, $\leq D^j$?, some $D \in \mathcal{R}$. Then, if $\delta \simeq 0$, $\gamma(\delta) \simeq 0$, $|\gamma(\delta)| \leq {}^*\sum_{j\in {}^*\mathcal{Z}_{\geq 0}} (D\delta)^j = \frac{D\delta}{1-D\delta} \simeq 0$. Taking $\epsilon = \delta$, we obtain $p_{4\eta,\epsilon}(\epsilon,\gamma(\epsilon))$, so that $p_{4\eta}(0,\gamma(\epsilon))$, and $det(\overline{D})(\gamma(\epsilon)) = 0$. As $\gamma(\epsilon) \simeq in$, we obtain the result.

□

Lemma 0.11. *If $(\Delta_\eta - \mu I)^r(f) = 0$, $f \neq 0$, μ finite, $r \in {}^*\mathcal{Z}_{\geq 0}$, then ${}^\circ\mu \in V^\Delta$.*

Proof. Assume first that $r \in \mathcal{Z}_{\geq 0}$. An easy adaptation of Lemma 0.5 in [**?**], shows that, if $G : {}^*[a,b] \times {}^*\mathcal{C}^n \rightarrow {}^*\mathcal{C}^n$ is internal and S-continuous, with the same hypotheses on ${}^\circ G$, then for $\overline{x}$ as defined, $\overline{x}'(t) = {}^\circ G(t,\overline{x}(t))$. For μ finite;

$+C_{i-1}^{2\eta-1}(1+(-1)^i) - C_i^{2\eta-2}(1+(-1)^{i+1}) + 2C_i^{2\eta-1}(1+(-1)^i) + 2C_{i-2}^{2\eta-2}(1+(-1)^i)]$

$+{}^*\sum_{i=3,i\ even}^{2\eta-3} \frac{\pi^i\mu^{\frac{i}{2}}}{2\eta^i}[C_{i+1}^{2\eta-2}(1+(-1)^i) - C_{i+1}^{2\eta-1}(1+(-1)^i) - C_{i-1}^{2\eta-2}(1+(-1)^i)$

$+C_{i-1}^{2\eta-1}(1+(-1)^i) - C_i^{2\eta-2}(1+(-1)^{i+1}) + 2C_i^{2\eta-1}(1+(-1)^i) + 2C_{i-2}^{2\eta-2}(1+(-1)^i)$

$+2C_{\frac{i}{2}-1}^{2\eta-1}(-1)^{\frac{i}{2}}]$

Evaluate remaining terms $i = 1, 2(even), 2\eta-2, 2\eta-1, 2\eta, {}^*\sum_{2\eta+1}^{4\eta}$.

...

Referring to $(*)$, we have that, for i even;

$${}^*\sum_{i=1}^{2\eta-2} \frac{\pi^{i-1}\mu^{\frac{i-1}{2}}C_i^{2\eta-2}}{2\eta^{i-1}} + {}^*\sum_{i=1}^{2\eta-2} \frac{\pi^{i-1}\mu^{\frac{i-1}{2}}C_i^{2\eta-2}(-1)^{i+1}}{2\eta^{i-1}}$$

$$-{}^*\sum_{i=1}^{2\eta-1} \frac{\pi^{i-1}\mu^{\frac{i-1}{2}}C_i^{2\eta-1}}{2\eta^{i-1}} - {}^*\sum_{i=1}^{2\eta-1} \frac{\pi^{i-1}\mu^{\frac{i-1}{2}}C_i^{2\eta-1}(-1)^{i+1}}{2\eta^{i-1}} = 0$$

and, for i odd;

$$= {}^*\sum_{i=1}^{2\eta-2} \frac{\pi^{i-1}\mu^{\frac{i-1}{2}}(C_i^{2\eta-2} - C_i^{2\eta-1})}{\eta^{i-1}} - \frac{\pi^{2\eta-2}\mu^{\eta-1}}{\eta^{2\eta-2}}$$

$$\simeq {}^*\sum_{i=1}^{2\eta-2} \frac{\pi^{i-1}\mu^{\frac{i-1}{2}}(-C_{i-1}^{2\eta-2})}{\eta^{i-1}} \quad (\dagger)$$

It follows that $\overline{D} = R_{4\eta}(\sqrt{\mu})$, where $R_{4\eta}$ is a $*$-polynomial of degree 4η, with coefficients $\{c_i\}_{0\leq i\leq 4\eta}$. Inspection of $(\dagger)$ and the other terms, shows that there exists $S \in \mathcal{R}$, with $|c_i| \leq S$, for $0 \leq i \leq 4\eta$.

$G(t, u_1, \ldots, u_{2r}) = (u_2, \ldots, u_{2r}, \sum_{j=0}^{r} u_{2(r-j)}\mu^j(-1)^{j+1}C_j^r)$

satisfies this criteria, hence, we can use Lemma 0.6 of **??** to obtain that $(\Delta - {}^\circ\mu I)^r({}^\circ f) = 0$ and ${}^\circ f \in C^\infty([-\pi, \pi])$, ([2]). Using Lemma 0.2, we obtain the result. If $r \in {}^*\mathcal{Z}_{\geq 0}$ is infinite, then we can assume that $f_s = (\Delta_\eta - \mu I)^s(f) \neq 0$, for $0 \leq s < r$, and then $(\Delta_\eta - \mu I)^{r-s}(f_s) = 0$. Taking $r - s$ to be finite, we can apply the first part of the lemma. □

Lemma 0.12. *Let $g, h : \overline{\mathcal{H}_\eta} \rightarrow {}^*\mathcal{C}$ be measurable. Then;*

(i). $\int_{\overline{\mathcal{H}_\eta}} g'(y)d\mu_\eta(y) = 0$

(ii). $(gh)' = g'h^{sh} + gh'$

(iii). $\int_{\overline{\mathcal{H}_\eta}} (g'h)(y)d\mu_\eta(y) = -\int_{\overline{\mathcal{H}_\eta}} g^{sh}h'd\mu_\eta(y)$

(iv). $\int_{\overline{\mathcal{H}_\eta}} g(y)d\mu_\eta(y) = \int_{\overline{\mathcal{H}_\eta}} g^{sh}(y)d\mu_\eta(y) = \int_{\overline{\mathcal{H}_\eta}} g^{rsh}(y)d\mu_\eta(y)$

(v). $(g')^{sh} = (g^{sh})'$

(vi). $\int_{\overline{\mathcal{H}_\eta}} (g''h)(y)d\mu_\eta(y) = \int_{\overline{\mathcal{H}_\eta}} (g^{sh^2}h'')(y)d\mu_\eta(y)$

Proof. In the first part, for (i), we have, using Definitions **??** and **??**, that;

$$\int_{\overline{\mathcal{H}_\eta}} g'(y)d\mu_\eta(y)$$

$$= \frac{\pi}{\eta}[{}^*\sum_{0 \leq j \leq 2\eta - 2} \frac{\eta}{\pi}[g(-\pi + \pi(\tfrac{j+1}{\eta})) - g(-\pi + \pi(\tfrac{j}{\eta}))]$$

$$+\frac{\eta}{\pi}[g(-\pi) - g(\pi - \tfrac{\pi}{\eta}]] = 0$$

The proofs of $(ii), (iii)$ are as in Lemma 0.12 of [**?**]. (iv) is clear. (v) follows easily from Definitions **??** and (vi) follows, repeating the result of (iii), and applying (v).

□

Definition 0.13. *We let $S : V(\overline{\mathcal{H}_\eta}) \rightarrow V(\overline{\mathcal{H}_\eta})$ be defined by;*

[2]Observe the extension of Lemma 0.6 to the endpoint π, taking the initial condition as $f(\pi)$

$S(f) = \Delta_\eta(f)^{rsh^2} - \Delta_\eta(f)$

Lemma 0.14. *Δ_η is almost self adjoint, in the sense that $\Delta_\eta * = \Delta_\eta + S$.*

Proof. We have, using $(iv), (v), (vi)$ above, that, if $\{f, g\} \subset V(\overline{\mathcal{H}_\eta})$;

$< \Delta_\eta(f), g >$

$=< f^{sh^2}, \Delta_\eta(g) >$

$=< f, (\Delta_\eta(g))^{rsh^2} >$

$=< f, \Delta_\eta(g) > + < f, \Delta_\eta(g)^{rsh^2} - \Delta_\eta(g)) >$

$=< f, (\Delta_\eta + S)(g) >$

□

Lemma 0.15. $\Delta_\eta = \Delta_{\eta,1} + \Delta_{\eta,2}$

where $\Delta_{\eta,1}$ is self adjoint, $\Delta_{\eta,2}$ is anti self adjoint, and $\Delta_{\eta,1} = \Delta_\eta + \frac{S}{2}$.

Proof. We have that;

$\Delta_{\eta,1} = \frac{(\Delta_\eta + \Delta_\eta^*)}{2} = \frac{(\Delta_\eta + \Delta_\eta + S)}{2} = \Delta_\eta + \frac{S}{2}$

$\Delta_{\eta,2} = \frac{(\Delta_\eta - \Delta_\eta^*)}{2}$

□

Lemma 0.16. *If $\mu_1 \neq \mu_2$, with μ_2 finite, $\Delta(f_1) = \mu_1 f_1$, $\Delta(f_2) = \mu_2 f_2$, and $||f_1|| = ||f_2|| = 1$, then $< f_1, f_2 > \simeq 0$;*

Proof. Without loss of generality, we can assume that $\mu_1 \neq 0$, $\mu_2 \neq 0$. Then;

$< \mu_1 f_1, f_2 >$

$=< \Delta(f_1), f_2 >$

$=< f_1, (\Delta + S)(f_2) >$

$=< f_1, \mu_2 f_2 > + < f_1, S(f_2) >$

By Lemma 0.11, we have that $S(f_2)(x) \simeq 0$, for $x \in \mathcal{H}_\eta$, hence, $< f_1, S(f_2) >= \epsilon \simeq 0$. It follows that $< f_1, f_2 >= \frac{\epsilon}{\mu_1 - \overline{\mu_2}} \simeq 0$. □

Lemma 0.17. *There exists a basis* $\{e_i : 1 \leq i \leq \eta\}$ *for* $V(\overline{\mathcal{H}_\eta})$, *with corresponding eigenvalues* $\{\lambda_i : 1 \leq i \leq \eta\} \subset {}^*\mathcal{C}$, *such that* $\Delta(e_i) = \mu_i e_i$, *and* μ_i *finite, for finite* i, *with the decay rate;*

$< f, e_i > | \leq \frac{C_{\Delta^t f}}{|\mu_i|^t}$

where, for $t \in {}^*\mathcal{Z}_{\geq 1}$, $C_{\Delta^t f} = max\{|\Delta^t f(x)| : x \in \mathcal{H}_\eta :\}$.

Proof. By transfer of the Theorem on Jordan Canonical form, there exists a basis $\{e_i : 1 \leq i \leq \eta\}$ of generalised eigenvectors for Δ, with the index i of each generalised eigenspace V_{μ_i} corresponding to the factor $(x - \mu_i)^i$ in $det(\overline{D})$. By Lemma **??**, (resultant calculation), $i = 1$, for each μ_i, so we obtain a basis of eigenvectors for Δ, $\Delta(e_i) = \mu_i e_i$. We can assume that $||e_i|| = 1$, for $1 \leq i \leq \eta$. Then;

$< \Delta^t(f^{rsh^{2t}}), e_i >$

$=< f, \Delta^t(e_i) >= \overline{\mu_i^t} < f, e_i >$

$| < f, e_i > |$

$\leq \frac{1}{|\mu_i^t|} < \Delta(f^{rsh^{2t}}), e_i >$

$\leq \frac{C_{\Delta^t(f^{rsh^{2t}})}}{|\mu_i|^t} = \frac{C_{\Delta^t(f)}}{|\mu_i|^t}$.

□

Lemma 0.18. *If* $W \subseteq V(\overline{\mathcal{H}_\eta})$, *spanned by* $\{e_1, \ldots, e_\kappa\}$, *then the orthogonal projection* $pr_W(f)$ *is given by;*

${}^*\sum_{1 \leq i \leq \kappa} \lambda_i e_i$

where

$$\overline{\lambda} = \begin{pmatrix} \lambda_1 \\ . \\ . \\ \lambda_j \\ . \\ . \\ \lambda_n \end{pmatrix}$$

$$\overline{A} = \begin{pmatrix} A_{11} & \dots & A_{1j} & \dots & A_{1n} \\ \dots & \dots & \dots & & \\ A_{i1} & \dots & A_{ij} & \dots & A_{in} \\ \dots & \dots & \dots & & \\ A_{n1} & \dots & A_{nj} & \dots & A_{nn} \end{pmatrix}$$

$$\overline{t} = \begin{pmatrix} < e_1, f > \\ . \\ . \\ < e_j, f > \\ . \\ . \\ < e_n, f > \end{pmatrix}$$

and $\overline{A\lambda} = \overline{t}$, $A_{ij} =< e_i, e_j >$, *for* $1 \leq i \leq j \leq n$.

.....

Lemma 0.19. *If* $f \in V(\overline{\mathcal{H}_\eta})$ *and* $C_{f''}$ *is finite, then;*

$f \simeq \sum_{1 \leq i \leq \eta} < f, e_i > e_i$

Proof. We can apply the Gramm-Schmidt orthogonalisation procedure to the basis $\{e_i : 1 \leq i \leq \eta\}$, beginning with e_η, to obtain an orthonormal basis $\{\overline{e_i} : 1 \leq i \leq \eta\}$ such that;

$f = {}^* \sum_{1 \leq i \leq \eta} < f, \overline{e_i} > \overline{e_i}$

and $\overline{e_{\eta-i}} = {}^* \sum_{0 \leq j \leq i} \lambda_{i,j} \overline{e_{\eta-j}}$, with $|\lambda_{i,j}| \leq 1$, [3]. We claim that;

$\overline{e_{\eta-i}} = {}^* \sum_{0 \leq j \leq i} \lambda'_{i,j} e_{\eta-j}$, with $|\lambda'_{i,j}| \leq 2^{\frac{i-1}{2}}$, $|\lambda'_{0,0}| = 1$, [4].

[3]Explicitly, for $j < i$ we can take;

$\lambda_{i,j} = \frac{-<\overline{e_{\eta-j}}, e_{\eta-i}>}{\sqrt{1+^* \sum_{j<i} |<\overline{e_{\eta-j}}, e_{\eta-i}>|^2}}$, $\lambda_{i,i} = \frac{1}{\sqrt{1+^* \sum_{j<i} |<\overline{e_{\eta-j}}, e_{\eta-i}>|^2}}$

[4]This can be shown using induction. The base case is trivial. Assume the result is true for the coefficients $\{\lambda'_{i_0,j} : 0 \leq j \leq i_0\}$. We have that

It follows that $| < f, \overline{e_{\eta-i}} > |$

$\leq {}^* \sum_{0\leq j\leq i} |\lambda'_{i,j}|| < f, e_{\eta-j} > |$

$\leq 2^{\frac{i-1}{2}} {}^* \sum_{0\leq j\leq i} \frac{C_{\Delta^t(f)}}{|\mu_{\eta-j}|^t}$

$\leq 2^{\frac{i-1}{2}} \frac{C_{\Delta^t(f)}}{|\mu_{\eta-i}|^{t-1}}$

(Assuming 1-spacing on infinite eigenvalues). We have that;

$f = {}^* \sum_{1\leq i\leq \eta} < f, \overline{e_i} > \overline{e_i}$

$= {}^* \sum_{1\leq i\leq \eta} < f, (\overline{e_i} - e_i) + e_i > ((\overline{e_i} - e_i) + e_i)$

$= {}^* \sum_{1\leq i\leq \eta} < f, (\overline{e_i} - e_i) > (\overline{e_i} - e_i)$

$+ {}^* \sum_{1\leq i\leq \eta} < f, (\overline{e_i} - e_i) > e_i$

$+ {}^* \sum_{1\leq i\leq \eta} < f, e_i > (\overline{e_i} - e_i)$

$+ {}^* \sum_{1\leq i\leq \eta} < f, e_i > e_i$

Lemma 0.20. *If $f \in V(\overline{\mathcal{H}_\eta})$ and $C_{f''}$ is finite, then, for $n \in \mathcal{Z}_{\geq 1}$;*

$f \simeq \sum_{1\leq i\leq n} < f, e_i > e_i + \overline{w}_{n+1}$

where $\overline{w_{n+1}} \in span(\{e_i\}_{n+1\leq i\leq \eta})$

and $lim_{n\rightarrow\infty} ||\overline{w_{n+1}}|| = 0$

Proof. □

□

References

Mathematics Department, Harrison Building, Streatham Campus, University of Exeter, North Park Road, Exeter, Devon, EX4 4QF, United Kingdom

E-mail address: t.depiro@curvalinea.net

$|\lambda'_{i_0+1,0}| \leq |{}^* \sum_{0\leq k\leq i_0} \lambda_{i_0+1,k} \lambda'_{k,0}|$

$\leq \sqrt{({}^* \sum_{0\leq k\leq i_0-1} 2^k) + 1} = 2^{\frac{i_0}{2}}$. A similar argument works for the coefficients $\{\lambda'_{i_0+1,k} : 0 \leq j \leq i_0 + 1\}$, with fewer steps.