NONSTANDARD MARTINGALES, MARKOV CHAINS AND THE HEAT EQUATION

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Abstract. We construct a nonstandard martingale from a discrete Markov chain. This is shown to be useful for solving the heat equation with a non smooth initial condition. We show that the nonstandard solution to the heat equation with a smooth initial condition specialises to the classical solution

One of the most fundamental results in the theory of Markov chains is the following;

Theorem 0.1. *Let P be the transition matrix of an irreducible, aperiodic,positive recurrent Markov chain, $\{X_n\}_{n\geq 0}$, with invariant distribution π. Then, for any initial distribution, $P(X_n = j) \rightarrow \pi_j$, as $n \rightarrow \infty$. In particular;*

$p_{ij}^{(n)} \rightarrow \pi_j$, *for all states* i, j, *as* $n \rightarrow \infty$

Proof. A good reference for this result is [3]. However, we give the proof as it is used and modified later. Let the initial distribution be λ, and let I be the state space. Choose $\{Y_n\}_{n\geq 0}$, such that $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ are independent, with $\{Y_n\}_{n\geq 0}$ Markov (π, P). Let $T = inf\{n \geq 1 : X_n = Y_n\}$. We claim that $P(T < \infty) = 1$, $(*)$. Let $W_n = (X_n, Y_n)$. Then $\{W_n\}_{n\geq 0}$ is a Markov chain on $I \times I$. By independence, it has transition probabilities given by;

$\overline{p}_{(i,j)(k,l)} = p_{ik}p_{jl}$

and initial distribution $\mu_{(i,j)} = \lambda_i \pi_j$. A simple calculation shows that;

$\overline{p}^{(n)}_{(i,j)(k,l)} = p^{(n)}_{ik} p^{(n)}_{jl}$ for fixed states i, j, k, l

As P is irreducible and aperiodic, we have that $min(p_{ik}^{(n)}, p_{jl}^{(n)}) > 0$, for sufficiently large n. Hence, for such n, $\overline{p}_{(i,j)(k,l)}^{(n)} > 0$ and $\overline{P}$ is irreducible. A similar straightforward calculation gives that the distribution $\mu_{(i,j)}$ is invariant for $\overline{P}$. By well known results, this implies that $\overline{P}$ is positive recurrent. Fix a state b, and let $S = inf\{n \geq 1 : X_n = Y_n = b\}$. Then S is the first passage time in the system $\{W_n\}_{n\geq 0}$ to (b, b), and $P(S < \infty) = 1$ follows by known results, and the fact that $\overline{P}$ is irreducible and recurrent. Clearly $P(S < \infty) \leq P(T < \infty)$, so $(*)$ follows. We now calculate;

$$P(X_n = j) = P(X_n = j, n \geq T) + P(X_n = j, n < T)$$

$$= P(Y_n = j, n \geq T) + P(X_n = j, n < T)$$

by definition of T and the fact that $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ have the same transition matrix. Then;

$$P(X_n = j) = P(Y_n = j, n \geq T) + P(Y_n = j, n < T)$$

$$- P(Y_n = j, n < T) + P(X_n = j, n < T)$$

$$= P(Y_n = j) - P(Y_n = j, n < T) + P(X_n = j, n < T)$$

$$= \pi_j - P(Y_n = j, n < T) + P(X_n = j, n < T) \ (**)$$

We have that $P(Y_n = j, n < T) \leq P(n < T)$ and $P(n < T) \rightarrow P(T = \infty) = 0$ as $n \rightarrow \infty$, using $(*)$. Similarly, $P(X_n = j, n < T) \rightarrow 0$ as $n \rightarrow \infty$. It follows that $P(X_n = j) \rightarrow \pi_j$, using $(**)$, as required. The final claim is a consequence of the fact that $p_{ij}^{(n)} = P(X_n = j)$ where the initial distribution of X_0 is the dirac function δ_i.

□

We now establish a rate of convergence result.

Lemma 0.2. *Let P be the transition matrix for a finite irreducible aperiodic Markov chain. Then there exists $m \geq 1$ and $\rho \in (0, 1)$, such that;*

$$|p_{ij}^{(n)} - \pi_j| \leq (1 - \rho)^{\frac{n}{m}-1}, \text{ for all states } i, j$$

Proof. From Theorem 0.1, taking the initial distribution of X_0 to be δ_i, we have that;

$P(X_n = j) = \pi_j - P(Y_n = j, n < T) + P(X_n = j, n < T)$

Hence;

$|p_{ij}^{(n)} - \pi_j| \leq P(n < T)$

As P is irreducible and aperiodic, we have that $p_{kl}^{(n)} > 0$ for all sufficiently large n, and all states k, l. As P is finite, there exists an $m \geq 1$ such that $p_{kl}^{(m)} > 0$ for all k, l. In particular, there exists $\rho \in (0, 1)$ such that $p_{kl}^{(m)} > \rho$. We have that;

$P_{k,l}(T \leq m) \geq \sum_u \overline{p}_{(k,l),(u,u)}^{(m)} = \sum_u p_{ku}^{(m)} p_{lu}^{(m)} \geq \rho \sum_u p_{ku}^{(m)} = \rho$

$P_{k,l}(T > m) \leq (1 - \rho)$

$P(T > m) = \sum_{(k,l)} P_{(k,l)}(T > m)\delta_{ik}\pi_l \leq (1 - \rho)$

Moreover;

$P(T > n) \leq P(T > [\frac{n}{m}]m)$

We claim that, for $k \geq 1$, $P(T > (k + 1)m | T > km) \leq 1 - \rho$. We have that;

$P(T > (k + 1)m | T > km)$

$= \sum_{i_{km} \neq j_{km}, i_{km-1} \neq j_{km-1}, \ldots, i, j} P(T > (k+1)m | W_{km} = (i_{km}, j_{km}), W_{km-1} = (i_{km-1}, j_{km-1}), \ldots, W_0 = (i, j))P(W_{km} = (i_{km}, j_{km}), W_{km-1} = (i_{km-1}, j_{km-1}), \ldots, W_0 = (i, j) | T > km)$

$= \sum_{i_{km} \neq j_{km}, i_{km-1} \neq j_{km-1}, \ldots, i, j} P(T > (k+1)m | W_{km} = (i_{km}, j_{km}))P(W_{km} = (i_{km}, j_{km}), W_{km-1} = (i_{km-1}, j_{km-1}), \ldots, W_0 = (i, j) | T > km)$

$\leq (1 - \rho) \sum_{i_{km} \neq j_{km}, i_{km-1} \neq j_{km-1}, \ldots, i, j} P(W_{km} = (i_{km}, j_{km}), W_{km-1} = (i_{km-1}, j_{km-1}), \ldots, W_0 = (i, j) | T > km)$

$= (1 - \rho)$

Inductively, we have that;

$P(T > km) = P(T > km, T > (k-1)m)$

$= P(T > km|T > (k-1)m)P(T > (k-1)m)$

$\leq P(T > km|T > (k-1)m)(1-\rho)^{k-1}$

$\leq (1-\rho)^k$

It follows that $|p_{ij}^{(n)} - \pi_j| \leq (1-\rho)^{[\frac{n}{m}]} \leq (1-\rho)^{\frac{n}{m}-1}$

$\square$

Lemma 0.3. *Let P define a Markov chain with N states, $\{0, 1, \ldots, N-1\}$ such that the transition probabilities of moving from state i to i-1,i,i+1 (mod N) respectively is $\frac{1}{3}$. Then P is irreducible and aperiodic, moreover, we can choose $m = N-1$ and $\rho = \frac{1}{3^{N-1}}$ in Lemma 0.2. It follows that;*

$|p_{ij}^{(n)} - \frac{1}{N}| \leq (\frac{3^{N-1}-1}{3^{N-1}})^{\frac{n}{N-1}-1} = \epsilon_n$

If N is odd, we can choose $m = \frac{N-1}{2}$ and $\rho = \frac{1}{3^{\frac{N-1}{2}}}$ in Lemma 0.2. It follows that;

$|p_{ij}^{(n)} - \frac{1}{N}| \leq (\frac{3^{\frac{N-1}{2}}-1}{3^{\frac{N-1}{2}}})^{\frac{2n}{N-1}-1} = \delta_n$

Moreover, for any initial probability distribution π_0, letting $\pi_j^n = P(X_n = j)$, we have that;

$|\pi_j^n - \frac{1}{N}| \leq \epsilon_n$, $0 \leq j \leq N-1$ $(*)$

and, similarly, with δ_n replacing ϵ_n, when N is odd. For any initial distribution λ_0 of positive numbers, with sum K, letting $\lambda_n = \lambda_0 P^n$, we have that;

$|\lambda_j^n - \frac{K}{N}| \leq K\epsilon_n$, $0 \leq j \leq N-1$

and, similarly, with δ_n replacing ϵ_n when N is odd.

Proof. The first two claims follow immediately by noting that for any 2 states k, l $p_{kl}^{((N-1)} \geq \frac{1}{3^{N-1}}$, and the transition probabilities $p_{kk} > 0$, for all states k. This also shows that we can choose $m = N-1$ and $\rho = \frac{1}{3^{N-1}}$. The final claim of the first part follows from Lemma 0.2.

The case when N is odd is similar. The penultimate claim follows by noting that $\pi_n = \pi_0 P^n$ and calculating;

$$\pi_j^n = \pi_0^0 p_{0j}^{(n)} + \pi_1^0 p_{1j}^{(n)} + \ldots + \pi_{N-1}^0 p_{N-1,j}^{(n)}$$

$$= (\pi_0^0 + \ldots + \pi_{N-1}^0)(\tfrac{1}{N}) + \pi_0^0 \epsilon_n^0 + \ldots + \pi_{N-1}^0 \epsilon_n^{N-1}$$

$$= \tfrac{1}{N} + \epsilon'_n$$

where $\epsilon_n^j \leq \epsilon_n$, for $0 \leq j \leq N-1$ and $\epsilon'_n \leq \epsilon_n$. The case N odd is the same. The final claim follows by observing that $\lambda_n = \lambda_0 P^n = K\pi_0 P^n$ for an initial probability distribution π_0. Then multiply the result $(*)$ through by K, similarly for N odd.

$\square$

Lemma 0.4. *Let P define a non standard Markov chain with η states, $\{0, 1, \ldots, \eta - 1\}$, for η infinite, such that the transition probabilities of moving from state i to i-1,i,i+1 (mod η) respectively is $\frac{1}{3}$. Then, if ϵ is an infinitesimal and*

$$n \geq (\eta - 1)\left(1 + \frac{log(\epsilon)}{log(3^{\eta-1}-1) - log(3^{\eta-1})}\right) \ (*)$$

we have for any initial probability distribution π_0, that;

$$\pi_j^n \simeq \tfrac{1}{\eta} \text{ for } 0 \leq j \leq \eta - 1 \ (**)$$

We obtain the same result, if η is odd, ϵ is an infinitesimal and;

$$n \geq \tfrac{\eta-1}{2}\left(1 + \frac{log(\epsilon)}{log(3^{\frac{\eta-1}{2}}-1) - log(3^{\frac{\eta-1}{2}})}\right) \ (***)$$

for any initial probability distribution π_0. If λ_0 is a nonstandard distribution with sum λ, possibly infinite, then if ϵ is an infinitesimal with $\lambda\epsilon \simeq 0$, and n satisfies $(*)$, we obtain that;

$$\pi_j^n \simeq \tfrac{\lambda}{\eta} \text{ for } 0 \leq j \leq \eta - 1 \ (****)$$

and, the same result holds when η is odd and n satisfies $(***)$.

Proof. Let $Seq_1 = \{f : \mathcal{N} \rightarrow \mathcal{R}\}$ and $Seq_2 = \{f : \mathcal{N}^2 \rightarrow \mathcal{R}\}$. We let;

$Prob_N = \{f \in Seq_1 : (\forall_{m\geq N+1} f(m) = 0) \wedge (\forall_{1\leq m\leq N} f(m) \geq 0) \wedge \sum_{1\leq m\leq N} f(m) = 1\}$.

encode probability vectors of length N. Let $G : \mathcal{N} \rightarrow Seq_2$ be defined by;

$G(N,1,1) = G(N,1,2) = G(N,1,N) = \frac{1}{3}$, $(\forall_{m\neq 1,2,N} G(N,1,m) = 0$

$(\forall_{1\leq k\leq N-1}\forall_{2\leq m\leq N} G(N,k+1,1) = G(N,k,N),\ G(N,k+1,m) = G(N,k,m-1)$

$\forall_{k\geq N+1}\forall_{m\geq N+1} G(N,k,m) = 0$

G encodes the transition matrices for the given Markov chain with N states. Let $H : \mathcal{N}^2 \rightarrow Seq_2$ be defined by;

$(\forall_{1\leq i,j\leq N})H(1,N,i,j) = G(N,i,j)$

$(\forall_{i,j>N})H(1,N,i,j) = 0$

$(\forall_{1\leq i,j\leq N})H(n,N,i,j) = \sum_{1\leq k\leq N} H(n-1,i,k)G(N,k,j)$

$(\forall_{i,j>N})H(n,N,i,j) = 0$

H encodes the powers $G(N)^n$ of the transition matrices. We define maps $L(N,n) : Prob_N \rightarrow Prob_N$ by;

$(\forall_{1\leq j\leq N})L(N,n)(f)(j) = \sum_{1\leq k\leq N} f(k)H(n,N,k,j)$

$(\forall_{j>N})L(N,n)(f)(j) = 0$

$L(N,n)(f)$ encodes the probability vectors π^n for an initial distribution π_0 represented by f.

By a simple rearrangement, we have that the bound in $|\pi^n_j - \frac{1}{N}|$, from lemma 0.3, can be formulated in first order logic as;

$\forall N \in \mathcal{N}\forall\pi \in Prob_N\forall\epsilon \in \mathcal{R}_{>0}\forall n \in \mathcal{N}(n \geq (N-1)(1+\frac{log(\epsilon)}{log(3^{N-1}-1)-log(3^{N-1})}) \rightarrow (|L(n,N)(\pi)(j) - \frac{1}{N}| \leq \epsilon, 0 \leq j \leq N-1)$

By transfer, we obtain a corresponding result, quantifying over ${}^*\mathcal{N}$. Taking ϵ to be an infinitesimal and η to be an infinite natural number, we obtain the first result. Observe that by construction of G, H, L, the nonstandard Markov chain with η states evolves by the usual nonstandard matrix multiplication by the transition matrix, of the initial probability distribution. The remaining claims are similar and left to the reader.

□

Definition 0.5. *We let $\eta, \lambda \subset {}^*N \setminus \mathcal{N}$, and let ν satisfy the bound $(*)$ in Lemma , where ϵ is an infinitesimal with with $\lambda\epsilon \simeq 0$.*

We let $\overline{\Omega_\eta} = \{x \in {}^*\mathcal{R} : 0 \leq x < 1\}$

and $\overline{\mathcal{T}_\nu} = \{t \in {}^*\mathcal{R} : 0 \leq x \leq 1\}$

We let $\mathcal{C}_\eta$ consist of internal unions of the intervals $[\frac{i}{\eta}, \frac{i+1}{\eta})$, for $0 \leq i \leq \eta - 1$, and let $\mathcal{D}_\nu$ consist of internal unions of $[\frac{i}{\nu}, \frac{i+1}{\nu})$ and $\{1\}$, for $0 \leq i \leq \nu - 1$.

We define counting measures μ_η and λ_ν on $\mathcal{C}_\eta$ and $\mathcal{D}_\nu$ respectively, by setting $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$, $\lambda_\nu([\frac{i}{\nu}, \frac{i+1}{\nu})) = \frac{1}{\nu}$, for $0 \leq i \leq \eta - 1$, $0 \leq i \leq \nu - 1$ respectively, and $\lambda_\nu(\{1\}) = 0$.

We let $(\overline{\Omega_\eta}, \mathcal{C}_\eta, \mu_\eta)$ and $(\overline{\mathcal{T}}_\nu, \mathcal{D}_\nu, \lambda_\nu)$ be the resulting $*$-finite measure spaces, in the sense of [2]. We let $(\overline{\Omega_\eta} \times \overline{\mathcal{T}}_\nu, \mathcal{C}_\eta \times \mathcal{D}_\nu, \mu_\eta \times \lambda_\nu)$ denote the corresponding product space.

Definition 0.6. *Let $f : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{R}_{\geq 0}$ be measurable with respect to the $*\sigma$-algebra $\mathcal{C}_\eta$, in the sense of [2], and suppose that ${}^*\sum_{0 \leq i \leq \eta - 1} f(\frac{i}{\eta}) = \lambda$. We define $F : \overline{\Omega_\eta} \times \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{R}_{\geq 0}$ by;*

$F(\frac{i}{\eta}, \frac{j}{\nu}) = (\pi_f K^j)(i)$, for $0 \leq i \leq \eta - 1$, $0 \leq j \leq \nu$

$F(x, y) = F(\frac{[x\eta]}{\eta}, \frac{[y\nu]}{\nu})$, $(x, y) \in \overline{\Omega_\eta} \times \overline{\mathcal{T}_\nu}$

where π_f is the nonstandard distribution vector corresponding to f, K is the transition matrix of the above Markov chain with η states, and K^j denotes a nonstandard power.

Lemma 0.7. *Let F be as defined in Definition 0.6, then F is measurable with respect to $\mathcal{C}_\eta \times \mathcal{D}_\nu$, and, moreover $F(x, 1) \simeq C$ where*

$C=\int_{\overline{\Omega_\eta}} f d\mu_\eta$.

Proof. The first proposition follows by observing that the defining schema for F is internal and by transfer from the result for finite measures spaces, see Lemma for the mechanics of this transfer process. For the second proposition, observe that, by definition of the nonstandard integral, see [5], and the assumptions on f, we have that $C=\frac{\lambda}{\eta}$. The result then follows by the choice of ν and the result of Lemma .

□

Remarks 0.8. *F defines the evolution of a stochastic process, which we can think of as the density of a number of gas particles each moving independently and at random. This idea is made more precise in* [6]*. The final density, which we refer to as the equilibrium density is close to being constant.*

Definition 0.9. *Let $(\overline{\Omega_\eta},\mathcal{E}_\eta,\gamma_\eta)$ be a nonstandard $*$-finite measure space. We define a reverse filtration on $\overline{\Omega_\eta}$ to be an internal collection of $*\sigma$-algebras $\mathcal{E}_{\eta,i}$, indexed by $0\leq i\leq\nu$, such that;*

(i). $\mathcal{E}_{\eta,0}=\mathcal{E}_\eta$

(ii). $\mathcal{E}_{\eta,i}\subseteq\mathcal{E}_{\eta,j}$, if $i\geq j$.

We say that $F:\overline{\Omega_\eta}\times\overline{\mathcal{T}_\nu}\rightarrow {}^*\mathcal{R}$ is adapted to the filtration if F is measurable with respect to $\mathcal{E}_\eta\times\mathcal{D}_\nu$ and $F_{\frac{i}{\nu}}:\overline{\Omega_\eta}\rightarrow {}^*\mathcal{R}$ is measurable with respect to $\mathcal{E}_{\eta,i}$, for $0\leq i\leq\nu$.

If $f:\overline{\Omega_\eta}\rightarrow {}^*\mathcal{R}$ is measurable with respect to $\mathcal{E}_{\eta,j}$ and $0\leq j\leq i\leq\nu$, we define the conditional expectation $E_\eta(f|\mathcal{E}_{\eta,i})$ to be the unique $g:\overline{\Omega_\eta}\rightarrow {}^*\mathcal{R}$ such that g is measurable with respect to $\mathcal{E}_{\eta,i}$ and;

$\int_U g d\gamma_\eta=\int_U f d\gamma_\eta$

for all $U\in\mathcal{E}_{\eta,i}$. We say that $F:\overline{\Omega_\eta}\times\overline{\mathcal{T}_\nu}\rightarrow {}^*\mathcal{R}$ is a reverse martingale if;

(i). F is adapted to the reverse filtration on $\overline{\Omega_\eta}$

(ii). $E_\eta(F_{\frac{j}{\nu}}|\mathcal{E}_{\eta,i})=F_{\frac{i}{\nu}}$ for $0\leq j\leq i\leq\nu$

Given $F : \overline{\Omega_\eta} \times \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{R}$ measurable with respect to $\mathcal{E}_\eta \times \mathcal{D}_\nu$, we define the cumulative density function; $P : {}^*\mathcal{R} \times [0,1] \rightarrow {}^*\mathcal{R}$ by;

$$P(x,t) = \gamma_\eta(F_t \leq x)$$

We say that F_1 and F_2 are equivalent in distribution, if their respective cumulative density functions P_1 and P_2 coincide.

Theorem 0.10. *Let F be as in Definition 0.6, then there exists a reverse filtration on $\overline{\Omega_\eta}$ and an extension of F to $\overline{F}$ such that $\overline{F}$ is a reverse martingale. Moreover the processes F and $\overline{F}$ are equivalent in distribution.*

Proof. We define the reverse filtration, by setting $\mathcal{E}_{\eta,i}$ to be internal unions of the intervals $[\frac{j}{3^{\nu-i}\eta}, \frac{j+1}{3^{\nu-i}\eta})$ for $0 \leq j \leq 3^{\nu-i}\eta - 1$, $0 \leq i \leq \nu$. Clearly, this is an internal collection. It follows that $\mathcal{E}_\eta = \mathcal{E}_{\eta,0}$ consists of internal unions of the intervals $[\frac{j}{3^\nu\eta}, \frac{j+1}{3^\nu\eta})$ for $0 \leq j \leq 3^\nu\eta - 1$, and we define the corresponding measure γ_η by setting $\gamma_\eta([\frac{j}{3^\nu\eta}, \frac{j+1}{3^\nu\eta})) = \frac{1}{3^\nu\eta}$. Observe that $\mathcal{E}_{\eta,\nu} = \mathcal{C}_\eta$, the original $*\sigma$-algebra. We define special points of the first and second kind on $\overline{\Omega_\eta} \times \overline{\mathcal{T}_\nu}$ inductively, as follows;

Base case; the points $\{(\frac{i}{\eta}, 1) : 0 \leq i \leq \eta - 1\}$ are special points of the first kind.

Inductive hypothesis; Suppose the special points of the first and second kind on $\overline{\Omega_\eta} \times [\frac{\nu-i}{\nu}, 1]$ have been defined for $0 \leq i \leq \nu - 1$

Then a point $(x, \frac{\nu-i-1}{\nu})$ is special of the first kind if $(x, \frac{\nu-i}{\nu})$ is special of the first or second kind, in addition, the points $(x + \frac{k}{3^{i+1}\eta}, \frac{\nu-i-1}{\nu})$, $1 \leq k \leq 2$, are special of the second kind, when $(x, \frac{\nu-i}{\nu})$ is special of the first kind.

We call a point of the form $(\frac{i}{\eta}, \frac{j}{\nu})$ for $0 \leq i \leq \eta - 1$, $0 \leq j \leq \nu$ original. We associate original points to special points inductively as follows;

Base case; We associate the original point $(\frac{i}{\eta}, 1)$ to the special point $(\frac{i}{\eta}, 1)$, for $0 \leq i \leq \eta - 1$

Inductive hypothesis: Suppose we have associated original points to special points on $\overline{\Omega_\eta} \times [\frac{\nu-i}{\nu}, 1]$,for $0 \leq i \leq \nu-1$.

Then if $(x, \frac{\nu-i-1}{\nu})$ is special of the first kind, and $(\frac{i_x}{\eta}, \frac{\nu-i}{\nu})$ is associated to $(x, \frac{\nu-i}{\nu})$, we associate $(\frac{i_x}{\eta}, \frac{\nu-i-1}{\nu})$ to $(x, \frac{\nu-i-1}{\nu})$. We match original points to special points of the second kind, by associating $(\frac{i_x-1}{\eta}, \frac{\nu-i-1}{\nu})$ to $(x+\frac{1}{3^{i+1}\eta}, \frac{\nu-i-1}{\nu})$ and $(\frac{i_x+1}{\eta}, \frac{\nu-i-1}{\nu})$ to $(x+\frac{2}{3^{i+1}\eta}, \frac{\nu-i-1}{\nu})$, where $(x, \frac{\nu-i-1}{\nu})$ is special of the first kind. We adopt the convention that $\frac{-1}{\eta} = \frac{\eta-1}{\eta}$ and $\frac{\eta}{\eta} = \frac{0}{\eta}$.

We define $\overline{F}$ on $\overline{\Omega_\eta} \times \overline{\mathcal{T}_\nu}$ by setting $\overline{F}(z) = F(\Gamma(z))$, $(*)$, where z is a special point of $\overline{\Omega_\eta} \times \overline{\mathcal{T}_\nu}$ and $\Gamma(z)$ is the associated original point. We then set $\overline{F}(x,y) = \overline{F}(\frac{[x3^{\nu-[\nu y]}\eta]}{3^{\nu-[\nu y]}\eta}, \frac{[\nu y]}{\nu})$, $(**)$.

We claim that $\overline{F}$ is a reverse martingale. By the definition $(*)$, $\overline{F}$ is adapted to the reverse filtration, (i). To verify (ii) of the definition of a reverse martingale in Definition 0.9, by the tower law for conditional expectation, it is sufficient to prove that $E_\eta(\overline{F}_{\frac{i}{\nu}}|\mathcal{E}_{i+1}) = \overline{F}_{\frac{i+1}{\nu}}$, for $0 \leq i \leq \nu-1$. We have that;

$$E_\eta(\overline{F}_{\frac{i}{\nu}}|\mathcal{E}_{i+1})(\tfrac{j}{3^{\nu-i-1}\eta})$$

$$= 3^{\nu-i-1}\eta \int_{[\frac{j}{3^{\nu-i-1}\eta}, \frac{j+1}{3^{\nu-i-1}\eta})} E_\eta(\overline{F}_{\frac{i}{\nu}}|\mathcal{E}_{i+1}) d\gamma_\eta$$

$$= 3^{\nu-i-1}\eta \int_{[\frac{j}{3^{\nu-i-1}\eta}, \frac{j+1}{3^{\nu-i-1}\eta})} \overline{F}_{\frac{i}{\nu}} d\gamma_\eta$$

$$= \frac{3^{\nu-i-1}\eta}{3^{\nu-i}\eta}(\textstyle\sum_{k=0}^{2} \overline{F}_{\frac{i}{\nu}}(\frac{3j+k}{3^{\nu-i}\eta}))$$

$$= \tfrac{1}{3}(F_{\frac{i}{\nu}}(\tfrac{i_x}{\eta}) + F_{\frac{i}{\nu}}(\tfrac{i_x-1}{\eta}) + F_{\frac{i}{\nu}}(\tfrac{i_x+1}{\eta}))$$

$$= F_{\frac{i+1}{\nu}}(\tfrac{i_x}{\eta}) = \overline{F}_{\frac{i+1}{\nu}}(\tfrac{j}{3^{\nu-i-1}\eta})$$

where $(\frac{i_x}{\eta}, \frac{i+1}{\nu})$ is associated to the special point $(\frac{j}{3^{\nu-i-1}\eta}, \frac{i+1}{\nu}) = (x, \frac{i+1}{\nu})$

We now claim that if $(\frac{i}{\eta}, \frac{\nu-j}{\nu})$ is an original point, then it is associated to 3^j special points of the form $(y, \frac{\nu-j}{\nu})$. We prove this by induction.

Base Case; $j=0$, then the original points $\{(\frac{i}{\eta},1):0\leq i\leq\eta-1\}$ are in bijection with the same special points.

Inductive hypothesis; Suppose that, for $0\leq i\leq\eta-1$, each original point $(\frac{i}{\eta},\frac{\nu-j}{\nu})$ is associated to 3^j special points of the form $(y,\frac{\nu-j}{\nu})$.

If the original point $(\frac{i}{\eta},\frac{\nu-j-1}{\nu})$ is associated to a special point $(x,\frac{\nu-j-1}{\nu})$, there are 3 cases, either $(\frac{i}{\eta},\frac{\nu-j}{\nu})$ is associated to $(x,\frac{\nu-j}{\nu})$, or, $(\frac{i-1}{\eta},\frac{\nu-j}{\nu})$ is associated to $(x-\frac{1}{3^{j+1}\eta},\frac{\nu-j}{\nu})$, or $(\frac{i+1}{\eta},\frac{\nu-j}{\nu})$ is associated to $(x-\frac{2}{3^{j+1}\eta},\frac{\nu-j}{\nu})$. These cases are disjoint and all occur, so we obtain a total of $3.3^j=3^{j+1}$ assignments.

Using this result and the definition of $\overline{F}$, for a given $\alpha\in{}^*\mathcal{R}$;

$$3^{\nu-[\nu t]}{}^*Card(\{i:0\leq i\leq\eta-1,F(\tfrac{i}{\eta},\tfrac{[\nu t]}{\nu})=\alpha\})$$

$$={}^*Card(\{i:0\leq i\leq 3^{\nu-[\nu t]}\eta-1,\overline{F}(\tfrac{i}{3^{\nu-[\nu t]}\eta},\tfrac{[\nu t]}{\nu})=\alpha\})$$

We now calculate;

$$\gamma_\eta(F_t=\alpha)$$

$$=\tfrac{1}{\eta}{}^*Card(\{i:0\leq i\leq\eta-1,F(\tfrac{i}{\eta},\tfrac{[\nu t]}{\nu})=\alpha\})$$

$$=\tfrac{1}{3^{\nu-[\nu t]}\eta}{}^*Card(\{i:0\leq i\leq 3^{\nu-[\nu t]}\eta-1,\overline{F}(\tfrac{i}{3^{\nu-[\nu t]}\eta},\tfrac{[\nu t]}{\nu})=\alpha\})$$

$$=\gamma_\eta(\overline{F}_t=\alpha)$$

Using the measurability of $F_{\frac{[\nu t]}{\nu}}$ and $\overline{F}_{\frac{[\nu t]}{\nu}}$ with respect to the algebras $\mathcal{C}_\eta$ and $\mathcal{E}_{\eta,\nu-[\nu t]}$ respectively.

□

Remarks 0.11. *The advantage of working with a reverse martingale to analyse the cumulative density function of F is that we have available a nonstandard martingale representation theorem, Ito's Lemma and a strategy to obtain a Fokker-Planck type equation. This is work in progress, see* [4].

We make some considerations in connection with the heat equation.

Definition 0.12. *We let $S^1(1)$ denote the circle of radius 1, which we identify with the closed interval $[-\pi,\pi]$, via $\mu:[-\pi,\pi]\to S^1(1)$, $\mu(\theta)=$*

$e^{i\theta}$. We let $C([-\pi,\pi]) = \{\mu^*(g) : g \in C^\infty(S^1)\}$ and $C^\infty([-\pi,\pi]) = \{\mu^*(g) : g \in C^\infty(S^1)\}$. We let $T = [-\pi,\pi] \times \mathcal{R}_{\geq 0}$ and $T^0 = (-\pi,\pi) \times \mathcal{R}_{>0}$ denote its interior. We let $C^\infty(T) = \{G \in C(T) : G_t \in C^\infty([-\pi,\pi])$, for $t \in \mathcal{R}_{\geq 0}, G|T^0 \in C^\infty(T^0)\}$. If $h \in C([-\pi, pi])$, we define its Fourier transform by;

$\mathcal{F}(h)(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-imx}dx$

If $g \in C(T)$, we define its Fourier transform in space by;

$\mathcal{F}(g)(m,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,t)e^{-imx}dx$

Lemma 0.13. *If $g \in C^\infty([-\pi,\pi])$, there exists a unique $G \in C^\infty(T)$, with $G_0 = g$, such that G satisfies the heat equation;*

$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2}$ $(*)$

on T^0.

Proof. Suppose, first, there exists such a solution G, then, applying $\mathcal{F}$ to $(*)$, we must have that;

$\mathcal{F}(\frac{\partial G}{\partial t} - \frac{\partial^2 G}{\partial x^2})(m,t) = 0$ $(t > 0, m \in \mathcal{Z})$

Differentiating under the integral sign, we have that;

$\mathcal{F}(\frac{\partial G}{\partial t}) = \frac{\partial \mathcal{F}(G)}{\partial t}(m,t)$, for $t > 0, m \in \mathcal{Z}$

Integrating by parts and using the fact that $G_t \in C^\infty([-\pi,\pi])$, for $t > 0$, we have that;

$\mathcal{F}\frac{\partial^2 G}{\partial x^2} = -m^2\mathcal{F}(G)(m,t)$, for $t > 0, m \in \mathcal{Z}$

We thus obtain the sequence of ordinary differential equations, indexed by $m \in \mathcal{Z}$;

$\frac{\partial \mathcal{F}(G)}{\partial t} + m^2\mathcal{F}(G)(m,t) = 0$ $(t > 0)$

with initial condition, given by;

$\mathcal{F}(G)(m,0) = \mathcal{F}(g)(m)$

By Picard's Theorem, this has the unique solution, given by;

$\mathcal{F}(G)(m,t) = e^{-m^2 t}\mathcal{F}(g)(m)\ (t \geq 0)$

As $G_t \in C^\infty([-\pi,\pi])$, its Fourier series converges absolutely to G_t and, in particular, G_t is determined by its Fourier coefficients, for $t > 0$. It follows that G is a unique solution.

If $g \in C^\infty([-\pi,\pi])$, its Fourier series converges absolutely to g, hence, the series;

$\sum_{m\in\mathcal{Z}} e^{-m^2 t}\mathcal{F}(g)(m)e^{imx}$

are absolutely convergent for $t > 0$. It follows that G defined by;

$G(x,t) = \sum_{m\in\mathcal{Z}} e^{-m^2 t}\mathcal{F}(g)(m)e^{imx}$

is a solution of the required form. □

Remarks 0.14. *Observe that as* $t \rightarrow \infty$, $G(x,t) \rightarrow \mathcal{F}(g)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g dx$, *a phenomenon we observed in Lemma 0.7. This suggests that the two processes are connected, we make this observation more precise below.*

Definition 0.15. *If* $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$, *we let* $\overline{\mathcal{V}_\eta} = {}^*\bigcup_{0\leq i\leq 2\eta-1}[-\pi+\pi\frac{i}{\eta}, -\pi+\pi\frac{i+1}{\eta})$, *so that* $\overline{\mathcal{V}_\eta} = {}^*[-\pi,\pi)$. *We let* $\mathcal{D}_\eta$ *denote the associated* $*$*-finite algebra, generated by the intervals* $[-\pi+\pi\frac{i}{\eta}, -\pi+\pi\frac{i+1}{\eta})$, *for* $0 \leq i \leq 2\eta - 1$, *and* μ_η *the associated counting measure defined by* $\mu_\eta([-\pi+\pi\frac{i}{\eta}, -\pi+\pi\frac{i+1}{\eta})) = \frac{\pi}{\eta}$. *We let* $(\overline{\mathcal{V}_\eta}, L(\mathcal{D}_\eta), L(\mu_\eta))$ *denote the associated Loeb space, see.... If* $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$, *we let* $\overline{\mathcal{T}_\nu} = {}^*\bigcup_{0\leq i\leq \nu^2-1}[\frac{i}{\nu}, \frac{i+1}{\nu})$, *so that* $\overline{\mathcal{T}_\nu} = [0,\nu) \subset {}^*\mathcal{R}_{\geq 0}$. *We let* $\mathcal{C}_\eta$ *denote the associated* $*$*-finite algebra, generated by the intervals* $[\frac{i}{\nu}, \frac{i+1}{\nu})$, *for* $0 \leq i \leq \nu^2 - 1$, *and* λ_ν *the associated counting measure defined by* $\lambda_\nu([\frac{i}{\nu}, \frac{i+1}{\nu})) = \frac{1}{\nu}$. *We let* $(\overline{\mathcal{T}_\nu}, L(\mathcal{C}_\nu), L(\lambda_\nu))$ *denote the associated Loeb space.*

We let $([-\pi,\pi], \mathfrak{D}, \mu)$ *denote the interval* $[-\pi,\pi]$, *with the completion* $\mathfrak{D}$ *of the Borel field, and* μ *the restriction of Lebesgue measure. We let* $(\mathcal{R}_{\geq 0} \cup \{+\infty\}, \mathfrak{C}, \lambda)$ *denote the extended real half line, with the completion* $\mathfrak{C}$ *of the extended Borel field, and* λ *the extension of Lebesgue measure, with* $\lambda(+\infty) = \infty$, *see....*

We let $(\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, \mathcal{D}_\eta \times \mathcal{C}_\nu, \mu_\eta \times \lambda_\nu)$ *be the associated product space and* $(\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta \times \mathcal{C}_\eta), L(\mu_\eta \times \lambda_\nu))$ *be the corresponding Loeb space.*

$(\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta)\times L(\mathcal{C}_\nu), L(\mu_\eta)\times L(\lambda_\nu))$ is the complete product of the Loeb spaces $(\overline{\mathcal{V}_\eta}, L(\mathcal{D}_\eta), L(\mu_\eta))$ and $(\overline{\mathcal{T}_\nu}, L(\mathcal{C}_\nu), L(\lambda_\nu))$. Similarly, $([-\pi,\pi]\times(\mathcal{R}_{\geq 0}\cup\{+\infty\}, \mathfrak{D}\times\mathfrak{C}, \mu\times\lambda)$ is the complete product of $([-\pi,\pi], \mathfrak{D}, \mu)$ and $(\mathcal{R}_{\geq 0}\cup\{+\infty\}, \mathfrak{C}, \lambda)$.

We let $({}^*\mathcal{R}, {}^*\mathfrak{E})$ denote the hyperreals, with the transfer of the Borel field $\mathfrak{C}$ on $\mathcal{R}$. A function $f:(\overline{\mathcal{V}_\eta}, \mathcal{D}_\eta)\rightarrow({}^*\mathcal{R}, {}^*\mathfrak{E})$ is measurable, if $f^{-1}: {}^*\mathfrak{E}\rightarrow\mathcal{D}_\eta$. The same definition holds for $\mathcal{T}_\nu$. Similarly, $f:(\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, \mathcal{D}_\eta\times\mathcal{C}_\nu)\rightarrow({}^*\mathcal{R}, {}^*\mathfrak{E})$ is measurable, if $f^{-1}: {}^*\mathfrak{E}\rightarrow\mathcal{D}_\eta\times\mathfrak{C}_\nu$. Observe that this is equivalent to the definition given in [?]. We will abbreviate this notation to $f:\overline{\mathcal{V}_\eta}\rightarrow{}^*\mathcal{R}$, $f:\overline{\mathcal{V}_\eta}\rightarrow{}^*\mathcal{R}$ or $f:\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}\rightarrow{}^*\mathcal{R}$ is measurable, $(*)$. The same applies to $({}^*\mathcal{C}, {}^*\mathfrak{E})$, the hyper complex numbers, with the transfer of the Borel field $\mathfrak{E}$, generated by the complex topology. Observe that $f:\overline{\mathcal{V}_\eta}\rightarrow{}^*\mathcal{C}$, $f:\overline{\mathcal{T}_\nu}\rightarrow{}^*\mathcal{C}$ $f:\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}\rightarrow{}^*\mathcal{C}$ is measurable, in this sense, iff $Re(f)$ and $Im(f)$ are measurable in the sense of $(*)$.

We let $\overline{\mathcal{S}_{\eta,\nu}}=\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}$ and;

$$V(\overline{\mathcal{V}_\eta})=\{f:\overline{\mathcal{V}_\eta}\rightarrow{}^*\mathcal{C},\ f \text{ measurable } d(\mu_\eta)\}$$

and, similarly, we define $V(\overline{\mathcal{T}_\nu})$. Let;

$$V(\overline{\mathcal{S}_{\eta,\nu}})=\{f:\overline{\mathcal{S}_{\eta,\nu}}\rightarrow{}^*\mathcal{C},\ f \text{ measurable } d(\mu_\eta\times\lambda_\nu)\}$$

Lemma 0.16. *The identity;*

$$i:(\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta\times\mathcal{C}_\nu), L(\mu_\eta\times\lambda_\nu))$$

$$\rightarrow(\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta)\times L(\mathcal{C}_\nu), L(\mu_\eta)\times L(\lambda_\nu))$$

and the standard part mapping;

$$st:(\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta)\times L(\mathcal{C}_\nu), L(\mu_\eta)\times L(\lambda_\nu))\rightarrow[-\pi,\pi]\times\mathcal{R}_{\geq 0}\cup\{+\infty\}$$

are measurable and measure preserving.

Proof. The proof is similar to Lemma 0.2 in [7], using Caratheodory's Extension Theorem and Theorem 22 of [1].

∎

Definition 0.17. *Discrete Partial Derivatives*

Let $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ *be measurable. As in* [6], *we define the discrete derivative* f' *to be the unique measurable function satisfying;*

$f'(-\pi + \pi\frac{i}{\eta}) = \frac{\eta}{2\pi}(f(-\pi + \pi\frac{i+1}{\eta}) - f(-\pi + \pi\frac{i-1}{\eta}));$

for $i \in {}^*\mathcal{N}_{1 \leq i \leq 2\eta - 2}$.

$f'(\pi - \frac{\pi}{\eta}) = \frac{\eta}{2\pi}(f(-\pi) - f(\pi - \pi\frac{2}{\eta}))$

$f'(-\pi) = \frac{\eta}{2\pi}(f(-\pi + \frac{\pi}{\eta}) - f(\pi - \frac{\pi}{\eta}))$

Let $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ *be measurable. As in* [6], *we define the discrete derivative* f' *to be the unique measurable function satisfying;*

$f'(\frac{i}{\nu}) = \nu(f(\frac{i+1}{\nu}) - f(\frac{i}{\nu}));$

for $i \in {}^*\mathcal{N}_{0 \leq i \leq \nu^2 - 2}$.

$f'(\frac{\nu-1}{\nu}) = 0;$

If $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ *is measurable, then we define the shift (left, right);*

$f^{lsh}(-\pi + \pi\frac{j}{\eta}) = f(-\pi + \pi\frac{j+1}{\eta})$ *for* $0 \leq j \leq 2\eta - 2$

$f^{lsh}(\eta - \frac{\pi}{\eta}) = f(-\pi)$

$f^{rsh}(-\pi + \pi\frac{j}{\eta}) = f(-\pi + \pi\frac{j-1}{\eta})$ *for* $1 \leq j \leq 2\eta - 1$

$f^{rsh}(-\pi) = f(\pi - \frac{\pi}{\eta})$

If $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ *is measurable, then we define the shift (left, right);*

$f^{lsh}(\frac{j}{\nu}) = f(\frac{j+1}{\nu})$ *for* $0 \leq j \leq \nu^2 - 2$

$f^{lsh}(\nu - \frac{1}{\nu}) = f(0)$

$f^{rsh}(\frac{j}{\nu}) = f(\frac{j-1}{\nu})$ *for* $1 \leq j \leq \nu^2 - 1$

$f^{rsh}(0) = f(\nu - \frac{1}{\nu})$

If $f:\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}\rightarrow {}^*\mathcal{C}$ is measurable. Then we define $\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial t}\}$ to be the unique measurable functions satisfying;

$\frac{\partial f}{\partial x}(-\pi+\pi\frac{i}{\eta},t)=\frac{\eta}{2\pi}(f(-\pi+\pi\frac{i+1}{\eta},t)-f(-\pi+\pi\frac{i-1}{\eta},y))$;

for $i\in {}^*\mathcal{N}_{1\leq i\leq 2\eta-2}, t\in\overline{\mathcal{T}_\nu}$

$\frac{\partial f}{\partial x}(\pi-\frac{\pi}{\eta},t)=\frac{\eta}{2\pi}(f(-\pi,t)-f(\pi-\pi\frac{2}{\eta},t))$

$\frac{\partial f}{\partial x}(-\pi,t)=\frac{\eta}{2\pi}(f(-\pi+\frac{\pi}{\eta},t)-f(\pi-\frac{\pi}{\eta},t))$

$\frac{\partial f}{\partial t}(x,\frac{j}{\nu})=(f(x,\frac{j+1}{\nu})-f(x,\frac{j}{\nu}))$;

for $j\in {}^*\mathcal{N}_{0\leq j\leq \nu^2-2}, x\in\overline{\mathcal{H}_\eta}$

$\frac{\partial f}{\partial t}(x,\nu-\frac{1}{\nu})=0$

We define $\{f^{lsh_x},f^{lsh_t},f^{rsh_x},f^{rsh_t}\}$ *by;*

$f^{lsh_x}(x_0,t_0)=(f_t)^{lsh}(x_0)$

$f^{lsh_t}(x_0,t_0)=(f_x)^{lsh}(t_0)$

$f^{rsh_x}(x_0,t_0)=(f_t)^{rsh}(x_0)$

$f^{rsh_t}(x_0,t_0)=(f_x)^{rsh}(t_0)$

where, if $(x_0,t_0)\in\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}$;

$f_{t_0}(x_0)=f_{x_0}(t_0)=f(\pi\frac{[\frac{\eta x_0}{\pi}]}{\eta},\frac{\nu t_0}{\nu})$

Remarks 0.18. *If f is measurable, then so are;*

$\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial t},\frac{\partial^2 f}{\partial x^2},f_x,f_t,f^{lsh_x},f^{lsh_t},f^{rsh_x},f^{rsh_t},f^{sh_x^2},f^{sh_t^2},f^{rsh_x^2},f^{rsh_t^2}\}$

This follows immediately, by transfer, from the corresponding result for the discrete derivatives and shifts of discrete functions $f:\mathcal{H}_n\times\mathcal{T}_m\rightarrow\mathcal{C}$, where $n,m\in\mathcal{N}$, see Definition 0.15 and Definition 0.18 of [8].

Lemma 0.19. *Let $g,h:\overline{\mathcal{V}_\eta}\rightarrow {}^*\mathcal{C}$ be measurable. Then;*

(i). $\int_{\overline{\mathcal{V}_\eta}} g'(y)d\mu_\eta(y) = 0$

(ii). $(gh)' = g'h^{lsh} + g^{rsh}h'$

(iii). $\int_{\overline{\mathcal{V}_\eta}}(g'h)(y)d\mu_\eta(y) = -\int_{\overline{\mathcal{V}_\eta}} gh'd\mu_\eta(y)$

(iv). $\int_{\overline{\mathcal{V}_\eta}} g(y)d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}} g^{lsh}(y)d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}} g^{rsh}(y)d\mu_\eta(y)$

(v). $(g')^{rsh} = (g^{rsh})'$, $(g')^{lsh} = (g^{lsh})'$

(vi). $\int_{\overline{\mathcal{V}_\eta}}(g''h)(y)d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}}(gh'')(y)d\mu_\eta(y)$

([1])

Proof. In the first part, for (i), we have, using Definition 0.17, that;

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[1]Similar results hold for $\{sh_x, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\}$. Namely, if $g, h : \overline{\mathcal{S}_{\eta,\nu}} \rightarrow {}^*\mathcal{C}$ are measurable. Then;

(i). $\int_{\overline{\mathcal{S}_{\eta,\nu}}} \frac{\partial g}{\partial x} d(\mu_\eta \times \lambda_\nu) = 0$

(ii). $\frac{\partial gh}{\partial x} = \frac{\partial g}{\partial x} h^{lsh_x} + g^{rsh_x} \frac{\partial h}{\partial x}$

(iii). $\int_{\overline{\mathcal{S}_{\eta,\nu}}} \frac{\partial g}{\partial x} h d(\mu_\eta \times \lambda_\nu) = -\int_{\overline{\mathcal{S}_{\eta,\nu}}} g \frac{\partial h}{\partial x} d(\mu_\eta \times \lambda_\nu)$

(iv). $\int_{\overline{\mathcal{S}_{\eta,\nu}}} g d(\mu_\eta \times \lambda_\nu) = \int_{\overline{\mathcal{S}_{\eta,\nu}}} g^{lsh_x} d(\mu_\eta \times \lambda_\nu) = \int_{\overline{\mathcal{S}_{\eta,\nu}}} g^{rsh_x} d(\mu_\eta \times \lambda_\nu)$

(v). $(\frac{\partial g}{\partial x})^{lsh_x} = \frac{\partial (g^{lsh_x})}{\partial x}$, and, similarly, with rsh_x replacing lsh_x.

(vi). $\int_{\overline{\mathcal{S}_{\eta,\nu}}} (\frac{\partial^2 g}{\partial x^2} h) d(\mu_\eta \times \lambda_\nu) = \int_{\overline{\mathcal{S}_{\eta,\nu}}} (g \frac{\partial^2 h}{\partial x^2}) d(\mu_\eta \times \lambda_\nu)$ $(*)$

For (i), using (i) from the argument in the main proof, we have;

$\int_{\overline{\mathcal{S}_{\eta,\nu}}} \frac{\partial g}{\partial x} d(\mu_\eta \times \lambda_\nu)$

$= \int_{\overline{\mathcal{V}_\eta}} (\int_{\overline{\mathcal{T}_\nu}} (\frac{\partial g}{\partial x})_t d\mu_\eta) d\lambda_\nu(t)$

$= \int_{\overline{\mathcal{V}_\eta}} (\int_{\overline{\mathcal{T}_\nu}} (\frac{\partial g_t}{\partial x}) d\mu_\eta) d\lambda_\nu(t)$

$= \int_{\overline{\mathcal{T}_\nu}} 0 d\lambda_\nu(t) = 0$

The proofs of $(ii), (iii), (iv)$ are similar to the main proof, relying on the result of (i). (v) follows easily from Definitions 0.17 and (vi) follows, repeating the result of (iii), and applying (v).

$\int_{\overline{\mathcal{V}_\eta}} g'(y)d\mu_\eta(y)$

$= \frac{\pi}{\eta}[{}^*\sum_{1\leq j\leq 2\eta-2} \frac{\eta}{2\pi}[g(-\pi+\pi(\frac{j+1}{\eta}))-g(-\pi+\pi(\frac{j-1}{\eta}))]$

$+\frac{\eta}{2\pi}[g(-\pi+\frac{\pi}{\eta})-g(\pi-\frac{\pi}{\eta})]+\frac{\eta}{2\pi}[g(-\pi)-g(\pi-2\frac{\pi}{\eta})]]=0$

For (ii), we calculate;

$(gh)'(-\pi+\pi\frac{j}{\eta})=$

$=\frac{\eta}{2\pi}(gh(-\pi+\pi\frac{j+1}{\eta})-gh(-\pi+\pi\frac{j-1}{\eta}))$

$=\frac{\eta}{2\pi}(gh(-\pi+\pi\frac{j+1}{\eta})-g(-\pi+\pi\frac{j-1}{\eta})h(-\pi+\pi\frac{j+1}{\eta})$

$+g(-\pi+\pi\frac{j-1}{\eta})h(-\pi+\pi\frac{j+1}{\eta})-gh(-\pi+\pi\frac{j-1}{\eta}))$

$=g'(-\pi+\pi\frac{j}{\eta})h(-\pi+\pi\frac{j+1}{\eta})+g(-\pi+\pi\frac{j-1}{\eta})h'(-\pi+\pi\frac{j}{\eta})$

$=(g'h^{lsh}+g^{rsh}h')(-\pi+\pi\frac{j}{\eta})$

Combining $(i),(ii)$, we have;

$0=\int_{\overline{\mathcal{V}_\eta}}(gh)'(x)d\mu_\eta(x)$

$=\int_{\overline{\mathcal{V}_\eta}}(g'h^{lsh}+g^{rsh}h')(x)d\mu_\eta(x)$

and, rearranging, that;

$\int_{\overline{\mathcal{V}_\eta}}(g'h^{lsh})d\mu_\eta=-\int_{\overline{\mathcal{H}_\eta}}(g^{rsh}h')d\mu_\eta$

For (iv), we have that;

$\int_{\overline{\mathcal{V}_\eta}} g^{rsh}(y)d\mu_\eta(y)$

$=\frac{\pi}{\eta}({}^*\sum_{0\leq j\leq 2\eta-1} g^{rsh}(-\pi+\pi\frac{j}{\eta}))$

$=\frac{\pi}{\eta}({}^*\sum_{1\leq j\leq 2\eta-2} g(-\pi+\pi\frac{j-1}{\eta})+g(\pi-\frac{\pi}{\eta}))$

$=\frac{\pi}{\eta}({}^*\sum_{0\leq j\leq 2\eta-1} g(-\pi+\pi\frac{j}{\eta})$

$=\int_{\overline{\mathcal{V}_\eta}} g(y)d\mu_\eta(y)$

A similar calculation holds with g^{lsh}. For (v), we have for $2 \leq j \leq 2\eta - 2$;

$(g')^{rsh}(-\pi + \pi\frac{j}{\eta})$

$= g'(-\pi + \pi\frac{j-1}{\eta})$

$= \frac{\eta}{2\pi}(g(-\pi + \pi\frac{j}{\eta}) - g(-\pi + \pi\frac{j-2}{\eta}))$

$(g^{rsh})'(-\pi + \pi\frac{j}{\eta})$

$= \frac{\eta}{2\pi}(g^{rsh}(-\pi + \pi\frac{j+1}{\eta}) - g^{rsh}(-\pi + \pi\frac{j-1}{\eta}))$

$= \frac{\eta}{2\pi}(g(-\pi + \pi\frac{j}{\eta}) - g(-\pi + \pi\frac{j-2}{\eta}))$

Similar calculations hold for the remaining j to give that $(g')^{rsh} = (g^{rsh})'$, and the calculation $(g')^{lsh} = (g^{lsh})'$ is also similar.

It follows that;

$\int_{\overline{\mathcal{V}_\eta}}(g'h)d\mu_\eta$

$= \int_{\overline{\mathcal{V}_\eta}}(g'(h^{rsh})^{lsh})d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(g^{rsh}(h^{rsh})')d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(g^{rsh}(h')^{rsh})d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(gh'))d\mu_\eta$

which gives (vi), using $(iv), (v)$.

$\square$

Definition 0.20. *If η is even, we define a restriction $\overline{()} : \overline{\mathcal{V}_\eta} \rightarrow \overline{\mathcal{V}_{\frac{\eta}{2}}}$. Namely;*

$\overline{f}(-\pi + \pi\frac{2i}{\eta}) = f(-\pi + \pi\frac{2i}{\eta})$;

for $i \in {}^*\mathcal{N}_{0 \leq i \leq \eta-1}$.

Lemma 0.21. *Let notation be as in Definitions 0.20 and 0.17, then;*

$\overline{f'}(-\pi+\pi\frac{2i}{\eta})=\frac{\eta}{2\pi}(f(-\pi+\pi\frac{2i+1}{\eta})-f(-\pi+\pi\frac{2i-1}{\eta}))$;

for $i\in {}^{*}\mathcal{N}_{1\leq i\leq\eta-1}$.

$\overline{f'}(-\pi)=\frac{\eta}{2\pi}(f(-\pi+\frac{\pi}{\eta})-f(\pi-\frac{\pi}{\eta}))$

and;

$\overline{f^{lsh}}(-\pi+\pi\frac{2j}{\eta})=f(-\pi+\pi\frac{2j+1}{\eta})$ *for* $0\leq j\leq\eta-1$

$\overline{f^{rsh}}(-\pi+\pi\frac{2j}{\eta})=f(-\pi+\pi\frac{2j-1}{\eta})$ *for* $1\leq j\leq\eta-1$

$\overline{f^{rsh}}(-\pi)=f(\pi-\frac{\pi}{\eta})$

Proof. The proof is an immediate consequence of Definitions 0.20 and 0.17

□

Remarks 0.22. *It is important to note that, in general* $\overline{f'}\neq\overline{f}'$ *and, similarly, for* lsh,rsh.

Lemma 0.23. *Let* $\{g,h\}\subset V(\overline{\mathcal{V}_{\eta}})$ *be measurable, then;*

(i). $\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{g'}(y)d\mu_{\frac{\eta}{2}}(y)=0$

(ii). $\overline{(gh)'}=\overline{g'}\,\overline{h^{lsh}}+\overline{g^{rsh}}\,\overline{h'}$

(iii). $\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{(g'h)}(y)d\mu_{\frac{\eta}{2}}(y)=-\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{g^{rsh}(h')^{rsh}}d\mu_{\eta}(y)$

Proof. For (i), we have that;

$\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{g'}(y)d\mu_{\frac{\eta}{2}}(y)$

$=\frac{2\pi}{\eta}[{}^{*}\sum_{1\leq j\leq\eta-1}\frac{\eta}{2\pi}[g(-\pi+\pi(\frac{2j+1}{\eta}))-g(-\pi+\pi(\frac{2j-1}{\eta}))]$

$+\frac{\eta}{2\pi}[g(-\pi+\frac{\pi}{\eta})-g(\pi-\frac{\pi}{\eta})]]=0$

(ii) is clear from the main proof and taking restrictions.

For (iii), integrating both sides of (ii) and using (i), we have that;

$\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{g' h^{lsh}} d\mu_{\frac{\eta}{2}}(y) = -\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{g^{rsh} h'} d\mu_{\frac{\eta}{2}}(y)$ $(*)$

Then;

$\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{g' h} d\mu_{\frac{\eta}{2}}(y)$

$= \int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{g' (h^{rsh})^{lsh}} d\mu_{\frac{\eta}{2}}(y)$

$= -\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{g^{rsh} (h^{rsh})'} d\mu_{\frac{\eta}{2}}(y)$ by $(*)$

$= -\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{g^{rsh} (h')^{rsh}} d\mu_{\frac{\eta}{2}}(y)$ by the main proof

$\square$

Lemma 0.24. *Given a measurable boundary conditions $f \in V(\overline{\mathcal{V}_\eta})$, there exists a unique measurable $F \in V(\overline{\mathcal{S}_{\eta,\nu}})$, satisfying the nonstandard heat equation;*

$\frac{\partial F}{\partial t} = \frac{\partial^2 f}{\partial x^2}$

on $(\overline{\mathcal{T}_\nu} \setminus [\nu - \frac{1}{\nu}, \nu)) \times \overline{\mathcal{V}_\eta}$

with $F(0,x) = f(x)$, *for* $x \in \overline{\mathcal{V}_\eta}$, $(*)$.

Moreover, if $\eta \leq \sqrt{2\pi}\nu$, *and, there exists* $M \in \mathcal{R}$, *with* $max\{f, f', f''\} \leq M$, *then* $max\{F, \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}\} \leq M$.

Proof. Observe that, by Definition 0.17, if $F : \overline{\mathcal{S}_{\eta,\nu}} \rightarrow {}^*\mathcal{C}$ is measurable, then;

$\frac{\partial^2 F}{\partial x^2}(-\pi + \pi\frac{i}{\eta}, t) = \frac{\eta^2}{4\pi^2}(F(-\pi + \pi\frac{i+2}{\eta}, t) - 2F(-\pi + \pi\frac{i}{\eta}, t) + F(-\pi + \pi\frac{i-2}{\eta}, t))$

$(2 \leq i \leq 2\eta - 3), t \in \overline{\mathcal{T}_\nu}$, with similar results for the remaining i.

Therefore, if F satisfies $(*)$, we must have;

$F(0,x) = f(x)$, $(x \in \overline{\mathcal{V}_\eta})$

$F(\frac{i+1}{\nu},-\pi+\pi\frac{j}{\eta})$

$=F(\frac{i}{\nu},-\pi+\pi\frac{j}{\eta})+\frac{\eta^2}{4\pi^2\nu}(F(\frac{i}{\nu},-\pi+\pi\frac{j+2}{\eta})-2F(\frac{i}{\nu},-\pi+\pi\frac{j}{\eta})+F(\frac{i}{\nu},-\pi+\pi\frac{j-2}{\eta}))$

$=\frac{\eta^2}{4\pi^2\nu}F(\frac{i}{\nu},-\pi+\pi\frac{j+2}{\eta})+(1-\frac{\eta^2}{2\pi^2\nu})(F(\frac{i}{\nu},-\pi+\pi\frac{j}{\eta})+\frac{\eta^2}{4\pi^2\nu}F(\frac{i}{\nu},-\pi+\pi\frac{j-2}{\eta}))$ $(*)$

$(1\leq i\leq \nu^2-2, 0\leq j\leq 2\eta-1)$

See also the proof of Lemma 0.5 in [7]. The choice of η ensures that $1-\frac{\eta^2}{2\pi^2\nu}\geq 0$. Hence, inductively, if $|F_{\frac{i}{\nu}}|\leq M$, then, by $(*)$;

$|F_{\frac{i+1}{\nu}}|\leq M(\frac{\eta^2}{4\pi^2\nu}+(1-\frac{\eta^2}{2\pi^2\nu})+\frac{\eta^2}{4\pi^2\nu})=M.$

We can differentiate $(*)$ and replace F with $\frac{\partial F}{\partial x}$ or $\frac{\partial^2 F}{\partial x^2}$. The same argument, and the assumption on the initial conditions, gives the required bound.

$\square$

Lemma 0.25. *If $f\in C^\infty[-\pi,\pi]$, and f_η is defined on $\overline{\mathcal{V}_\eta}$ by;*

$f_\eta(-\pi+\pi\frac{j}{\eta})=f^*(-\pi+\pi\frac{j}{\eta})$

$f_\eta(x)=f(-\pi+\frac{\pi}{\eta}[\frac{\eta(x+\pi)}{\pi}]$

where f^* is the transfer of f to ${}^*[-\pi,\pi)$, then there exists a constant $M\in\mathcal{R}$, such that $max\{f_\eta,f'_\eta,f''_\eta\}\leq M$. In particular, if F solves the nonstandard heat equation, with initial condition f_η, then, $max\{F,\frac{\partial F}{\partial x},\frac{\partial^2 F}{\partial x^2}\}\leq M$ as well.

Proof. We have, for $x\in[-\pi,\pi)$, using Taylor's Theorem, that;

$|\frac{1}{2h}(f(x+h)-f(x-h))-f'(x)|$

$=\frac{1}{2h}(f(x)+hf'(x)+\frac{h^2}{2}f''(c)-f(x)+hf'(x)-\frac{h^2}{2}f''(c'))-f'(x)|$

$\leq hK$

where $K=max_{[-\pi,\pi)}f''$. By transfer, it follows, that, for infinite η, $(f_\eta)'\simeq(f')_\eta$. Clearly $(f')_\eta$ is bounded, as f' is, which gives the result for $(f_\eta)'$. The case for $(f_\eta)''$ is similar. The final result is immediate from Lemma 0.24. $\square$

Definition 0.26. *We let* $\overline{\mathcal{Z}_\eta} = \{m \in {}^*\mathcal{Z} : -\eta \leq m \leq \eta - 1\}$ *Given a measurable* $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$*, we define, for* $m \in \mathcal{Z}_\eta$*, the* $m'th$ *discrete Fourier coefficient to be;*

$$\hat{f}_\eta(m) = \frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f(y)exp_\eta(-iym)d\mu_\eta(y)$$

Transposing Lemma 0.9 of [9], *(*[2]*);*

$$f(x) = \sum_{m\in\mathcal{Z}_\eta} \hat{f}_\eta(m)exp_\eta(ixm) \ (*)$$

Given a measurable $f : \overline{\mathcal{S}_{\eta,\nu}} \rightarrow {}^*\mathcal{C}$*, we define the nonstandard vertical Fourier transform* $\hat{f} : \overline{\mathcal{T}_\nu} \times \overline{\mathcal{Z}_\eta} \rightarrow {}^*\mathcal{C}$ *by;*

$$\hat{f}(t,m) = \frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f(t,x)exp_\eta(-ixm)d\mu_\eta(x)$$

and, given a measurable $g : \overline{\mathcal{T}_\nu} \times \overline{\mathcal{Z}_\eta} \rightarrow {}^*\mathcal{C}$*, we define the nonstandard inverse vertical Fourier transform by;*

$$\check{g}(t,x) = \sum_{m\in\mathcal{Z}_\eta} g(t,m)exp_\eta(ixm)$$

so that, by $(*)$*,* $f = \check{\hat{f}}$

Similar to Definition 0.20 of [8], *for* $f \in \overline{\mathcal{V}_\eta}$*, we let* $\phi_\eta : \overline{\mathcal{Z}_\eta} \rightarrow {}^*\mathcal{C}$ *be defined by;*

$$\phi_\eta(m) = \frac{\eta}{2\pi}(exp_\eta(-im\frac{\pi}{\eta}) - exp_\eta(im\frac{\pi}{\eta}))$$

We let $\psi_\eta : \overline{\mathcal{Z}_{\frac{\eta}{2}}} \rightarrow {}^*\mathcal{C}$ *be defined by;*

$$\psi_\eta(m) = \frac{\eta}{2\pi}(1 - exp_\eta(im\frac{2\pi}{\eta}))$$

and, we let $U_\eta : \overline{\mathcal{Z}_{\frac{\eta}{2}}} \rightarrow {}^*\mathcal{C}$ *be defined by;*

$$U_\eta(m) = exp_\eta(-im\frac{2\pi}{\eta}))$$

[2] *We have there that the measure on* $\overline{\mathcal{S}_\eta} = \lambda_\eta$. The result follows using the scalar map $p : \overline{\mathcal{V}_\eta} \rightarrow \overline{\mathcal{S}_\eta}$, $p(x) = \frac{x}{\pi}$, and the fact that $p_*(\mu_\eta) = \lambda_\eta$

The following is the analogue of Lemma 0.14 in [9], using the definition of the discrete derivative in Definition 0.17 and the discrete Fourier coefficients from Definition 0.26;

Lemma 0.27. *Let $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ be measurable; then, for $m \in \mathcal{Z}_\eta$,*

$\hat{f''}(m) = \phi_\eta^2(m)\hat{f}(m)$

Proof. We have, using Lemma 0.19(iii), that;

$(\hat{f'})(m) = \frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f'(x)exp_\eta(-ixm)d\mu_\eta(y)$

$= -\frac{1}{2\pi}\int_{\overline{\mathcal{V}_\eta}} f(x)(exp_\eta)'(-ixm)d\mu_\eta(x)$

A simple calculation shows that;

$(exp_\eta)'(-ixm) = exp_\eta(-ixm)\phi_\eta(m)$

Therefore;

$(\hat{f'})(m) = -\phi_\eta(m)\hat{f}(m)$

Then $\hat{f''}(m)$

$= -\phi_\eta(m)\hat{f'}(m)$

$= \phi_\eta^2(m)\hat{f}(m)$

as required. □

Lemma 0.28. *If $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ is measurable, then, for $m \in \mathcal{Z}_{\frac{\eta}{2}}$, we have that;*

$\hat{\overline{f''}}(m) = \psi_\eta(m)^2 U_\eta(m)(\hat{\overline{f}}(m))$

Proof. we have, using (iii) in footnote 3, that;

$\hat{\overline{f'}}(m) = \frac{1}{2\pi}\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{f'}(x)exp_{\frac{\eta}{2}}(-ixm)d\mu_{\frac{\eta}{2}}(x)$

$= \frac{1}{2\pi}\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{f'}(x)\overline{exp_\eta}(-ixm)d\mu_{\frac{\eta}{2}}(x)$

$= \frac{1}{2\pi}\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{f^{rsh}}(x)\overline{{exp'_\eta}^{rsh}}(-ixm)d\mu_{\frac{\eta}{2}}(x)$

We calculate;

$\overline{{exp'_\eta}^{rsh}}(-im(-\pi+\pi\frac{2j}{\eta}))$

$= exp'_\eta(-im(-\pi+\pi\frac{2j-1}{\eta}))$

$= \frac{\eta}{2\pi}(exp_\eta(-im(-\pi+\pi\frac{2j}{\eta}))-exp_\eta(-im(-\pi+\pi\frac{2j-2}{\eta})))$

$= exp_\eta(-im(-\pi+\pi\frac{2j}{\eta}))[\frac{\eta}{2\pi}(1-exp_\eta(im(\frac{2\pi}{\eta})))]$

$= \psi_\eta(m)\overline{exp_\eta}(-im(-\pi+\pi\frac{2j}{\eta}))$

$\hat{\overline{f'}}(m) = \frac{1}{2\pi}\psi_\eta(m)\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{f^{rsh}}(x)\overline{exp_\eta}(-ixm)d\mu_{\frac{\eta}{2}}(x)$

$= \psi_\eta(m)(\hat{\overline{f^{rsh}}}(m))$

It follows that;

$\hat{\overline{f''}}(m) = \psi_\eta(m)(\hat{\overline{(f')^{rsh}}}(m))$

$= \psi_\eta(m)(\hat{\overline{(f^{rsh})'}}(m))$

$= \psi_\eta(m)^2(\hat{\overline{f^{rsh^2}}}(m))$

We calculate;

$\hat{\overline{f^{rsh^2}}}(m)$

$= \frac{1}{2\pi}\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{f^{rsh^2}}(x)\overline{exp_\eta}(-ixm)d\mu_{\frac{\eta}{2}}(x)$

$= \frac{1}{2\pi}exp_\eta(\frac{-2\pi im}{\eta})\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{f^{rsh^2}}(x)\overline{exp_\eta^{rsh^2}}(-ixm)d\mu_{\frac{\eta}{2}}(x)$

$= \frac{1}{2\pi}U_\eta(m)\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}}\overline{f}(x)\overline{exp_\eta}(-ixm)d\mu_{\frac{\eta}{2}}(x)$

$= U_\eta(m)\hat{\overline{f}}(m)$

Hence;

$\hat{\overline{f''}}(m) = \psi_\eta(m)^2 U_\eta(m)(\hat{\overline{f}}(m))$

as required.

$\square$

Lemma 0.29. *If $f \in V(\overline{\mathcal{V}_\eta})$, with f'' bounded, then, there exists a constant $F \in \mathcal{R}$, with;*

$|\hat{\overline{f}}(m)| \leq \frac{F}{m^2}$, *for $m \in \mathcal{Z}_{\frac{\eta}{2}}$.*

Moreover;

$({}^\circ\overline{f})(x) = \sum_{m\in\mathcal{Z}} {}^\circ((\hat{\overline{f}})(m))exp(im^\circ x)$, $x \in \overline{\mathcal{V}_\eta}$.

Proof. Using results of [9], we have that $m \leq |\psi_\eta(m)| \leq 2m$, and $|U_\eta(m)| = 1$ for $|m| \leq \frac{\eta}{2}$. As f'' is bounded, so is $\overline{f''}$, so $|\hat{\overline{f''}}(m)| \leq F \in \mathcal{R}$. This implies, by the result of Lemma 0.28 that;

$|\hat{\overline{f}}(m)| \leq \frac{F}{m^2}$. $(*)$

for $m \in \mathcal{Z}_{\frac{\eta}{2}}$, as required. Using the Inversion Theorem from Definition 0.26, we have that;

$\overline{f}(x) = {}^*\sum_{m\in\mathcal{Z}_{\frac{\eta}{2}}} \hat{\overline{f}}(m)exp_{\frac{\eta}{2}}(ixm) = M$, $(x \in \overline{\mathcal{V}_\eta})$

Let $L = \sum_{m\in\mathcal{Z}} {}^\circ(\hat{\overline{f}}(m))exp(im^\circ x)$

If $\epsilon > 0$, we have, using the result of 0.29, and the fact that $exp_{\frac{\eta}{2}}$ is S-continuous for $m \in \mathcal{Z}$, that;

$|M - L| \leq |M - M_n| + |M_n - L_n| + |L - L_n|$

$\leq {}^*\sum_{n+1\leq|m|\leq\frac{\eta}{2}} \frac{F}{m^2} + \sum_{m=1}^n \delta_i + \sum_{|m|\geq n+1} \frac{F+1}{m^2}$ $(\delta_i \simeq 0)$

$\leq 2F(\frac{1}{n} - \frac{2}{\eta}) + \delta + \frac{2(F+1)}{n}$ $(\delta \simeq 0)$

$\leq \frac{4(F+1)}{n} < \epsilon$

for $n > \frac{\epsilon}{4(F+1)}$, $n \in \mathcal{N}$. As ϵ was arbitrary, we obtain the result.

$\square$

Lemma 0.30. *If F solves the nonstandard heat equation, with initial condition f, bounded and S-continuous, such that ${}^\circ f(x) = g({}^\circ x)$, where g is continuous and bounded on $[-\pi, \pi]$, then;*

$${}^\circ(\hat{\overline{F}}(m,t)) = e^{-m^2 {}^\circ t}(\hat{g})(m)$$

for $m \in \mathcal{Z}$ and finite t.

Proof. As $\frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial x^2} = 0$, we have, taking restrictions, that;

$$\overline{\frac{\partial F}{\partial t}} - \overline{\frac{\partial^2 F}{\partial x^2}} = 0$$

Taking Fourier coefficients, for $m \in \mathcal{Z}$, and, using Lemma 0.28;

$$\frac{d\hat{\overline{F}}(m,t)}{dt} - \theta_\eta(m)\hat{\overline{F}}(m,t) = 0$$

where $\theta_\eta(m) = \psi_\eta^2(m)U_\eta(m)$. Then;

$$\nu(\hat{\overline{F}}(m, t + \tfrac{1}{\nu}) - \overline{F}(\hat{m}, t)) = \theta_\eta(m)\hat{\overline{F}}(m,t)$$

Rearranging, we obtain;

$$\hat{\overline{F}}(m, t + \tfrac{1}{\nu}) = (1 + \tfrac{\theta_\eta(m)}{\nu})\hat{\overline{F}}(m,t)$$

and, solving the recurrence;

$$\hat{\overline{F}}(m,t) = (1 + \tfrac{\theta_\eta(m)}{\nu})^{[\nu t]}\hat{\overline{F}}(m,0)$$

Taking standard parts, and using the facts that $lim_{n\rightarrow\infty}(1+\frac{x}{n})^n = e^x$, and ${}^\circ\theta_\eta(m) = -m^2$, we obtain;

$${}^\circ(\hat{\overline{F}}(m,t)) = (e^{-m^2 {}^\circ t}){}^\circ(\hat{\overline{f}}(m))$$

for finite t. As f is bounded and S-continuous, so is $\overline{f}$, and ${}^\circ f = {}^\circ\overline{f}$ is integrable. We have that;

$${}^\circ\int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{f}(x) exp_{\frac{\eta}{2}}(-imx) d\mu_{\frac{\eta}{2}} = \int_{-\pi}^{\pi} {}^\circ f({}^\circ x)e^{-ixm}dx = \int_{-\pi}^{\pi} g(x)e^{-ixm}dx$$

Hence;

$^\circ(\hat{\overline{F}}(m,t)) = (e^{-m^2 {}^\circ t}\hat{g}(m))$

as required.

□

Theorem 0.31. *Let $g \in C^\infty([-\pi,\pi])$, and G be as in Lemma 0.13. Let $f = g_\eta$, and let F be as in Lemma 0.24. Then, for finite t, and $(x,t) \in \overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}$, ${}^\circ F(x,t) = G({}^\circ x, {}^\circ t)$*

Proof. By Lemma 0.25, we have that $\frac{\partial^2 F}{\partial x^2}$ is bounded. By Lemma 0.29;

${}^\circ\overline{F}(x,t) = \sum_{m\in\mathcal{Z}} {}^\circ(\hat{\overline{F}}(m,t))exp(im^\circ x)$ $(*)$

By Lemma 0.30;

${}^\circ(\hat{\overline{F}}(m,t)) = e^{-m^2 {}^\circ t}\mathcal{F}(g)(m)$ $(**)$

Comparing $(*), (**)$, with the expression;

$G({}^\circ x, {}^\circ t) = \sum_{m\in\mathcal{Z}} e^{-m^2 {}^\circ t}\mathcal{F}(g)(m)e^{im^\circ x}$

obtained in Lemma 0.13, gives the result that ${}^\circ\overline{F}(x,t) = G({}^\circ x, {}^\circ t)$. However, $\overline{F}_t$ is S-continuous, for finite t, by the fact that $(\frac{\partial F}{\partial x})_t$ is bounded, from Lemma 0.25, hence;

${}^\circ\overline{F}(x,t) = {}^\circ F(x,t) = G({}^\circ x, {}^\circ t)$

as required.

□

Theorem 0.32. *Let $g \in C^\infty([-\pi,\pi])$, and G be as in Lemma 0.13. Let $f = g_\eta$, and let F be as in Lemma 0.24. Then, for infinite t, and $(x,t) \in \overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}$, $F(x,t) \simeq \int_{\overline{\mathcal{V}_\eta}} f d\mu_\eta$.*

Proof. Again, using Lemma 0.25 and lemma 0.29, we have, using the proof of 0.30, that;

$\overline{F}(x,t) \simeq \sum_{m\in\mathcal{Z}} exp_\nu^{-\theta_\eta(m)t}\hat{f}(m)exp_{\frac{\eta}{2}}^{imx}$

Taking standard parts, using lemma 0.29, and the fact that $exp_{\nu}^{-\theta_{\eta}(m)t} \simeq 0$, for finite m and infinite t, we see that all the coefficients vanish, except when $m = 0$, that is;
$\overline{F}(x,t) \simeq \hat{f}(0) = \int_{\overline{\mathcal{V}_{\frac{\eta}{2}}}} \overline{f} d\mu_{\frac{\eta}{2}} \simeq \int_{\overline{\mathcal{V}_{\eta}}} f d\mu_{\eta}$

The result follows for $F(x,t)$ by S-continuity.

$\square$

Remarks 0.33. *When $\eta = \frac{2\pi\sqrt{\nu}}{\sqrt{3}}$, we obtain, by Lemma 0.24, the iterative scheme for the nonstandard Markov chain with transition probabilities $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$. By Lemma 0.32, we obtain convergence to equilibrium after at least $\nu^2 = \frac{9\eta^4}{16\pi^2}$ steps, which is polynomial in the number of states η. This is a considerable improvement over the result in Lemma 0.4, which is exponential in η. The discrepancy results from the choice of a "smooth" initial distribution. The method of reverse martingales is useful to consider other "nonsmooth" cases, for which a Fourier analysis is impossible.*

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