OSCILLATORY INTEGRALS

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Abstract.

Lemma 0.1. *For $g \in S(\mathcal{R})$, with corresponding $g_\eta \in V(\mathcal{R}_\eta)$, and $\sqrt{c} \leq \eta$, we have that;*

$\int_{\overline{\mathcal{R}_{\eta c}}} g_{\eta,\sqrt{c}} d\mu_{\eta c} \simeq \sqrt{c} \int_{\overline{\mathcal{R}_\eta}} g_\eta d\mu_\eta$

$\int_{\overline{\mathcal{R}_{\eta c}}} |g_{\eta,\sqrt{c}}| d\mu_{\eta c} \simeq \sqrt{c} \int_{\overline{\mathcal{R}_\eta}} |g_\eta| d\mu_\eta$

In particular, for $3\frac{\sqrt{\eta c}}{4} < |y| < \sqrt{\eta c}$, and for δ infinitesimal, $0 \leq |y| < \eta^\delta$, $y \in \overline{\mathcal{R}_{\eta\rfloor}}$, we have that;

$|\mathcal{F}_{\eta c}(g_{\eta,\sqrt{c}})(y)| \leq E\sqrt{c}$, *for some $E \in \mathcal{R}$*

where $D = \int_{\mathcal{R}_\eta} f_\eta d\mu_\eta$

Proof. We have that;

$\int_{\overline{\mathcal{R}_{\eta c}} \setminus [\eta c - c, \eta c)} g_{\eta,\sqrt{c}} d\mu_{\eta c}$

$= \frac{1}{\sqrt{\eta c}}{}^* \sum_{-\eta c \leq i \leq \eta c - c - 1} g_{\eta,\sqrt{c}}(\frac{i}{\sqrt{\eta c}})$

$= \frac{1}{\sqrt{\eta c}}{}^* \sum_{-\eta c \leq i \leq \eta c - c - 1} g_\eta(\frac{i}{\sqrt{\eta c}})$

$= \frac{\sqrt{c}}{\sqrt{\eta}}{}^* \sum_{-\eta \leq i \leq \eta - 2} g_\eta(\frac{i}{\sqrt{\eta}})$

$= \sqrt{c} \int_{\overline{\mathcal{R}_\eta} \setminus [\eta - 1, \eta)} g_\eta d\mu_\eta$

Then;

$| \int_{\overline{\mathcal{R}_{\eta c}}} g_{\eta,\sqrt{c}} d\mu_{\eta c} - \sqrt{c} \int_{\overline{\mathcal{R}_\eta}} g_\eta d\mu_\eta |$

$= |(\int_{\overline{\mathcal{R}_{\eta c}} \setminus [\eta c - c, \eta c)} g_{\eta,\sqrt{c}} d\mu_{\eta c} + \int_{[\eta c - c, \eta c)} g_{\eta,\sqrt{c}} d\mu_{\eta c})$

$-\sqrt{c}(\int_{\overline{\mathcal{R}_\eta}\setminus[\eta-1,\eta)} g_\eta d\mu_\eta + \int_{[\eta-1,\eta)} g_\eta d\mu_\eta)|$

$= |\int_{[\eta c-c,\eta c)} g_{\eta,\sqrt{c}} d\mu_{\eta c} - \sqrt{c}\int_{[\eta-1,\eta)} g_\eta d\mu_\eta|$

$\leq \frac{D(c-1)}{\sqrt{\eta c}\eta^2} + \frac{D\sqrt{c}}{\eta^2} \simeq 0$

The same proof works for $|g_{\eta,\sqrt{c}}|$.

Then, for $3\frac{\sqrt{\eta c}}{4} < |y| < \sqrt{\eta c}$, and $0 \leq |y| < \eta^\delta$, $y \in \overline{\mathcal{R}_{\eta\rfloor}}$, we have

$|\mathcal{F}_{\eta c}(g_{\eta,\sqrt{c}})(y)|$

$\leq \int_{\overline{\mathcal{R}_{\eta c}}} |g_{\eta,\sqrt{c}}| d\mu_{\eta c}$

$\simeq \sqrt{c}\int_{\overline{\mathcal{R}_\eta}} |g_\eta| d\mu_\eta$

$= D\sqrt{c}$

Taking $E = D+1$ gives the result.

□

Lemma 0.2. *For $\frac{\eta c}{4} \leq |r| \leq \eta c$, we have that;*

$0 < \{\mu x : x > 0, cos_{\eta c}(\frac{2\pi r x}{\sqrt{\eta c}}) = 0\} \leq \frac{1}{\sqrt{\eta c}}$

Proof. We have that;

${}^*cos(2\pi(\frac{(\frac{\eta c}{4})}{\sqrt{\eta c}})\frac{1}{\sqrt{\eta c}})$

$= cos_{\eta c}(2\pi(\frac{(\frac{\eta c}{4})}{\sqrt{\eta c}})\frac{1}{\sqrt{\eta c}}) = 0$

when c is even, as $\frac{\eta c}{4} \in {}^*\mathcal{Z}$, the result is then clear.

□

Lemma 0.3. *For $0 < \gamma < \frac{1}{2}$, $\gamma \in \mathcal{R}$, $\sqrt{c}(\eta^{\frac{1}{2}-\gamma}) \leq |y| \leq \frac{\sqrt{\eta c}}{4}$;*

$\mathcal{F}_{\eta c}(f_{\eta,\sqrt{c}})(y) \simeq 0$

Proof. We prove the result for $Re(f_\eta)$. Let $\epsilon > 0$. As $Re(f)|_{[-1-\epsilon,1+\epsilon]}$ is continuous, without loss of generality, it has finitely many zeroes at

$\{x_1,\ldots,x_n\}$, with $-1\leq x_1\leq\ldots\leq x_n\leq 1$. As $Re(f)|_{[x_i,x_{i+1}]}$ is differentiable, it has finitely many maxima and minima, $\{x_{i,1},\ldots,x_{i,n(i)}\}$, with $x_i\leq x_{i,1}\leq\ldots\leq x_{i,n(i)}\leq x_{i+1}$. It follows that $Re(f)|_{[x_{i,j},x_{i,j+1}]}$ is monotone for $1\leq j\leq n(i)-1$. Without loss of generality, there are five cases to consider, $f\equiv 0$ on $[x_{i,j},x_{i,j+1}]$, $f>0$ on $[x_{i,j},x_{i,j+1}]$, with $f(x_{i,j})<f(x_{i,j+1})$, $f>0$ on $[x_{i,j},x_{i,j+1}]$, with $f(x_{i,j})>f(x_{i,j+1})$, $f|_{[x_{i,j},x_{i,j+1}]}>0$, with $f(x_{i,j})=0$, and $f|_{[x_{i,j},x_{i,j+1}]}>0$, with $f(x_{i,j+1})=0$. The cases for $f|_{[x_{i,j},x_{i,j+1}]}\leq 0$ follow by considering $-Re(f)$. Let $y_1=\frac{[-\sqrt{\eta c}]}{\sqrt{\eta c}}$, $y_2=\frac{[\sqrt{\eta c}]+1}{\sqrt{\eta c}}$, $[y_1,y_2]\subset {}^*[-1-\epsilon,1+\epsilon]$, for $\epsilon>0$, and consider $Re(f_\eta)|_{[y_1,y_2]}$.

By transfer, there are again five cases to consider, $Re(f_\eta)\equiv 0$ on $[x_{i,j},x_{i,j+1}]$, $Re(f_\eta)>0$ on $[x_{i,j},x_{i,j+1}]$, with $Re(f_\eta)(x_{i,j})<Re(f_\eta)(x_{i,j+1})$, $Re(f_\eta)>0$ on $[x_{i,j},x_{i,j+1}]$, with $Re(f_\eta)(x_{i,j})>Re(f_\eta)(x_{i,j+1})$, $Re(f_\eta)|_{[x_{i,j}+\frac{1}{\sqrt{\eta}},x_{i,j+1}]}>0$, $Re(f_\eta)(x_{i,j+1})>\epsilon>0$, $\epsilon\in\mathcal{R}$, with $Re(f_\eta)(x_{i,j})\simeq 0$, and $Re(f_\eta)|_{[x_{i,j},x_{i,j+1}-\frac{1}{\sqrt{\eta}}]}>0$, $Re(f_\eta)(x_{i,j})>\epsilon>0$, $\epsilon\in\mathcal{R}$, $Re(f_\eta)(x_{i,j+1})\simeq 0$.

There are again five cases to consider, $Re(f_{\eta,\sqrt{c}})\equiv 0$ on $[\sqrt{c}x_{i,j},\sqrt{c}x_{i,j+1}]$, $Re(f_{\eta,\sqrt{c}})>0$ on $[\sqrt{c}x_{i,j},\sqrt{c}x_{i,j+1}]$, with $Re(f_{\eta,\sqrt{c}})(\sqrt{c}x_{i,j})<Re(f_{\eta,\sqrt{c}})(\sqrt{c}x_{i,j+1})$, $Re(f_{\eta,\sqrt{c}})>0$ on $[\sqrt{c}x_{i,j},\sqrt{c}x_{i,j+1}]$, with $Re(f_{\eta,\sqrt{c}})(\sqrt{c}x_{i,j})>Re(f_{\eta,\sqrt{c}})(\sqrt{c}x_{i,j+1})$, $Re(f_{\eta,\sqrt{c}})|_{[\sqrt{c}x_{i,j}+\frac{\sqrt{c}}{\sqrt{\eta}},\sqrt{c}x_{i,j+1}]}>0$, with $Re(f_{\eta,\sqrt{c}})|_{[\sqrt{c}x_{i,j},\sqrt{c}x_{i,j}+\frac{\sqrt{c}}{\sqrt{\eta}})}\simeq 0$, and $Re(f_{\eta,\sqrt{c}})|_{[\sqrt{c}x_{i,j},\sqrt{c}x_{i,j+1}-\frac{\sqrt{c}}{\sqrt{\eta}}]}>0$, with $Re(f_{\eta,\sqrt{c}})|_{(\sqrt{c}x_{i,j+1}-\frac{\sqrt{c}}{\sqrt{\eta}},\sqrt{c}x_{i,j+1}]}\simeq 0$.

Finitely many elements in sum over cases.
Case 1. No contribution.
Case 3, we prove that;

Let $\{r_{ijk}:0\leq k\leq v(i,j)\}$ and $\{s_{ijk}:0\leq k\leq w(i,j)\}$ enumerate the zeroes of ${}^*cos(-2\pi ixy)$ and ${}^*sin(-2\pi ixy)$ on $[\sqrt{c}x_{ij},\sqrt{c}x_{i,j+1}]$, then;

$$\int_{[\sqrt{c}x_{i,j},\sqrt{c}x_{i,j+1}]}f_{\eta,\sqrt{c}}exp_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)$$

$$=\int_{[\sqrt{c}x_{i,j},\sqrt{c}x_{i,j+1}]}f_{\eta,\sqrt{c}}cos_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)+i\int_{[\sqrt{c}x_{i,j},\sqrt{c}x_{i,j+1}]}f_{\eta,\sqrt{c}}sin_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)$$

$$=\int_{[\sqrt{c}x_{i,j},r_{ij0})}f_{\eta,\sqrt{c}}cos_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)$$

$$+^*\sum_{k=0}^{v(i,j)-1}\int_{[r_{ijk},r_{ij(k+1)})}f_{\eta,\sqrt{c}}cos_{\eta c}(-2\pi ixy)d\mu_{\eta c}(x)$$

$+\int_{[r_{ijv(i,j)},\sqrt{c}x_{i,j+1}]} f_{\eta,\sqrt{c}} cos_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)$

$+i(\int_{[\sqrt{c}x_{i,j},s_{ij0})} f_{\eta,\sqrt{c}} sin_{\eta c}(-2\pi xy) d\mu_{\eta c}(x)$

$+^{*}\sum_{k=0}^{w(i,j)-1}\int_{[s_{ijk},s_{ij(k+1)})} f_{\eta,\sqrt{c}} sin_{\eta c}(-2\pi xy) d\mu_{\eta c}(x)$

$+\int_{[s_{ijw(i,j)},\sqrt{c}x_{i,j+1}]} f_{\eta,\sqrt{c}} sin_{\eta c}(-2\pi xy) d\mu_{\eta c}(x))$

Let $\theta_{ij}(k)=\int_{[r_{ijk},r_{ij(k+1)})} f_{\eta,\sqrt{c}} cos_{\eta c}(-2\pi xy) d\mu_{\eta c}(x)$. Then, as $f_{\eta,\sqrt{c}}|_{[r_{ijk},r_{ij(k+1)}]}>0$, and;

$$\frac{cos_{\eta c}|_{[r_{ijk},r_{ij(k+1)}]}}{|cos_{\eta c}|_{[r_{ijk},r_{ij(k+1)}]}|}=-\frac{cos_{\eta c}|_{[r_{ij(k+1)},r_{ij(k+2)}]}}{|cos_{\eta c}|_{[r_{ij(k+1)},r_{ij(k+2)}]}|}$$

the $*$-sequence $\{\theta_{ij}(k):0\leq k\leq v(i,j)-1\}$ is alternating. As $f_{\eta,\sqrt{c}}|_{[r_{ijk},r_{ij(k+1)}]}>0$ is decreasing, we have that;

$|\theta_{ij}(k)|\leq|\theta_{ij}(k+1)|$, for $0\leq k\leq v(i,j)-1$

Assuming, that $0\leq\theta_{ij}(0)\leq c_{ij}$, $c_{ij}\in {}^{*}\mathcal{R}$, and $v(i,j)$ is odd, it follows that;

$0\leq {}^{*}\sum_{k=0}^{v(i,j)-1}\theta_{ij}(k)\leq c_{ij}$

as $\theta_{ij}(2l)+\theta_{ij}(2l-1)\leq 0$ for $1\leq l\leq\frac{v(i,j)-1}{2}$, $\theta_{ij}(0)=c_{ij}$

and $\theta_{ij}(2l)+\theta_{ij}(2l+1)\geq 0$, for $0\leq l\leq\frac{v(i,j)-3}{2}$, $\theta_{ij}(v(i,j)-1)\geq 0$

and when $v(i,j)$ is even.

$0\leq {}^{*}\sum_{k=0}^{v(i,j)-1}\theta_{ij}(k)\leq c_{ij}$

as $\theta_{ij}(2l)+\theta_{ij}(2l-1)\leq 0$, for $1\leq l\leq\frac{v(i,j)-2}{2}$, and

$\theta_{ij}(0)=c_{ij},\theta_{ij}(v(i,j)-1)\leq 0$

$\theta_{ij}(2l)+\theta_{ij}(2l+1)\geq 0$, for $0\leq l\leq\frac{v(i,j)-2}{2}$

We have that;

$|\theta_{ij}(0)|=|\int_{[r_{ij0},r_{ij1})} f_{\eta,\sqrt{c}} cos_{\eta c}(2\pi xy) d\mu_{\eta c}(x)|$

$\leq \int_{[r_{ij0},r_{ij1})} |f_{\eta,\sqrt{c}}| d\mu_{\eta c}$

$= D\mu_{\eta c}([[r_{ij0}, r_{ij1})))$

$\leq D\sqrt{c}w$

where $w = \mu x\{x > 0, {}^*cos(2\pi xy) = 0\}$

$= D\frac{\sqrt{c}}{|y|}$

$\leq \frac{D\sqrt{c}\pi}{\sqrt{c}(\eta^{\frac{1}{2}-\gamma})} \simeq 0$

Case 2. Reverse the sequences from Case 3.
Cases 4 and 5 are similar to Cases 2 and 3, and left to the reader.

This proves $|\int_{[-\sqrt{c},\sqrt{c}]} f_{\eta,\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)| \simeq 0$

Similarly, one can show that;

$|\int_{[-n\sqrt{c},n\sqrt{c}]} f_{\eta,\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)| \simeq 0$

for $n \in \mathcal{N}$. Let $\epsilon > 0$ and define θ_ϵ, by;

$\theta_\epsilon(n) = \int_{[-n\sqrt{c},n\sqrt{c}]} f_{\eta,\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)$

By overflow, there exists $n_\epsilon \in {}^*\mathcal{N}$ infinite, with $|\theta_\epsilon(n_\epsilon)| < \epsilon$, so that;

$|\int_{[-n_\epsilon\sqrt{c},n_\epsilon\sqrt{c}]} f_{\eta,\sqrt{c}} exp_{\eta c}(-2\pi ixy) d\mu_{\eta c}(x)| < \epsilon$

We have that;

$|\int_{|x|>n_{\epsilon c}\cap\overline{\mathcal{R}_{\eta c}}} f^{\sqrt{c}} exp_{\eta c}(-2\pi ixy)(x) d\mu_{\eta c}|$

$\leq \frac{2c}{n_\epsilon c}$

$= \frac{1}{n_\epsilon} \simeq 0$

so that;

$|\mathcal{F}_{\eta c}(f_{\eta,\sqrt{c}})(y)|$

$= |\int_{\overline{\mathcal{R}_{\eta c}}} f_{\eta,\sqrt{c}} exp_{\eta c}(-2\pi i x y) d\mu_{\eta c}(x)|$

$\leq |\int_{[-n_\epsilon\sqrt{c}, n_\epsilon\sqrt{c}]} f_{\eta,\sqrt{c}} exp_{\eta c}(-2\pi i x y) d\mu_{\eta c}(x)|$

$+|\int_{|x|>n_\epsilon \cap \overline{\mathcal{R}_{\eta c}}} f_{\sqrt{c}} exp_{\eta c}(-2\pi i x y)(x) d\mu_{\eta c}|$

$< 2\epsilon$

As this holds for all $\epsilon > 0$, we obtain the result.

The final part follows from Lemma 0.1.

□

Lemma 0.4. *For $\kappa \in \mathcal{R}$, $0 < \kappa < 1$, $c = \eta^{\frac{2}{3}}$, $\kappa\sqrt{\eta c} < |y| < \sqrt{\eta c}$*

$\int_{(|x|>y)\cap\overline{\mathcal{R}_{\eta c}}} f_{\eta,\sqrt{c}} \simeq 0$

Proof. We have that;

$|\int_{(|x|>y)\cap\overline{\mathcal{R}_{\eta c}}} f_{\sqrt{c}}(x) d\mu_{\eta c}|$

$\simeq ? 2\sqrt{c} |\int_{(|x|>\frac{y}{\sqrt{c}})\cap\overline{\mathcal{R}_{\eta c}}} f_{\sqrt{c}}(x) d\mu_{\eta c}|$

$\leq 2\sqrt{c}\int_{\frac{y}{\sqrt{c}}}^{\sqrt{\eta}} \frac{dx}{x^2}$ (by transfer)

$= 2\sqrt{c}[\frac{-1}{x}]_{\frac{y}{\sqrt{c}}}^{\sqrt{\eta c}}$

$= 2\sqrt{c}(\frac{\sqrt{c}}{y} - \frac{1}{\sqrt{\eta c}})$.

$\leq \frac{2c}{y}$

$= \frac{2c}{\lambda\sqrt{\eta c}} \simeq 0$

for $\kappa \leq \lambda \leq 1$, $\lambda \in {}^*\mathcal{R}$, $c = \eta^{\frac{2}{3}}$.

□

Lemma 0.5. *For δ infinitesimal, $|y| \leq \eta^\delta$, and $\sqrt{c} = \eta^{\frac{1}{3}}$, we have that;*

$|(-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (-4\pi^2 i)\frac{[\sqrt{\eta c} y]^2}{\eta c^2}| \leq 4\pi^2(2\eta^{\delta-\frac{1}{2}}(\frac{(1+c)}{c^{\frac{3}{2}}}) + \frac{1}{\eta} + \frac{1}{\eta c^2}) \simeq 0$

and $|(\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)(-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)(-4\pi^2 i)\frac{[\sqrt{\eta c} y]^2}{\eta c^2}|$

$\leq E\sqrt{c}(4\pi^2(2\eta^{\delta-\frac{1}{2}}(\frac{(1+c)}{c^{\frac{3}{2}}})+\frac{1}{\eta}+\frac{1}{\eta c^2}))\simeq 0$

Proof. Let $\frac{\sqrt{\eta}y}{\sqrt{c}}=m+\delta$, with $m\in {}^*\mathcal{Z}$, $|\delta|<1$

Then;

$\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}$

$=\frac{m^2}{\eta}$

$=\frac{(\frac{\sqrt{\eta}y}{\sqrt{c}}-\delta)^2}{\eta}$

Let $\frac{\sqrt{\eta c}y}{\eta c^2}=n+\delta'$, with $n\in {}^*\mathcal{Z}$, $|\delta'|<1$

Then;

$\frac{[\sqrt{\eta c}y]^2}{\eta c^2}$

$=\frac{n^2}{\eta c^2}$

$=\frac{(\sqrt{\eta c}y-\delta')^2}{\eta c^2}$

We have that;

$|\frac{(-4\pi^2 i)[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}-(-4\pi^2 i)\frac{[\sqrt{\eta c}y]^2}{\eta c^2}|$

$=4\pi^2|\frac{(\sqrt{\frac{\eta y}{\sqrt{c}}}-\delta)^2}{\eta}-\frac{(\sqrt{\eta c}-\delta')^2}{\eta c^2}|$

$=4\pi^2|\frac{(\frac{\eta y^2}{c}-\frac{2\delta\sqrt{\eta}y}{\sqrt{c}}+\delta^2)}{\eta}-\frac{(\eta c y^2-2\sqrt{\eta c}y\delta'+\delta'^2)}{\eta c^2}|$

$=4\pi^2|(\frac{y^2}{c}-\frac{2\delta y}{\sqrt{\eta c}}+\frac{\delta^2}{\eta})-(\frac{y^2}{c}-\frac{2\delta' y}{\sqrt{\eta}c^{\frac{3}{2}}}+\frac{\delta'^2}{\eta c^2})|$

$=4\pi^2|\frac{2\delta' y}{\sqrt{\eta}c^{\frac{3}{2}}}-\frac{2\delta y}{\sqrt{\eta c}}+\frac{\delta^2}{\eta}-\frac{\delta'^2}{\eta c^2}|$

$\leq 4\pi^2(\frac{2|y|}{\sqrt{\eta}c^{\frac{3}{2}}}+\frac{2|y|}{\sqrt{\eta c}}+\frac{1}{\eta}+\frac{1}{\eta c^2})$

$\leq 4\pi^2(\frac{2\eta^{\delta}}{\eta^{\frac{1}{2}}c^{\frac{3}{2}}}+\frac{2\eta^{\delta}}{\sqrt{\eta c}}+\frac{1}{\eta}+\frac{1}{\eta c^2})$

$\leq 4\pi^2(2\eta^{\delta-\frac{1}{2}}(\frac{(1+c)}{c^{\frac{3}{2}}})+\frac{1}{\eta}+\frac{1}{\eta c^2})\simeq 0$

We have that;

$|(\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)(-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)(-4\pi^2 i)\frac{[\sqrt{\eta c}y]^2}{\eta c^2}|$

$\leq |(\mathcal{F}_{\eta c}(g_{\sqrt{c}})(y)|(-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta} - (-4\pi^2 i)\frac{[\sqrt{\eta c}y]^2}{\eta c^2}|$

$\leq E\sqrt{c}(4\pi^2(2\eta^{\delta-\frac{1}{2}}(\frac{(1+c)}{c^{\frac{3}{2}}}) + \frac{1}{\eta} + \frac{1}{\eta c^2})) \simeq 0$

using Lemma 0.1 and the hypotheses on c.

□

Lemma 0.6. *For $f \in S(\mathcal{R})$ and corresponding $\{f_\eta, f_{\eta,\sqrt{c}}\}$, we have that;*

$$((-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}\chi_{|y|>\frac{\sqrt{\eta c}}{4}} - (-4\pi^2 i)\frac{[\sqrt{\eta c}y]^2}{\eta c^2}\chi_{|y|>\frac{\sqrt{\eta c}}{4}})\mathcal{F}_{\eta c}(f_{\eta,\sqrt{c}}) \simeq 0$$

Proof. For $|y \leq \eta^\delta$, the result follows from Lemma 0.5, for $0 < \gamma < \frac{1}{2}$, $\sqrt{c}\eta^{\frac{1}{2}-\gamma} \leq |y| < \frac{\sqrt{\eta c}}{4}$, $(*)$, the result follows from Lemma 0.3, for $\frac{\sqrt{\eta c}}{4} < |y| \leq \sqrt{\eta c}$, the result follows from the definition of $\chi_{|y|>\frac{\sqrt{\eta c}}{4}}$, ?fill in details for $\eta^\delta < |y| < \frac{\sqrt{\eta c}}{4}$, from $(*)$.

□

Lemma 0.7. *We have that;*

$F_\eta^{-1}(gF_\eta(f_\eta))(\frac{m}{\sqrt{\eta}})$

$\simeq F_{\eta c}^{-1}(g_{\sqrt{c}}F_{\eta c}(f_\eta^{\sqrt{c}}))(\frac{m}{\sqrt{\eta c}})$ *(Lemma 0.69)*

$? \simeq F_{\eta c}^{-1}(g_{\sqrt{c}}\chi_{|y|>\frac{\sqrt{\eta c}}{4}}F_{\eta c}(f_{\eta,\sqrt{c}}))(\frac{m}{\sqrt{\eta c}})$ *(tricky part) (adapt Lemma 0.69 for truncated)*

$? \simeq F_{\eta c}^{-1}(time evolution for K_{\eta c}\chi_{|y|>\frac{\sqrt{\eta c}}{4}}F_{\eta c}(f_{\eta,\sqrt{c}}))(\frac{m}{\sqrt{\eta c}})$, *using* $(*)$

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$\mathcal{F}_{\eta c}^{-1}(((-4\pi^2 i)\frac{[\sqrt{\eta}(\frac{y}{\sqrt{c}})]^2}{\eta}\chi_{|y|>\frac{\sqrt{\eta c}}{4}})\mathcal{F}_{\eta c}(f_{\eta,\sqrt{c}}))(\frac{m}{\sqrt{\eta c}})$

$\simeq \mathcal{F}_{\eta c}^{-1}((-4\pi^2 i)\frac{[\sqrt{\eta c}y]^2}{\eta c^2}\chi_{|y|>\frac{\sqrt{\eta c}}{4}}\mathcal{F}_{\eta c}(f_{\eta,\sqrt{c}}))(\frac{m}{\sqrt{\eta c}})$ $(*)$

....

relate propagators, for convolution equation with $K_{\eta c}\chi_{|y|>\frac{\sqrt{\eta c}}{4}}$ *(Lemma 0.80), and then go back to original equation* $K_{\eta c}$*.*

Lemma 0.8. *Relate the propogators using the relation* $G_c(t, \frac{y}{\sqrt{c}}) = G(t, y)$*, for initial conditions* $\{f_{\eta,\sqrt{c}}, f_\eta\}$*, from Lemma* **??***, and apply method in note.*

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