

# SIMPLE PROOFS OF THE RIEMANN-LEBESGUE LEMMAS USING NONSTANDARD ANALYSIS

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ABSTRACT. We give simple proofs of the Riemann-Lebesgue Lemmas for Fourier series and Fourier transforms using nonstandard analysis.

**Definition 0.1.** We let  $\eta \in {}^*\mathcal{N}$  be infinite, and  $\overline{V}_\eta = {}^*[-1, 1)$ , with the  $*$ -algebra  $\mathfrak{B}_\eta$  generated by;

$$\{[\frac{i}{\eta}, \frac{i+1}{\eta}) : -\eta \leq i \leq \eta - 1\}$$

and measure  $\mu_\eta$  defined by  $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$ , for  $-\eta \leq i \leq \eta - 1$

We let  $\{e_i : -\eta \leq i \leq \eta - 1\}$  be the standard orthonormal basis defined by;

$$e_i(\frac{j}{\eta}) = \sqrt{\eta} \delta_{ij}, \text{ for } -\eta \leq j \leq \eta - 1$$

with respect to the inner product  $\langle, \rangle_\eta$ , defined by;

$$\langle f, g \rangle_\eta = \int_{\overline{V}_\eta} f \overline{g} d\mu_\eta, \text{ for } \{f, g\} \subset V(\overline{V}_\eta)$$

We recall the inversion theorem from [2], that for  $f \in V(\overline{V}_\eta)$ ;

$$f(x) = \frac{1}{2} {}^* \sum_{m \in \mathfrak{Z}_\eta} (\mathcal{F}_\eta(f))(m) \exp_\eta(\pi i m x)$$

where  $\mathfrak{Z}_\eta = {}^*\mathcal{Z} \cap [-\eta, \eta)$

$$\exp_\eta(\pi i m x) = {}^* \exp(\pi i m \frac{[x\eta]}{\eta}), \text{ for } -\eta \leq m \leq \eta - 1$$

and  $(\mathcal{F}_\eta(f))(m)$

$$= \int_{\overline{V}_\eta} f(x) \exp_\eta(-\pi i m x) d\mu_\eta(x)$$

$= \langle f, \exp_\eta(\pi imx) \rangle_\eta$ , for  $-\eta \leq m \leq \eta - 1$

We define the nonstandard derivative on  $V(\overline{V}_\eta)$  by;

$$f'(\frac{j}{\eta}) = \eta(f(\frac{j+1}{\eta}) - f(\frac{j}{\eta})), \text{ for } -\eta \leq j \leq \eta - 2$$

$$f'(\frac{\eta-1}{\eta}) = \eta(f(-1) - f(\frac{\eta-1}{\eta}))$$

**Lemma 0.2.** *If  $f \in C([-1, 1])$ , with corresponding  $f_\eta \in V(\overline{V}_\eta)$ , then, for infinite  $m \in \mathfrak{Z}_\eta$ , we have;*

$$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$$

*Proof.* Let  $\{a, b\} \subset \overline{V}_\eta$ , and let  $\chi_{[a,b],\eta} \in V(\overline{V}_\eta)$  be defined by;

$$\chi_{[a,b],\eta}(x) = 1 \text{ if } \frac{[a\eta]}{\eta} \leq x < \frac{[b\eta]+1}{\eta}$$

$$\chi_{[a,b],\eta}(x) = 0 \text{ otherwise}$$

We claim that, for infinite  $m$ ,  $\mathcal{F}_\eta(\chi_{[a,b]})(m) \simeq 0$ , (\*)

We have, as in [2], that for  $m \in \mathfrak{Z}_\eta$ ,  $x \in \overline{V}_\eta$ , that;

$$(\exp_\eta(-\pi imx))'$$

$$= \exp_\eta(-\pi imx)\theta_\eta(m)$$

$$\text{where } \theta_\eta(m) = \eta(\exp_\eta(-\pi i \frac{m}{\eta}) - 1)$$

and;

$$\exp_\eta(-\pi imx) = \frac{(\exp_\eta(-\pi imx))'}{\theta_\eta(m)}$$

It follows that;

$$(\mathcal{F}_\eta(\chi_{[a,b]}))(m)$$

$$= \int_{\overline{V}_\eta} \chi_{[a,b]}(x) \exp_\eta(-\pi imx) d\mu_\eta(x)$$

$$= \int_{\overline{V}_\eta} \chi_{[a,b]}(x) \frac{(\exp_\eta(-\pi imx))'}{\theta_\eta(m)} d\mu_\eta(x)$$

$$\begin{aligned}
 &= \frac{1}{\theta_\eta(m)} \int_{\sqrt{\eta}} \chi_{[a,b]}(x) (\exp_\eta(-\pi i m x))' d\mu_\eta(x) \\
 &= \frac{1}{\eta \theta_\eta(m)} * \sum_{i=\frac{[a\eta]}{\eta}}^{\frac{[b\eta]+1}{\eta}} \chi_{[a,b]}(\frac{i}{\eta}) (\exp_\eta(-\pi i m \frac{i}{\eta}))' \\
 &= \frac{1}{\eta \theta_\eta(m)} * \sum_{i=\frac{[a\eta]}{\eta}}^{\frac{[b\eta]+1}{\eta}} \eta (\exp_\eta(-\pi i m \frac{i+1}{\eta}) - (\exp_\eta(-\pi i m \frac{i}{\eta}))) \\
 &= \frac{1}{\theta_\eta(m)} (\exp_\eta(-\pi i m (\frac{[b\eta]+2}{\eta})) - (\exp_\eta(-\pi i m (\frac{[a\eta]}{\eta}))))
 \end{aligned}$$

As in [2], Lemma 0.12, for  $m \in \mathfrak{Z}_\eta$ , that;

$$2|m| \leq |\theta_\eta(m)|$$

It follows that, for  $m$  infinite;

$$\begin{aligned}
 |(\mathcal{F}_\eta(\chi_{[a,b]}))(m)| &\leq \frac{1}{2|m|} |(exp_\eta(-\pi i m (\frac{[b\eta]+2}{\eta})) - (exp_\eta(-\pi i m (\frac{[a\eta]}{\eta}))))| \\
 &\leq \frac{2}{2|m|} = \frac{1}{|m|} \simeq 0
 \end{aligned}$$

Hence, (\*) is proved. As  $f \in C([-1, 1])$ , by Darboux's Theorem, there exists a sequence of step functions  $\{g_n : n \in \mathcal{N}\}$ , such that;

$$\int_{-1}^1 |f - g_n| d\mu < \frac{1}{n}$$

where  $\mu$  denotes Lebesgue measure. We have that;

$$\begin{aligned}
 g_{n,\eta} &= (\sum_{k=1}^{m(n)-1} c_k \chi_{[b_{kn}, b_{(k+1)n}]} )_\eta \\
 &= (\sum_{k=1}^{m(n)-1} c_k (\chi_{[b_{kn}, b_{(k+1)n}]} )_\eta \\
 &= (\sum_{k=1}^{m(n)-1} c_k \chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]}), (**)
 \end{aligned}$$

for a partition  $-1 \leq b_{1n} \leq \dots \leq b_{m(n)n} \leq 1$  and  $c_k \in \mathcal{R}$ , for  $1 \leq k \leq m(n) - 1$ . We have that, for infinite  $m \in \mathfrak{Z}_\eta$ , using (\*), (\*\*), that;

$$\begin{aligned}
 &\mathcal{F}_\eta(g_{n,\eta})(m) \\
 &= \mathcal{F}_\eta((\sum_{k=1}^{m(n)-1} c_k \chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]}))(m) \\
 &= \sum_{k=1}^{m(n)-1} c_k \mathcal{F}_\eta(\chi_{[\frac{[b_{kn}\eta]+1}{\eta}, \frac{[b_{(k+1)n}\eta]}{\eta}]})(m)
 \end{aligned}$$

$$\simeq 0$$

Then, for infinite  $m$ , and  $n \in \mathcal{N}$ ;

$$\begin{aligned} & |\mathcal{F}_\eta(f_\eta)(m)| \\ &= |\mathcal{F}_\eta(f_\eta - g_{n,\eta} + g_{n,\eta})(m)| \\ &\leq |\mathcal{F}_\eta(g_{n,\eta})(m)| + |\mathcal{F}_\eta(f_\eta - g_{n,\eta})(m)| \\ &\leq \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta, (***) \end{aligned}$$

As  $f \in C[-1, 1]$ , we have that,  $f_\eta$  is  $S$ -continuous, bounded and  $S$ -integrable, and  ${}^\circ f_\eta = st^* f$ , where  $st : \overline{V}_\eta \rightarrow [-1, 1]$  is the standard part mapping.  $g_{n,\eta}$  is bounded and  $S$ -integrable,  $|(f_\eta - g_{n,\eta})|$  is bounded and  $S$ -integrable. It follows, using the  $S$ -integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

$$\begin{aligned} & {}^\circ(\int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta) \\ &= \int_{\overline{V}_\eta} |{}^\circ(f_\eta - g_{n,\eta})| dL(\mu_\eta) \\ &= \int_{\overline{V}_\eta} |(st^*(f) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta) \\ &\leq \int_{\overline{V}_\eta} |(st^*(f) - st^*g_n)| dL(\mu_\eta) \\ &+ \int_{\overline{V}_\eta} |(st^*(g_n) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta) \\ &\simeq \int_{-1}^1 |f - g_n| d\mu < \frac{1}{n} \end{aligned}$$

Therefore, using (\*\*);

$$\begin{aligned} & |\mathcal{F}_\eta(f_\eta)(m)| \\ &< \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \end{aligned}$$

As this holds for all  $n \in \mathcal{N}$ , using countable comprehension and overflow, see [1],  $(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

□

**Remarks 0.3.** *It is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

*If  $f \in C([-1, 1])$ , then;*

$$\lim_{|m| \rightarrow \infty} \mathcal{F}(f)(m) = 0$$

*It is sufficient to show that, given  $\epsilon > 0$ , there exists  $M(\epsilon)$ , such that;*

*$|\mathcal{F}(f)(m)| < \epsilon$ , for all  $m \in \mathcal{Z}$ ,  $m \geq M(\epsilon)$ . As, for all infinite  $m \in \mathfrak{Z}_\eta$ ;*

$$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$$

*it follows by underflow, that, for all  $|m| \geq M(\epsilon)$ ,  $m \in \mathcal{Z}$ ;*

$$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$$

*The result then follows from the fact, that, for finite  $m \in \mathcal{Z}$ ;*

$$\circ(\mathcal{F}_{f_\eta}(m)) = \mathcal{F}(f)(m)$$

*as  $f_\eta(x)$  and  $\exp_\eta(-\pi imx)$  are  $S$ -continuous and  $S$ -integrable on  $\overline{V}_\eta$ . The proof above follows the structure of the standard result.*

**Definition 0.4.** *We recall the definitions from [2]. We let  $\eta \in {}^*\mathcal{N}$  be infinite and odd, we let;*

$$\overline{\mathcal{R}}_\eta = {}^*\bigcup_{-\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}} \left[ \frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}} \right)$$

*so that  $\overline{\mathcal{R}}_\eta = {}^*\left[-\frac{(\eta-1)}{2\sqrt{\eta}}, \frac{(\eta+1)}{2\sqrt{\eta}}\right)$ . We let  $\mathcal{D}_\eta$  denote the associated  $*$ -finite algebra, generated by the intervals  $\left[\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}}\right)$ , for  $-\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}$ , and  $\mu_\eta$  the associated counting measure defined by  $\mu_\eta\left(\left[\frac{i}{\sqrt{\eta}}, \frac{i+1}{\sqrt{\eta}}\right)\right) = \frac{1}{\sqrt{\eta}}$ . We let  $(\overline{\mathcal{R}}_\eta, L(\mathcal{D}_\eta), L(\mu_\eta))$  denote the associated Loeb space.*

*We let  $(\mathcal{R} \cup \{+\infty, -\infty\}, \mathfrak{D}, \mu)$  denote the extended real line, with the completion  $\mathfrak{D}$  of the extension of the Borel field, and  $\mu$  the extension of Lebesgue measure, with  $\mu(+\infty) = \mu(-\infty) = \infty$ . We let  $(\mathcal{R}_{\geq 0} \cup \{+\infty\}, \mathfrak{C}, \lambda)$  denote the extended real half line, with the completion  $\mathfrak{C}$  of the extended Borel field, and  $\lambda$  the extension of Lebesgue*

measure, with  $\lambda(+\infty) = \infty$ , see [?], Chapter 6.

Given a measurable  $f_\eta : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$ , we define the nonstandard Fourier transform  $\mathcal{F}(f_\eta) : \overline{\mathcal{R}_\eta} \rightarrow {}^*\mathcal{C}$  by;

$$\mathcal{F}(f_\eta)(y) = \int_{\overline{\mathcal{R}_\eta}} f_\eta(x) \exp_\eta(-2\pi ixy) d\mu_\eta(x)$$

With this definition, we have, see [5] that;

$$f_\eta(x) = \int_{\overline{\mathcal{R}_\eta}} \hat{f}_\eta(y) \exp_\eta(2\pi ixy) d\mu_\eta(y) \quad (*)$$

We define the nonstandard derivative on  $V(\overline{\mathcal{R}_\eta})$  by;

$$f'(\frac{j}{\sqrt{\eta}}) = \sqrt{\eta}(f(\frac{j+1}{\sqrt{\eta}}) - f(\frac{j}{\sqrt{\eta}})), \text{ for } -\frac{(\eta-1)}{2} \leq j \leq \frac{\eta-3}{2}$$

$$f'(\frac{\eta-1}{2\sqrt{\eta}}) = \sqrt{\eta}(f(-\frac{(\eta-1)}{2\sqrt{\eta}}) - f(\frac{\eta-1}{\sqrt{\eta}}))$$

**Lemma 0.5.** *If  $f \in S(\mathcal{R})$ , with corresponding  $f_\eta \in V(\overline{\mathcal{V}_\eta})$ , then, for infinite  $y \in \overline{\mathcal{R}_\eta}$ , we have;*

$$(\mathcal{F}_\eta(f_\eta))(y) \simeq 0$$

*Proof.* Let  $n \in \mathcal{N}$ , we first prove that;

$$|\int_{(|x| > \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta \exp_\eta(-2\pi ixy) d\mu_\eta(x)| \leq \frac{E}{(n-1)}, \quad (*)$$

where  $E \in \mathcal{R}$ .

We have that;

$$\begin{aligned} & |\int_{(|x| \geq \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta \exp_\eta(-2\pi ixy) d\mu_\eta(x)| \\ & \leq \frac{1}{\sqrt{\eta}} * \sum_{|k|=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} |f_\eta(\frac{k}{\sqrt{\eta}})| \\ & = \frac{1}{\sqrt{\eta}} * \sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} |f^*(\frac{k}{\sqrt{\eta}})| \end{aligned}$$

As  $f \in S(\mathcal{R})$ , we have that,  $|f^*(x)| \leq \frac{C}{|x|^2}$ , for  $|x| \geq 1$ ,  $C \in \mathcal{R}$ . It follows that;

$$|f^*|(\frac{k}{\sqrt{\eta}}) \leq \frac{C}{|\frac{k}{\sqrt{\eta}}|^2} = \frac{C\eta}{|k|^2}, \text{ for } |k| \geq [n\sqrt{\eta}]$$

Then;

$$\begin{aligned} & \frac{1}{\sqrt{\eta}} * \sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} |f^*|(\frac{k}{\sqrt{\eta}}) \\ & \leq \frac{1}{\sqrt{\eta}} * \sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} \frac{C\eta}{|k|^2} \\ & = \frac{2\eta C}{\sqrt{\eta}} * \sum_{k=[n\sqrt{\eta}]}^{\frac{\eta-1}{2}} \frac{1}{k^2} \\ & \leq \frac{2\eta C}{\sqrt{\eta}} \int_{[n\sqrt{\eta}]-1}^{\frac{\eta-1}{2}} \frac{dx}{x^2} \text{ (by transfer)} \\ & = 2C\sqrt{\eta} \left[ \frac{-1}{x} \right]_{[n\sqrt{\eta}]-1}^{\frac{\eta-1}{2}} \\ & = 2C\sqrt{\eta} \left( \frac{1}{[n\sqrt{\eta}]-1} - \frac{2}{\eta-1} \right) \\ & \leq 2C\sqrt{\eta} \frac{1}{n\sqrt{\eta}-2} \\ & \leq 2C\sqrt{\eta} \frac{1}{(n-1)\sqrt{\eta}} \\ & = \frac{2C}{(n-1)} \end{aligned}$$

which gives the result (\*), taking  $E = 2C$ . Let  $\bar{V}_{\eta,n} = (-\frac{[n\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \bar{R}_{\eta}$ .

Let  $\{a, b\} \subset \bar{V}_{\eta,n}$ , and let  $\chi_{[a,b],\eta} \in V(\bar{V}_{\eta,n})$  be defined by;

$$\chi_{[a,b],\eta}(x) = 1 \text{ if } \frac{[a\sqrt{\eta}]}{\sqrt{\eta}} \leq x < \frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}$$

$$\chi_{[a,b],\eta}(x) = 0 \text{ otherwise}$$

We claim that, for infinite  $y$ ,  $\mathcal{F}_{\eta}(\chi_{[a,b]}) (y) \simeq 0$ , (\*\*)

We have, as in [5], that for  $y \in \bar{R}_{\eta}$ ,  $x \in \bar{V}_{\eta,n}$ , that;

$$(\exp_{\eta}(-2\pi i y x))' = \chi_{\eta}(y) \exp_{\eta}(-2\pi i y x)$$

where;

$$\chi_{\eta}(y) = \sqrt{\eta} (\exp_{\eta}(\frac{-2\pi i y}{\sqrt{\eta}}) - 1)$$

and;

$$\exp_\eta(-\pi iyx) = \frac{(\exp_\eta(-\pi iyx))'}{\chi_\eta(y)}$$

It follows that;

$$\begin{aligned} &= \int_{\overline{V}_{\eta,n}} \chi_{[a,b]}(x) \exp_\eta(-2\pi iyx) d\mu_\eta(x) \\ &= \int_{\overline{V}_{\eta,n}} \chi_{[a,b]}(x) \frac{(\exp_\eta(-2\pi iyx))'}{\chi_\eta(y)} d\mu_\eta(x) \\ &= \frac{1}{\chi_\eta(y)} \int_{\overline{V}_{\eta,n}} \chi_{[a,b]}(x) (\exp_\eta(-2\pi iyx))' d\mu_\eta(x) \\ &= \frac{1}{\sqrt{\eta}\chi_\eta(y)} * \sum_{i=\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}}^{\frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}} \chi_{[a,b]}(\frac{i}{\sqrt{\eta}}) (\exp_\eta(-2\pi iy\frac{i}{\sqrt{\eta}}))' \\ &= \frac{1}{\sqrt{\eta}\chi_\eta(y)} * \sum_{i=\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}}^{\frac{[b\sqrt{\eta}]+1}{\sqrt{\eta}}} \sqrt{\eta} (\exp_\eta(-2\pi iy\frac{i+1}{\sqrt{\eta}}) - (\exp_\eta(-2\pi iy\frac{i}{\sqrt{\eta}}))) \\ &= \frac{1}{\chi_\eta(y)} (\exp_\eta(-2\pi iy(\frac{[b\sqrt{\eta}]+2}{\sqrt{\eta}})) - (\exp_\eta(-2\pi iy(\frac{[a\sqrt{\eta}]}{\sqrt{\eta}})))) \end{aligned}$$

As in [5], Lemma 0.20, for  $y \in \overline{\mathcal{R}}_\eta$ , we have that;

$$4|y| \leq |\chi_\eta(y)|$$

It follows that, for  $y$  infinite;

$$\begin{aligned} &| \int_{\overline{V}_{\eta,n}} \chi_{[a,b]}(x) \exp_\eta(-2\pi iyx) d\mu_\eta(x) | \\ &\leq \frac{1}{4|y|} |(\exp_\eta(-2\pi iy(\frac{[b\sqrt{\eta}]+2}{\sqrt{\eta}})) - (\exp_\eta(-2\pi iy(\frac{[a\sqrt{\eta}]}{\sqrt{\eta}}))))| \\ &\leq \frac{2}{4|y|} = \frac{1}{2|y|} \simeq 0 \end{aligned}$$

Hence, (\*\*) is proved. As  $f \in C([-n, n])$ , by Darboux's Theorem, there exists a sequence of step functions  $\{g_r : r \in \mathcal{N}\}$ , such that;

$$\int_{-n}^n |f - g_r| d\mu < \frac{1}{r}$$

where  $\mu$  denotes Lebesgue measure. We have that;

$$\begin{aligned} g_{r,\eta} &= (\sum_{k=1}^{m(r)-1} c_k \chi_{[b_{kr}, b_{(k+1)r}]})_\eta \\ &= (\sum_{k=1}^{m(r)-1} c_k (\chi_{[b_{kr}, b_{(k+1)r}]}))_\eta \end{aligned}$$



$$= \left( \sum_{k=1}^{m(r)-1} c_k \chi_{\left[ \frac{\lfloor b_{kr}\sqrt{\eta} \rfloor + 1}{\sqrt{\eta}}, \frac{\lfloor b_{(k+1)r}\sqrt{\eta} \rfloor}{\sqrt{\eta}} \right]} \right), (***)$$

for a partition  $-n \leq b_{1r} \leq \dots \leq b_{m(r)r} \leq n$  and  $c_k \in \mathcal{R}$ , for  $1 \leq k \leq m(r) - 1$ . We have that, for infinite  $y \in \overline{\mathcal{R}}_\eta$ , using (\*\*), (\*\*\*), that;

$$\begin{aligned} & \int_{\overline{V}_{\eta,n}} g_{r,\eta}(x) \exp_\eta(-2\pi i y x) d\mu_\eta(x) \\ &= \int_{\overline{V}_{\eta,n}} \left( \left( \sum_{k=1}^{m(r)-1} c_k \chi_{\left[ \frac{\lfloor b_{kr}\sqrt{\eta} \rfloor + 1}{\sqrt{\eta}}, \frac{\lfloor b_{(k+1)r}\sqrt{\eta} \rfloor}{\sqrt{\eta}} \right]} \right) \right) \exp_\eta(-2\pi i y x) d\mu_\eta(x) \\ &= \sum_{k=1}^{m(r)-1} c_k \int_{\overline{V}_{\eta,n}} \left( \chi_{\left[ \frac{\lfloor b_{kr}\sqrt{\eta} \rfloor + 1}{\sqrt{\eta}}, \frac{\lfloor b_{(k+1)r}\sqrt{\eta} \rfloor}{\sqrt{\eta}} \right]} \right) \exp_\eta(-2\pi i y x) d\mu_\eta(x) \\ &\simeq 0 \end{aligned}$$

Then, for infinite  $y$ , and  $r \in \mathcal{N}$ ;

$$\begin{aligned} & \left| \int_{\overline{V}_{\eta,n}} f_\eta \exp_\eta(-2\pi i y x) d\mu_\eta(x) \right| \\ &= \left| \int_{\overline{V}_{\eta,n}} (f_\eta - g_{r,\eta} + g_{r,\eta}) \exp_\eta(-2\pi i y x) d\mu_\eta(x) \right| \\ &\leq \left| \int_{\overline{V}_{\eta,n}} g_{r,\eta} \exp_\eta(-2\pi i y x) d\mu_\eta(x) \right| + \left| \int_{\overline{V}_{\eta,n}} (f_\eta - g_{r,\eta}) \exp_\eta(-2\pi i y x) d\mu_\eta(x) \right| \\ &\leq \frac{1}{n} + \int_{\overline{V}_{\eta,n}} |(f_\eta - g_{r,\eta})| d\mu_\eta, (***) \end{aligned}$$

As  $f \in C[-n, n]$ , we have that,  $f_\eta$  is  $S$ -continuous, bounded and  $S$ -integrable, and  ${}^\circ f_\eta = st^* f$ , where  $st : \overline{V}_{\eta,n} \rightarrow [-n, n]$  is the standard part mapping.  $g_{r,\eta}$  is bounded and  $S$ -integrable,  $|(f_\eta - g_{r,\eta})|$  is bounded and  $S$ -integrable. It follows, using the  $S$ -integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

$$\begin{aligned} & {}^\circ \left( \int_{\overline{V}_{\eta,n}} |(f_\eta - g_{r,\eta})| d\mu_\eta \right) \\ &= \int_{\overline{V}_{\eta,n}} |{}^\circ(f_\eta - g_{r,\eta})| dL(\mu_\eta) \\ &= \int_{\overline{V}_{\eta,n}} |(st^*(f) - ({}^\circ g_{r,\eta}))| dL(\mu_\eta) \\ &\leq \int_{\overline{V}_{\eta,n}} |(st^*(f) - st^*g_r)| dL(\mu_\eta) \\ &+ \int_{\overline{V}_{\eta,n}} |(st^*(g_r) - ({}^\circ g_{r,\eta}))| dL(\mu_\eta) \end{aligned}$$

$$\simeq \int_{-1}^1 |f - g_r| d\mu < \frac{1}{r}$$

Therefore, using (\*\* \*\*);

$$\begin{aligned} & \left| \int_{\overline{V_{\eta,n}}} f_{\eta} \exp_{\eta}(-2\pi i y x) d\mu_{\eta}(x) \right| \\ & < \frac{1}{n} + \int_{\overline{V_{\eta}}} |(f_{\eta} - g_{r,\eta})| d\mu_{\eta}(x) < \frac{1}{n} + \frac{2}{r} \end{aligned}$$

Taking  $r \geq n$ , we obtain that, for infinite  $y$ , that;

$$\begin{aligned} & |\mathcal{F}_{\eta}(f_{\eta})(y)| \\ & \leq \left| \int_{\overline{V_{\eta,n}}} f_{\eta} \exp_{\eta}(-2\pi i y x) d\mu_{\eta}(x) \right| + \left| \int_{(|x| > \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_{\eta}}} f_{\eta} \exp_{\eta}(-2\pi i y x) d\mu_{\eta}(x) \right| \\ & < \frac{3}{n} + \frac{2C}{(n-1)} \end{aligned}$$

As this holds for all  $n \in \mathcal{N}$ , using countable comprehension and overflow, see [1],  $(\mathcal{F}_{\eta}(f_{\eta}))(y) \simeq 0$

□

**Remarks 0.6.** We let  $\mathcal{S}(\mathcal{R})$  denote the Schwartz space. If  $h \in \mathcal{S}(\mathcal{R})$ , we define its Fourier transform by;

$$\mathcal{F}(h)(y) = \int_{-\infty}^{\infty} h(x) e^{-2\pi i y x} dx$$

for  $y \in \mathcal{R}$ .

It is straightforward to deduce the standard Riemann-Lebesgue Lemma;

If  $f \in \mathcal{S}(\mathcal{R})$ , then;

$$\lim_{|y| \rightarrow \infty} \mathcal{F}(f)(y) = 0$$

It is sufficient to show that, given  $\epsilon > 0$ , there exists  $M(\epsilon)$ , such that;

$$|\mathcal{F}(f)(y)| < \epsilon, \text{ for all } y \in \mathcal{R}, y \geq M(\epsilon).$$

As, for all infinite  $y \in \overline{\mathcal{R}_{\eta}}$ ;

$$|(\mathcal{F}_{\eta}(f_{\eta}))(y)| < \epsilon$$

it follows by underflow, that, for all  $|y| \geq M(\epsilon)$ ,  $y \in \overline{\mathcal{R}_\eta}$ ;

$$|(\mathcal{F}_\eta(f_\eta))(y)| < \epsilon$$

The result then follows from the fact, that, for finite  $y \in \mathcal{R}$ ;

$${}^\circ(\mathcal{F}_{f_\eta}(y)) = \mathcal{F}(f)({}^\circ y) = \mathcal{F}(f)(y)$$

as  $f_\eta \exp_\eta(-\pi i m x)$  is  $S$ -continuous and  $S$ -integrable on  $\overline{V_\eta}$ , see [4].

**Lemma 0.7.** *If  $f$  is Riemann integrable on  $[-1, 1]$  and bounded, with corresponding  $f_\eta \in V(\overline{V_\eta})$ , then  $f_\eta$  is  $S$ -integrable on  $\overline{V_\eta}$  and there exist  $\{W_n : n \in \mathcal{N}\} \subset \mathfrak{B}_\eta$ , such that  $\mu_\eta(W_n) < \frac{1}{n}$  and  ${}^\circ(f_\eta) = st^*(f)$  on  $\overline{V_\eta} \setminus W_n$ . Moreover, for  $m$  infinite with  $m \in \mathfrak{I}_\eta$ , we have that;*

$$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$$

*Proof.* Without loss of generality, assume  $f$  is real valued. As  $f_n$  is uniformly bounded, it follows that  $f_\eta$  is bounded, by transfer, and, in particular,  $S$ -integrable. By Lebesgue's Theorem, there exists  $D \subset [-1, 1]$  with  $\mu(D) = 0$ , such that  $f$  is continuous on  $\{[-1, 1] \setminus D\}$ . Let  $C = st^*(D)$ , then  $L(\mu_\eta)(C) = 0$ , let  $E = st^*([-1, 1] \setminus D)$ , then  $E$  and  $C$  are disjoint and  $L(\mu_\eta)(E) = 2$ . Let  $x \in E$ , with corresponding  ${}^\circ x \in [-1, 1] \setminus D$ , then we claim that  $f_\eta(x) \simeq f({}^\circ x) = st^*(f)(x)$ , (\*). As  $f$  is continuous at  ${}^\circ x$ , we have, for  $n \in \mathcal{N}$ , that there exists intervals  $[{}^\circ x - \frac{1}{n}, {}^\circ x + \frac{1}{n}]$ , with  $a(n) \in \mathcal{N}$  increasing, such that  $|f(y) - f({}^\circ x)| < \frac{1}{a(n)}$ , for all  $y \in [{}^\circ x - \frac{1}{n}, {}^\circ x + \frac{1}{n}]$ . It follows, by transfer, as  $|x - {}^\circ x| < \frac{1}{n}$ , for all  $n \in \mathcal{N}$ , that  $|f_\eta(x) - f({}^\circ x)| < \frac{1}{a(\eta)} \simeq 0$ , hence (\*) is shown. It follows that  ${}^\circ(f_\eta) = st^*(f)$  on  $E$ . Using Theorem 3.4 of [?], there exist  $\{W_n : n \in \mathcal{N}\} \subset \mathfrak{B}_\eta$ , with  $W_n \supset C$ , such that  $\mu_\eta(W_n) < \frac{1}{n}$ . Clearly  ${}^\circ(f_\eta) = st^*(f)$  on  $\overline{V_\eta} \setminus W_n$  for  $n \in \mathcal{N}$ . This gives the first claim. Then, for infinite  $m$ , and  $n \in \mathcal{N}$ . Using the fact that  $f$  is Lebesgue integrable, we can find a sequence  $\{g_n : n \in \mathcal{N}\}$  as in Lemma 0.2. Then;

$$\begin{aligned} & |\mathcal{F}_\eta(f_\eta)(m)| \\ &= |\mathcal{F}_\eta(f_\eta - g_{n,\eta} + g_{n,\eta})(m)| \\ &\leq |\mathcal{F}_\eta(g_{n,\eta})(m)| + |\mathcal{F}_\eta(f_\eta - g_{n,\eta})(m)| \end{aligned}$$

$$\leq \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta, (***)$$

We have that,  $f_\eta$  is bounded and  $S$ -integrable, and  ${}^\circ f_\eta = st^* f$ , on  $\overline{V}_\eta \setminus W_n$  where  $st : \overline{V}_\eta \rightarrow [-1, 1]$  is the standard part mapping.  $g_{n,\eta}$  is bounded and  $S$ -integrable,  $|(f_\eta - g_{n,\eta})|$  is bounded and  $S$ -integrable. It follows, using the  $S$ -integrability criteria, see [1], and the fact the standard part mapping is measurable and measure preserving, that;

$$\begin{aligned} & {}^\circ \left( \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| d\mu_\eta \right) \\ &= \int_{\overline{V}_\eta} |({}^\circ f_\eta - ({}^\circ g_{n,\eta}))| dL(\mu_\eta) \\ &= \int_{\overline{V}_\eta \setminus W_n} |{}^\circ(f_\eta - g_{n,\eta})| dL(\mu_\eta) + \int_{W_n} |{}^\circ(f_\eta - g_{n,\eta})| dL(\mu_\eta) \\ &\leq \int_{\overline{V}_\eta \setminus W_n} |(st^*(f) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta) + R\mu_\eta(V_n) \\ &\leq \int_{\overline{V}_\eta \setminus W_n} |(st^*(f) - st^*g_n)| dL(\mu_\eta) + \frac{R}{n} \\ &\quad + \int_{\overline{V}_\eta \setminus W_n} |(st^*(g_n) - ({}^\circ g_{n,\eta}))| dL(\mu_\eta) \\ &\simeq \int_{-1}^1 |f - g_n| d\mu + \frac{R}{n} < \frac{R+2}{n} \end{aligned}$$

Therefore, using (\*\*\*);

$$\begin{aligned} & |\mathcal{F}_\eta(f_\eta)(m)| \\ &< \frac{1}{n} + \int_{\overline{V}_\eta} |(f_\eta - g_{n,\eta})| < \frac{1}{n} + \frac{R+1}{n} = \frac{R+2}{n} \end{aligned}$$

As this holds for all  $n \in \mathcal{N}$ , using countable comprehension and overflow, see [1],  $(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$

□

**Remarks 0.8.** *If  $h \in C([-1, 1])$ , we define its Fourier coefficient by;*

$$\mathcal{F}(h)(m) = \int_{-1}^1 h(x) e^{-\pi i m x} dx$$

for  $m \in \mathcal{Z}$ .

As above, it is straightforward to deduce the standard Riemann-Lebesgue Lemma, in the form;

If  $f$  is Riemann integrable on  $[-1, 1]$  and bounded, then;

$$\lim_{|m| \rightarrow \infty} \mathcal{F}(f)(m) = 0$$

It is sufficient to show that, given  $\epsilon > 0$ , there exists  $M(\epsilon)$ , such that;

$$|\mathcal{F}(f)(m)| < \epsilon, \text{ for all } m \in \mathcal{Z}, m \geq M(\epsilon).$$

As, for all infinite  $m \in \mathcal{Z}_\eta$ ;

$$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$$

it follows by underflow, that, for all  $|m| \geq M(\epsilon)$ ,  $m \in \mathcal{Z}_\eta$ ;

$$|(\mathcal{F}_\eta(f_\eta))(m)| < \epsilon$$

The result then follows from the fact, that, for finite  $m \in \mathcal{Z}$ ;

$$\circ(\mathcal{F}_{f_\eta}(m)) = \mathcal{F}(f)(m)$$

as  $f_\eta \exp_\eta(-\pi i m x)$  is  $S$ -continuous and  $S$ -integrable on  $\overline{V_\eta}$ , see [4].

We now give a different, but more geometric proof of Lemma 0.5, but we require an extra assumption;

**Lemma 0.9.** *If  $f \in S(\mathcal{R})$ , with  $Re(f)$  and  $Im(f)$  analytic, with corresponding  $f_\eta \in V(\overline{V_\eta})$ ,  $\eta$  prime, then, there exists  $N \in \overline{\mathcal{R}_\eta}$  infinite,  $N = \frac{\sqrt{\eta}}{4}$ , such that for infinite  $y \in \overline{\mathcal{R}_\eta}$ , with  $|y| \leq N$ , we have;*

$$(\mathcal{F}_\eta(f_\eta))(y) \simeq 0$$

*Proof.* The first part is the same as Lemma 0.5, for  $n \in \mathcal{N}$ ;

$$\left| \int_{(|x| > \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta \exp_\eta(-2\pi i x y) d\mu_\eta(x) \right| \leq \frac{E}{(n-1)}, (*)$$

where  $E \in \mathcal{R}$ .

We claim that there exists  $N \in \overline{\mathcal{R}_\eta}$  infinite, such that for  $y$  infinite, with  $|y| \leq N$ , and  $n \in \mathcal{N}$ ;

$$\int_{\bar{V}_{\eta,n}} f_{\eta} \exp_{\eta}(-2\pi i y x) d\mu_{\eta}(x) \simeq 0, (*)$$

$$\text{where } \bar{V}_{\eta,n} = \left(-\frac{[n\sqrt{\eta}]}{\sqrt{\eta}}, \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}\right) \cap \bar{R}_{\eta}.$$

As in [5], we have that;

$$\begin{aligned} & \int_{\bar{V}_{\eta,n}} f_{\eta} \exp_{\eta}(-2\pi i y x) d\mu_{\eta}(x) \\ &= \int_{\bar{V}_{\eta,n}} \operatorname{Re}(f_{\eta}) \cos_{\eta}(2\pi x y) d\mu_{\eta}(x) \\ &+ i \int_{\bar{V}_{\eta,n}} \operatorname{Im}(f_{\eta}) \cos_{\eta}(2\pi x y) d\mu_{\eta}(x) \\ &- i \int_{\bar{V}_{\eta,n}} \operatorname{Re}(f_{\eta}) \sin_{\eta}(2\pi x y) d\mu_{\eta}(x) \\ &+ \int_{\bar{V}_{\eta,n}} \operatorname{Im}(f_{\eta}) \sin_{\eta}(2\pi x y) d\mu_{\eta}(x), (\#) \end{aligned}$$

It is sufficient to prove that  $\int_{\bar{V}_{\eta,n}} \operatorname{Re}(f_{\eta}) \cos_{\eta}(2\pi x y) d\mu_{\eta}(x) \simeq 0$ , the remaining cases are left to the reader;

We have that;

$$\begin{aligned} & \int_{\bar{V}_{\eta,n}} \operatorname{Re}(f_{\eta}) \cos_{\eta}(2\pi x y) d\mu_{\eta}(x) \\ &= \frac{1}{\sqrt{\eta}} * \sum_{|l| < [n\sqrt{\eta}]} \operatorname{Re}(f^*)\left(\frac{l}{\sqrt{\eta}}\right) \theta_k\left(\frac{l}{\sqrt{\eta}}\right), \text{ where } y = \frac{k}{\sqrt{\eta}} \\ &\text{where } \theta_k\left(\frac{l}{\sqrt{\eta}}\right) = \cos_{\eta}\left(\frac{2\pi l k}{\eta}\right), \theta_k(x) = \cos^*\left(\frac{2\pi k x}{\sqrt{\eta}}\right) \end{aligned}$$

We compute an upper bound, for given  $k \in {}^* \mathcal{Z}$ , with  $-\frac{(\eta-1)}{2} \leq k \leq \frac{(\eta-1)}{2}$   $n \in \mathcal{N}$ , of;

$$\frac{1}{\sqrt{\eta}} * \sum_{|l| < n\sqrt{\eta}} \operatorname{Re}(f^*)\left(\frac{l}{\sqrt{\eta}}\right) \theta_k\left(\frac{l}{\sqrt{\eta}}\right)$$

by transfer of the result for;

$$\frac{1}{m} * \sum_{|l| < nm} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right)$$

where  $m \in \mathcal{R}_{>0}$ ,  $r \in \mathcal{Z}$ ,  $-\frac{(m^2-1)}{2} \leq r \leq \frac{(m^2-1)}{2}$   $n \in \mathcal{N}$ , and  $\theta_r\left(\frac{l}{m}\right) = \cos_{m^2}\left(\frac{2\pi l r}{m^2}\right)$

For  $r \in \mathcal{Z}$ ,  $-\frac{(m^2-1)}{2} \leq r \leq \frac{(m^2-1)}{2}$ ,  $r \neq 0$ ,  $m \in \mathcal{R}_{>0}$ ,  $x \in \mathcal{R}$ , we let;

$$\theta_r(x) = \cos\left(\frac{2\pi r x}{m}\right)$$

For  $x \in \mathcal{R}$ , we have that;

$$\theta_r(x) = \cos\left(\frac{2\pi r x}{m}\right) = 0$$

$$\text{iff } \frac{2\pi r x}{m} = \frac{\pi}{2} + t\pi, (t \in \mathcal{Z})$$

$$\text{iff } x = \frac{\frac{\pi m}{2} + \pi t m}{2\pi r}$$

$$\text{iff } x = \left(\frac{1}{4r} + \frac{t}{2r}\right)m, (\#\#)$$

$$\{r, t\} \subset \mathcal{Z}, m \in \mathcal{R}_{>0}$$

With the assumption that  $|\frac{r}{m}| \leq \frac{m}{4}$ , we have that  $|r| \leq \frac{m^2}{4}$ ,  $\frac{1}{|r|} \geq \frac{4}{m^2}$ ,  $\frac{m}{2|r|} \geq \frac{2}{m} > \frac{1}{m}$ , where  $\frac{m}{2|r|} = z_2 - z_1$ ,  $\theta_r(z_1) = 0$  and  $z_2 = \mu z(z > z_1 : \theta_r(z_2) = 0)$ . It follows, by transfer, with  $m$  corresponding to  $\sqrt{\eta}$  and  $r$  corresponding to  $k$ , that for  $|\frac{k}{\sqrt{\eta}}| \leq \frac{\sqrt{\eta}}{4}$ ,  $\frac{\sqrt{\eta}}{2|k|} \geq \frac{2}{\sqrt{\eta}} > \frac{1}{\sqrt{\eta}}$ , where  $\frac{\sqrt{\eta}}{2|k|} = z_2 - z_1$ ,  $\theta_k(z_1) = 0$  and  $z_2 = \mu z(z > z_1 : \theta_k(z_2) = 0)$ . We let  $N = \frac{\sqrt{\eta}}{4} < \frac{\sqrt{\eta}}{2}$ . Suppose  $y$  is infinite, with  $|y| \leq N$  and  $y = \frac{k}{\sqrt{\eta}}$ . Using the above calculation, we have, by transfer, that;

$$\frac{\sqrt{\eta}}{2|k|} = \frac{\sqrt{\eta}}{2|y\sqrt{\eta}|} = \frac{1}{2|y|} = z_2 - z_1 \simeq 0$$

With the assumption that  $m^2$  is prime, we have that  $m^2(1 + 2t)$  is odd, so that, for  $r \neq 0$   $\frac{m^2(1+2t)}{4r} \notin \mathcal{Z}$ , so if  $\theta_r(x_0) = 0$ , then  $m x_0 \notin \mathcal{Z}$ , ( $\dagger$ ).

As in [5], using the assumption that  $Re(f)$  is analytic, we have that  $Re(f)$  has finitely many zeroes at  $\{x_1, \dots, x_{a(n)}\}$ , with  $-n \leq x_1 \leq \dots \leq x_i \leq \dots \leq x_{a(n)} \leq n$ , and finitely many maxima and minima,  $\{x_{i,1}, \dots, x_{i,j}, \dots, x_{i,b(i)}\}$ , with  $x_i \leq x_{i,1} \leq x_{i,j} \leq \dots \leq x_{i,b(i)} \leq x_{i+1}$ ,  $0 \leq i \leq a(n)$ ,  $1 \leq j \leq b(i) - 1$ . As in [5], let  $x_{i,0} = x_i$ , for  $0 \leq i \leq a(n) + 1$ , then it follows that  $Re(f)|_{[x_{i,j}, x_{i,j+1}]}$  is monotone for  $0 \leq i \leq a(n)$ ,  $0 \leq j \leq b(i) - 1$  and  $Re(f)|_{[x_{i,b(i)}, x_{i+1,0}]}$  is monotone for  $0 \leq i \leq a(n)$ .

Again, as in [5], we have that, for sufficiently large  $m$ , with  $m \notin \mathcal{Z}$ ,  $m x_i \notin \mathcal{Z}$ , for  $1 \leq i \leq a(n) - 1$  and  $m x_{i,j} \notin \mathcal{Z}$ , for  $1 \leq i \leq a(n) - 1$ ,  $1 \leq j \leq b(i) - 1$ , ( $\dagger\dagger$ ).

Let  $\{r_{ijs} : 1 \leq s \leq e(i, j)\}$  enumerate the zeroes of  $\theta_r$  on  $[x_{i,j}, x_{i,j+1}]$ ,  $r_{ij0} = x_{i,j}$ ,  $r_{ij(e(i,j)+1)} = x_{i,j+1}$ , for  $0 \leq i \leq a(n)$ ,  $0 \leq j \leq b(i) - 1$ , and let  $\{r_{is} : 1 \leq s \leq e(i)\}$  enumerate the zeroes of  $\theta_r$  on  $[x_{i,b(i)}, x_{i+1,0}]$ ,  $r_{i0} = x_{i,b(i)}$ ,  $r_{i(e(i)+1)} = x_{i+1,0}$ ,  $0 \leq i \leq a(n)$ . Then, using  $(\dagger)$ ,  $(\dagger\dagger)$ , we have that;

$$\begin{aligned} & \frac{1}{m} * \sum_{|l| < nm} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \\ &= \frac{1}{m} * \sum_{i=0}^{a(n)} * \sum_{j=0}^{b(i)-1} * \sum_{s=0}^{e(i,j)} * \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \\ &+ \frac{1}{m} * \sum_{i=0}^{a(n)} * \sum_{s=0}^{e(i)} * \sum_{l=[mr_{is}]+1}^{[mr_{i(s+1)}]-1} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \end{aligned}$$

We compute  $\frac{1}{m} |* \sum_{s=1}^{e(i,j)-1} * \sum_{l=[mx_{ij(s)}]+1}^{[mx_{ij(s+1)}]-1} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right)|$ .

$$\text{Let } \theta_{i,j}(s) = \frac{1}{m} * \sum_{l=[mr_{ijs}]+1}^{[mr_{ij(s+1)}]-1} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right)$$

Considering Case 1 in [5], we have that;

$$0 \leq |* \sum_{s=1}^{e(i,j)-1} \theta_{i,j}(s)| \leq l_{i,j}, (*)$$

where;

$$\begin{aligned} l_{i,j} &= \frac{1}{m} * \sum_{l=[mr_{ij1}]+1}^{[mr_{ij2}]-1} |\operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right)| \\ &\leq \frac{1}{m} * \sum_{l=[mr_{ij1}]+1}^{[mr_{ij2}]-1} D, \text{ where } |\operatorname{Re}(f)| \leq D, \text{ and } 0 \leq \theta_r|_{[r_{ij1}, r_{ij2}]} \leq 1 \\ &\leq \frac{D}{m} (([mr_{ij2}] - 1) - ([mr_{ij1}] + 1) + 1) \\ &= \frac{D}{m} ([mr_{ij2}] - [mr_{ij1}] - 1) \\ &\leq \frac{D}{m} ((mr_{ij2} + 1) - (mr_{ij1} - 1) - 1) \\ &= D((r_{ij2} - r_{ij1}) + \frac{1}{m}) \end{aligned}$$

so that, using  $(*)$ ,  $(\#\#)$ ;

$$\begin{aligned} 0 &\leq \frac{1}{m} |* \sum_{s=1}^{e(i,j)-1} * \sum_{l=[mx_{ij(s)}]+1}^{[mx_{ij(s+1)}]-1} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right)| \leq l_{i,j} \\ &\leq D((r_{ij2} - r_{ij1}) + \frac{1}{m}) \end{aligned}$$



$$= D\left(\frac{m}{2|r|} + \frac{1}{m}\right)$$

Similarly, for Case 2 in [5]. In the other cases, reversing the sequences, we obtain;

$$\begin{aligned} & \frac{1}{m} \left| \sum_{s=1}^{e(i,j)-1} \sum_{l=[mx_{ijs}]_{+1}}^{[mx_{ij(s+1)}]_{-1}} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \right| \\ & \leq D\left((r_{ije(i,j)} - r_{ij(e(i,j)-1)}) + \frac{1}{m}\right) \\ & = D\left(\frac{m}{2|r|} + \frac{1}{m}\right) \end{aligned}$$

Considering all 4 cases, we obtain the same bound;

$$\frac{1}{m} \left| \sum_{s=1}^{e(i)-1} \sum_{l=[mr_{is}]_{+1}}^{[mr_{i(s+1)}]_{-1}} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \right| \leq D\left(\frac{m}{2|r|} + \frac{1}{m}\right)$$

Similarly;

$$\begin{aligned} & \max(A_{i,j}, B_{i,j}, C_i, D_i) \\ & \leq D\left(\frac{m}{2|r|} + \frac{1}{m}\right) \end{aligned}$$

where;

$$\begin{aligned} A_{i,j} &= \frac{1}{m} \left| \sum_{l=[mr_{ij0}]_{+1}}^{[mr_{ij1}]_{-1}} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \right| \\ B_{i,j} &= \frac{1}{m} \left| \sum_{l=[mr_{ije(i,j)}]_{+1}}^{[mr_{ij(e(i,j)+1)}]_{-1}} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \right| \\ C_i &= \frac{1}{m} \left| \sum_{l=[mr_{i0}]_{+1}}^{[mr_{i1}]_{-1}} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \right| \\ D_i &= \frac{1}{m} \left| \sum_{l=[mr_{ie(i)}]_{+1}}^{[mr_{i(e(i)+1)}]_{-1}} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \right| \end{aligned}$$

As in [5], we have that;

$$\begin{aligned} & \frac{1}{m} \left| \sum_{|l| < nm} \operatorname{Re}(f)\left(\frac{l}{m}\right) \theta_r\left(\frac{l}{m}\right) \right| \\ & \leq (w(n) + a(n) + 1) 3D\left(\frac{m}{2|r|} + \frac{1}{m}\right) \end{aligned}$$

where  $w(n) = \operatorname{Card}(\operatorname{Re}(f)'|_{[-n,n]} = 0)$ ,  $a(n) = \operatorname{Card}(\operatorname{Re}(f)|_{[-n,n]} = 0)$

It follows, by transfer, with the assumption that  $y$  is infinite,  $y = \frac{k}{\sqrt{\eta}}$  and  $y \leq N$ , using ( $\#\#$ );

$$\begin{aligned} & \left| \int_{(-\frac{n\sqrt{\eta}}{\sqrt{\eta}}, \frac{n\sqrt{\eta}}{\sqrt{\eta}})} \operatorname{Re}(f_\eta) \cos_\eta(2\pi xy) d\mu_\eta(x) \right| \\ &= \frac{1}{\sqrt{\eta}} \left| \sum_{|l| < n\sqrt{\eta}} \operatorname{Re}(f^*)\left(\frac{l}{\sqrt{\eta}}\right) \theta_k\left(\frac{l}{\sqrt{\eta}}\right) \right| \\ &\leq (w(n) + a(n) + 1) 3D \left( \frac{\sqrt{\eta}}{2|k|} + \frac{1}{\sqrt{\eta}} \right) \simeq 0, \text{ as } w(n), a(n) \text{ are finite and} \\ &\frac{\sqrt{\eta}}{2|k|} = \frac{1}{2|y|} \simeq 0. \end{aligned}$$

and, similarly, as in [5], considering the additional terms from ( $\#$ ), we have that;

$$\begin{aligned} & \left| \int_{(|x| < \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta \exp_\eta(-2\pi ixy) d\mu_\eta(x) \right| \\ &\leq (w(n) + a(n) + w'(n) + a'(n) + 2) 6D \left( \frac{\sqrt{\eta}}{2|k|} + \frac{1}{\sqrt{\eta}} \right) \simeq 0 \end{aligned}$$

again, as  $w(n), a(n), w'(n), a'(n)$  are finite and  $\frac{\sqrt{\eta}}{2|k|} = \frac{1}{2|y|} \simeq 0$ .

It follows that, for infinite  $y \in \overline{\mathcal{R}_\eta}$ , with  $|y| \leq N$ , that;

$$\begin{aligned} & \left| \int_{\overline{\mathcal{R}_\eta}} f_\eta \exp_\eta(-2\pi ixy) d\mu_\eta(x) \right| \\ &\leq \left| \int_{(|x| < \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta \exp_\eta(-2\pi ixy) d\mu_\eta(x) \right| + \left| \int_{(|x| \geq \frac{[n\sqrt{\eta}]}{\sqrt{\eta}}) \cap \overline{\mathcal{R}_\eta}} f_\eta \exp_\eta(-2\pi ixy) d\mu_\eta(x) \right| \\ &\leq \delta + \frac{E}{n-1} < \frac{E+1}{n-1} \end{aligned}$$

where  $\delta \simeq 0$ . As this holds for all  $n \in \mathcal{N}$ , we obtain that;

$$\int_{\overline{\mathcal{R}_\eta}} f_\eta \exp_\eta(-2\pi ixy) d\mu_\eta(x) \simeq 0$$

□

**Remarks 0.10.** *It is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

*If  $f \in S(\mathcal{R})$ , with  $\operatorname{Re}(f)$  and  $\operatorname{Im}(f)$  analytic, then;*

$$\lim_{|y| \rightarrow \infty} \mathcal{F}(f)(y) = 0$$

*Just follow the proof in Remark 0.6, the addition restraint that  $|y| \leq N$  doesn't effect the application of underflow.*

**Lemma 0.11.** *If  $f \in C[-1, 1]$ , with  $Re(f)$  and  $Im(f)$  analytic, and corresponding  $f_\eta \in V(\overline{V}_\eta)$ ,  $\eta$  prime, then, there exists  $N \in \mathcal{Z}_\eta$  infinite,  $N = \eta - 1$ , such that for infinite  $m \in \mathcal{Z}_\eta$ , with  $|m| \leq N$ , we have;*

$$(\mathcal{F}_\eta(f_\eta))(m) \simeq 0$$

*Proof.* We claim that there exists  $N \in \mathcal{Z}_\eta$  infinite, such that for  $m$  infinite, with  $|m| \leq N$ ;

$$\int_{\overline{V}_\eta} f_\eta(x) \exp_\eta(-\pi i m x) d\mu_\eta(x) \simeq 0, (*)$$

As above, we have that;

$$\begin{aligned} & \int_{\overline{V}_\eta} f_\eta \exp_\eta(-\pi i m x) d\mu_\eta(x) \\ &= \int_{\overline{V}_\eta} Re(f_\eta) \cos_\eta(\pi m x) d\mu_\eta(x) \\ &+ i \int_{\overline{V}_\eta} Im(f_\eta) \cos_\eta(\pi m x) d\mu_\eta(x) \\ &- i \int_{\overline{V}_\eta} Re(f_\eta) \sin_\eta(\pi m x) d\mu_\eta(x) \\ &+ \int_{\overline{V}_\eta} Im(f_\eta) \sin_\eta(\pi m x) d\mu_\eta(x), (\#) \end{aligned}$$

It is sufficient to prove that  $\int_{\overline{V}_\eta} Re(f_\eta) \cos_\eta(\pi m x) d\mu_\eta(x) \simeq 0$ , the remaining cases are left to the reader;

As  $Re(f_\eta)$  and  $\cos^*(\pi m \frac{[x\eta]}{\eta})$  are bounded,  $\frac{1}{\eta} Re(f_\eta)(-1) \cos^*(-\pi m) \simeq 0$ , so we have that;

$$\begin{aligned} & \int_{\overline{V}_\eta} Re(f_\eta) \cos_\eta(\pi m x) d\mu_\eta(x) \\ & \simeq \frac{1}{\eta} * \sum_{|l| < \eta-1} Re(f^*) \left(\frac{l}{\eta}\right) \cos_\eta\left(\frac{\pi m l}{\eta}\right) \\ &= \frac{1}{\eta} * \sum_{|l| < \eta-1} Re(f^*) \left(\frac{l}{\eta}\right) \theta_m\left(\frac{l}{\eta}\right) \end{aligned}$$

where  $\theta_m\left(\frac{l}{\eta}\right) = \cos_\eta\left(\frac{\pi m l}{\eta}\right)$ ,  $\theta_m(x) = \cos^*(\pi m x)$

We compute an upper bound, given  $m \in {}^* \mathcal{Z}$ ,  $-\eta \leq m \leq \eta - 1$ , for;

$$\frac{1}{\eta} * \sum_{|l| < \eta-1} Re(f^*) \left(\frac{l}{\eta}\right) \theta_m\left(\frac{l}{\eta}\right)$$

by transfer of the result for;

$$\frac{1}{n} * \sum_{|l| < n-1} Re(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right)$$

where  $n \in \mathcal{N}_{>0}$  is prime,  $r \in \mathcal{Z}$ ,  $-n \leq r \leq n-1$ , and  $\theta_r\left(\frac{l}{n}\right) = \cos\left(\frac{\pi r l}{n}\right)$

For  $r \in \mathcal{Z}$ ,  $-n \leq r \leq n-1$ ,  $r \neq 0$ ,  $n$  prime,  $x \in [-1, 1]$ , we let;

$$\theta_r(x) = \cos(\pi r x)$$

For  $x \in [-1, 1]$ , we have that;

$$\theta_r(x) = \cos(\pi r x) = 0$$

$$\text{iff } \pi r x = \frac{\pi}{2} + t\pi, (t \in \mathcal{Z})$$

$$\text{iff } x = \frac{\frac{\pi}{2} + \pi t}{\pi r}$$

$$\text{iff } x = \left(\frac{1}{2r} + \frac{t}{r}\right), (\#\#)$$

$$\{r, t\} \subset \mathcal{Z}, -n \leq r \leq n-1, -r - \frac{1}{2} \leq t \leq r - \frac{1}{2}, \text{ for } r > 0, \\ r - \frac{1}{2} \leq t \leq -r - \frac{1}{2}, \text{ for } r < 0$$

With the assumption that  $|r| \leq n-1$ , we have that,  $\frac{1}{|r|} \geq \frac{1}{n-1} > \frac{1}{n}$ , where  $\frac{1}{|r|} = z_2 - z_1$ ,  $\theta_r(z_1) = 0$  and  $z_2 = \mu z(z > z_1 : \theta_r(z_2) = 0)$ . It follows, by transfer, with  $n$  corresponding to  $\eta$  and  $r$  corresponding to  $m$ , that for  $|m| \leq \eta-1$ ,  $\frac{1}{|m|} \geq \frac{1}{\eta-1} > \frac{1}{\eta}$ , where  $\frac{1}{|m|} = z_2 - z_1$ ,  $\theta_m(z_1) = 0$  and  $z_2 = \mu z(z > z_1 : \theta_m(z_2) = 0)$ . We let  $N = \eta - 1$ . Suppose  $m$  is infinite, with  $|m| \leq \eta - 1$ . Clearly, we have that;

$$\frac{1}{|m|} = z_2 - z_1 \simeq 0$$

With the assumption that  $n$  is prime, we have that  $n(1+2t)$  is odd, so that, for  $r \neq 0$   $\frac{n(1+2t)}{2r} \notin \mathcal{Z}$ , so if  $\theta_r(x_0) = 0$ , then  $n x_0 \notin \mathcal{Z}$ , ( $\dagger$ ).

As in Lemma 0.9, using the assumption that  $Re(f)$  is analytic, we have that  $Re(f)$  has finitely many zeroes at  $\{x_1, \dots, x_{a(1)}\}$ , with  $-1 \leq x_1 \leq \dots \leq x_i \leq \dots \leq x_{a(n)} \leq 1$ , and finitely many maxima and minima,  $\{x_{i,1}, \dots, x_{i,j}, \dots, x_{i,b(i)}\}$ , with  $x_i \leq x_{i,1} \leq x_{i,j} \leq \dots \leq x_{i,b(i)} \leq x_{i+1}$ ,  $0 \leq i \leq a(1)$ ,  $1 \leq j \leq b(i) - 1$ . As in Lemma 0.9, let  $x_{i,0} = x_i$ , for  $0 \leq i \leq a(1) + 1$ , then it follows that  $Re(f)|_{[x_{i,j}, x_{i,j+1}]}$  is monotone

for  $0 \leq i \leq a(1)$ ,  $0 \leq j \leq b(i) - 1$  and  $Re(f)|_{[x_i, b(i), x_{i+1}, 0]}$  is monotone for  $0 \leq i \leq a(1)$ .

If  $w \in \mathcal{Q}$ , then, for any  $d \in \mathcal{R} \setminus \mathcal{Q}$ , we have that  $w + d \in \mathcal{R} \setminus \mathcal{Q}$ . Given a finite set  $\{w_1, \dots, w_i, \dots, w_m\} \subset \mathcal{R} \setminus \mathcal{Q}$ , let  $U_i = \{u \in \mathcal{R} \setminus \mathcal{Q} : w_i + u \in \mathcal{R} \setminus \mathcal{Q}\}$ , then  $U_i$  is dense in  $\mathcal{R}$ , hence, the finite intersection  $\bigcap_{1 \leq i \leq m} U_i$  is dense in  $\mathcal{R}$ , in particular nonempty. It follows that there exists  $d \in \mathcal{R}$ , with  $x_i + d \in \mathcal{R} \setminus \mathcal{Q}$ ,  $x_{i,j} + d \in \mathcal{R} \setminus \mathcal{Q}$ , for  $1 \leq i \leq a(1) - 1$ ,  $1 \leq j \leq b(i) - 1$ . By considering  $g(x) = f(x - \epsilon)$ , for sufficiently small  $\epsilon$ , with  $\epsilon \in \mathcal{R} \setminus \mathcal{Q}$ , we can assume that  $\{x_1, \dots, x_{a(1)}\} \subset \mathcal{R} \setminus \mathcal{Q}$ , for  $1 \leq i \leq a(n) - 1$ , and  $\{x_{i,1}, \dots, x_{i,j}, \dots, x_{i,b(i)}\} \subset \mathcal{R} \setminus \mathcal{Q}$ , for  $1 \leq i \leq a(n) - 1$ ,  $1 \leq j \leq b(i) - 1$ . In particular, it follows that for  $n$  prime,  $nx_i \notin \mathcal{Z}$ , for  $1 \leq i \leq a(n) - 1$  and  $nx_{i,j} \notin \mathcal{Z}$ , for  $1 \leq i \leq a(n) - 1$ ,  $1 \leq j \leq b(i) - 1$ , ( $\dagger\dagger$ ).

Let  $\{r_{ijs} : 1 \leq s \leq e(i, j)\}$  enumerate the zeroes of  $\theta_r$  on  $[x_{i,j}, x_{i,j+1}]$ ,  $r_{ij0} = x_{i,j}$ ,  $r_{ij(e(i,j)+1)} = x_{i,j+1}$ , for  $0 \leq i \leq a(1)$ ,  $0 \leq j \leq b(i) - 1$ , and let  $\{r_{is} : 1 \leq s \leq e(i)\}$  enumerate the zeroes of  $\theta_r$  on  $[x_{i,b(i)}, x_{i+1,0}]$ ,  $r_{i0} = x_{i,b(i)}$ ,  $r_{i(e(i)+1)} = x_{i+1}$ ,  $0 \leq i \leq a(1)$ . Then, using ( $\dagger$ ), ( $\dagger\dagger$ ), we have that;

$$\begin{aligned} & \frac{1}{n} * \sum_{|l| < n-1} Re(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right) \\ &= \frac{1}{n} * \sum_{i=0}^{a(1)*} \sum_{j=0}^{b(i)-1*} \sum_{s=0}^{e(i,j)*} \sum_{l=[nr_{ijs}]+1}^{[nr_{ij(s+1)}]-1} Re(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right) \\ &+ \frac{1}{n} * \sum_{i=0}^{a(1)*} \sum_{s=0}^{e(i)*} \sum_{l=[nr_{is}]+1}^{[nr_{i(s+1)}]-1} Re(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right) \end{aligned}$$

We compute  $\frac{1}{n} |* \sum_{s=1}^{e(i,j)-1*} \sum_{l=[nx_{ijs}]+1}^{[nx_{ij(s+1)}]-1} Re(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right)|$ .

$$\text{Let } \theta_{i,j}(s) = \frac{1}{n} * \sum_{l=[nr_{ijs}]+1}^{[nr_{ij(s+1)}]-1} Re(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right)$$

Considering Case 1 as above, we have that;

$$0 \leq |* \sum_{s=1}^{e(i,j)-1*} \theta_{i,j}(s)| \leq l_{i,j}, \quad (*)$$

where;

$$\begin{aligned} l_{i,j} &= \frac{1}{n} * \sum_{l=[nr_{ij1}]}^{[nr_{ij2}]} |Re(f)|\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right) \\ &\leq \frac{1}{n} * \sum_{l=[nr_{ij1}]+1}^{[nr_{ij2}]-1} D, \text{ where } |Re(f)| \leq D, \text{ and } 0 \leq \theta_r|_{[r_{ij1}, r_{ij2}]} \leq 1 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{D}{n} (([nr_{ij2}] - 1) - ([nr_{ij1}] + 1) + 1) \\
&= \frac{D}{n} ([nr_{ij2}] - [nr_{ij1}] - 1) \\
&\leq \frac{D}{n} ((nr_{ij2} + 1) - (nr_{ij1} - 1) - 1) \\
&= D((r_{ij2} - r_{ij1}) + \frac{1}{n})
\end{aligned}$$

so that, using (\*), ( $\#\#$ );

$$\begin{aligned}
0 &\leq \frac{1}{n} |* \sum_{s=1}^{e(i,j)-1} \sum_{l=[nx_{ijs}]+1}^{[nx_{ij(s+1)}]-1} \operatorname{Re}(f)(\frac{l}{n}) \theta_r(\frac{l}{n})| \leq l_{i,j} \\
&\leq D((r_{ij2} - r_{ij1}) + \frac{1}{n}) \\
&= D(\frac{1}{|r|} + \frac{1}{n})
\end{aligned}$$

Similarly, for Case 2 in [5]. In the other cases, reversing the sequences, we obtain;

$$\begin{aligned}
&\frac{1}{n} |* \sum_{s=1}^{e(i,j)-1} \sum_{l=[nx_{ijs}]+1}^{[nx_{ij(s+1)}]-1} \operatorname{Re}(f)(\frac{l}{n}) \theta_r(\frac{l}{n})| \\
&\leq D((r_{ije(i,j)} - r_{ij(e(i,j)-1)}) + \frac{1}{n}) \\
&= D(\frac{1}{|r|} + \frac{1}{n})
\end{aligned}$$

Considering all 4 cases, we obtain the same bound;

$$\frac{1}{n} |* \sum_{s=1}^{e(i)-1} \sum_{l=[nr_{is}]+1}^{[nr_{i(s+1)}]-1} \operatorname{Re}(f)(\frac{l}{n}) \theta_r(\frac{l}{n})| \leq D(\frac{1}{|r|} + \frac{1}{n})$$

Similarly;

$$\begin{aligned}
&\max(A_{i,j}, B_{i,j}, C_i, D_i) \\
&\leq D(\frac{1}{|r|} + \frac{1}{n})
\end{aligned}$$

where;

$$\begin{aligned}
A_{i,j} &= \frac{1}{n} |* \sum_{l=[nr_{ij0}]+1}^{[nr_{ij1}]-1} \operatorname{Re}(f)(\frac{l}{n}) \theta_r(\frac{l}{n})| \\
B_{i,j} &= \frac{1}{n} |* \sum_{l=[nr_{ije(i,j)}]+1}^{[nr_{ij(e(i,j)+1)}]-1} \operatorname{Re}(f)(\frac{l}{n}) \theta_r(\frac{l}{n})|
\end{aligned}$$

$$C_i = \frac{1}{n} |^* \sum_{l=[nr_{i0}]+1}^{[nr_{i1}]-1} \operatorname{Re}(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right) |$$

$$D_i = \frac{1}{n} |^* \sum_{l=[nr_{ie(i)}]+1}^{[nr_{i(e(i)+1)}]-1} \operatorname{Re}(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right) |$$

As in [5], we have that;

$$\begin{aligned} & \frac{1}{n} |^* \sum_{|l| < n-1} \operatorname{Re}(f)\left(\frac{l}{n}\right) \theta_r\left(\frac{l}{n}\right) | \\ & \leq (w(1) + a(1) + 1) 3D\left(\frac{1}{|r|} + \frac{1}{n}\right) \end{aligned}$$

where  $w(1) = \operatorname{Card}(\operatorname{Re}(f)'|_{[-1,1]} = 0)$ ,  $a(1) = \operatorname{Card}(\operatorname{Re}(f)|_{[-1,1]} = 0)$

It follows, by transfer, with the assumption that  $m$  is infinite and  $m \leq N$ , using ( $\#\#$ );

$$\begin{aligned} & \left| \int_{\overline{V}_\eta} \operatorname{Re}(f_\eta) \cos_\eta(\pi m x) d\mu_\eta(x) \right| \\ & \simeq \frac{1}{\eta} |^* \sum_{|l| < \eta-1} \operatorname{Re}(f^*)\left(\frac{l}{\eta}\right) \theta_m\left(\frac{l}{\eta}\right) | \\ & \leq (w(1) + a(1) + 1) 3D\left(\frac{1}{|m|} + \frac{1}{\eta}\right) \simeq 0, \text{ as } w(1), a(1) \text{ are finite and} \\ & \frac{1}{|m|} \simeq 0, \frac{1}{\eta} \simeq 0. \end{aligned}$$

and, similarly, as in [5], considering the additional terms from ( $\#$ ), we have that;

$$\begin{aligned} & \left| \int_{\overline{V}_\eta} f_\eta \exp_\eta(-\pi i m x) d\mu_\eta(x) \right| \\ & \simeq \frac{1}{\eta} |^* \sum_{|l| < \eta-1} \operatorname{Re}(f^*)\left(\frac{l}{\eta}\right) \theta_m\left(\frac{l}{\eta}\right) | \\ & \leq (w(1) + a(1) + w'(1) + a'(1) + 2) 6D\left(\frac{1}{|r|} + \frac{1}{n}\right) \simeq 0 \end{aligned}$$

again, as  $w(1), a(1), w'(1), a'(1)$  are finite and  $\frac{1}{|r|} \simeq 0, \frac{1}{\eta} \simeq 0$

It follows that, for infinite  $m \in \mathcal{Z}_\eta$ , with  $|m| \leq N$ , that;

$$\int_{\overline{V}_\eta} f_\eta \exp_\eta(-\pi i m x) d\mu_\eta(x) \simeq 0$$

□

**Remarks 0.12.** *Again, it is straightforward to deduce the standard Riemann-Lebesgue Lemma;*

If  $f \in C[-1, 1]$ , with  $Re(f)$  and  $Im(f)$  analytic, then;

$$\lim_{|m| \rightarrow \infty} \mathcal{F}(f)(m) = 0$$

Just follow the proof in Remark 0.6, the addition restraint that  $|m| \leq N$  doesn't effect the application of underflow.

#### REFERENCES

- [1] Loeb Measures in Practice: Recent Advances, Nigel Cutland, Springer, (1999).
- [2] A Simple Proof of the Pointwise Convergence of Tent Series using Nonstandard Analysis, Tristram de Piro, ,(2019).
- [3] Advances in Nonstandard Analysis, Tristram de Piro, (2018).
- [4] A Simple Proof of the Inversion Theorem for Fourier Transforms using Nonstandard Analysis, Tristram de Piro, ,(2019).
- [5] Nonstandard Methods for Solving Schrodinger's Equation, Tristram de Piro, (2019).

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