COMPUTING THE PROBABILITY DISTRIBUTION OF VELOCITIES IN SOME SOLUTIONS TO THE NONSTANDARD DIFFUSION EQUATION

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ABSTRACT. We compute the distribution and stochastic equation for the reverse martingale found in [4].

One of the most fundamental results in the theory of Markov chains is the following;

Theorem 0.1. *Let P be the transition matrix of an irreducible, aperiodic,positive recurrent Markov chain, $\{X_n\}_{n\geq 0}$, with invariant distribution π. Then, for any initial distribution, $P(X_n = j) \rightarrow \pi_j$, as $n \rightarrow \infty$. In particular;*

$p_{ij}^{(n)} \rightarrow \pi_j$, *for all states* i, j, *as* $n \rightarrow \infty$

Proof. A good reference for this result is [2]. However, we give the proof as it is used and modified later. Let the initial distribution be λ, and let I be the state space. Choose $\{Y_n\}_{n\geq 0}$, such that $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ are independent, with $\{Y_n\}_{n\geq 0}$ Markov (π, P). Let $T = inf\{n \geq 1 : X_n = Y_n\}$. We claim that $P(T < \infty) = 1$, $(*)$. Let $W_n = (X_n, Y_n)$. Then $\{W_n\}_{n\geq 0}$ is a Markov chain on $I \times I$. By independence, it has transition probabilities given by;

$\overline{p}_{(i,j)(k,l)} = p_{ik}p_{jl}$ (†)

and initial distribution $\mu_{(i,j)} = \lambda_i \pi_j$. A simple calculation, using (†), shows that;

$\overline{p}^{(n)}_{(i,j)(k,l)} = p^{(n)}_{ik} p^{(n)}_{jl}$ for fixed states i, j, k, l

As P is irreducible and aperiodic, we have that $min(p_{ik}^{(n)}, p_{jl}^{(n)}) > 0$, for sufficiently large n. Hence, for such n, $\overline{p}_{(i,j)(k,l)}^{(n)} > 0$ and $\overline{P}$ is irreducible. A similar straightforward calculation gives that the distribution $\pi_{(i,j)} = \pi_i\pi_j$ is invariant for $\overline{P}$. By well known results, this implies that $\overline{P}$ is positive recurrent. Fix a state $b \in I$, and let $S = inf\{n \geq 1 : X_n = Y_n = b\}$. Then S is the first passage time in the system $\{W_n\}_{n\geq 0}$ to (b, b), and $P(S < \infty) = 1$ follows by known results, and the fact that $\overline{P}$ is irreducible and recurrent. Clearly $P(S < \infty) \leq P(T < \infty)$, so $(*)$ follows. We now calculate;

$$P(X_n = j) = P(X_n = j, n \geq T) + P(X_n = j, n < T)$$

$$= P(Y_n = j, n \geq T) + P(X_n = j, n < T)$$

by definition of T and the fact that $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ have the same transition matrix. Then;

$$P(X_n = j) = P(Y_n = j, n \geq T) + P(Y_n = j, n < T)$$

$$- P(Y_n = j, n < T) + P(X_n = j, n < T)$$

$$= P(Y_n = j) - P(Y_n = j, n < T) + P(X_n = j, n < T)$$

$$= \pi_j - P(Y_n = j, n < T) + P(X_n = j, n < T) \ (**)$$

We have that $P(Y_n = j, n < T) \leq P(n < T)$ and $P(n < T) \rightarrow P(T = \infty) = 0$ as $n \rightarrow \infty$, using $(*)$. Similarly, $P(X_n = j, n < T) \rightarrow 0$ as $n \rightarrow \infty$. It follows that $P(X_n = j) \rightarrow \pi_j$, using $(**)$, as required. The final claim is a consequence of the fact that $p_{ij}^{(n)} = P(X_n = j)$ where the initial distribution of X_0 is the dirac function δ_i.

□

We now establish a rate of convergence result.

Lemma 0.2. *Let P be the transition matrix for a finite irreducible aperiodic Markov chain. Then there exists $m \geq 1$ and $\rho \in (0, 1)$, such that;*

$$|p_{ij}^{(n)} - \pi_j| \leq (1 - \rho)^{\frac{n}{m} - 1}, \text{ for all states } i, j$$

where π is the limiting distribution guaranteed by Theorem 0.1.

Proof. From Theorem 0.1, taking the initial distribution of X_0 to be δ_i, we have that;

$$P(X_n = j) = \pi_j - P(Y_n = j, n < T) + P(X_n = j, n < T)$$

Hence;

$$|p_{ij}^{(n)} - \pi_j| \leq P(n < T)$$

As P is irreducible and aperiodic, we have that $p_{kl}^{(n)} > 0$ for all sufficiently large n, and all states k, l. As P is finite, there exists an $m \geq 1$ such that $p_{kl}^{(m)} > 0$ for all k, l. In particular, there exists $\rho \in (0,1)$ such that $p_{kl}^{(m)} \geq \rho$. We have that;

$$P_{k,l}(T \leq m) \geq \sum_u \overline{p}_{(k,l),(u,u)}^{(m)} = \sum_u p_{ku}^{(m)} p_{lu}^{(m)} \geq \rho \sum_u p_{ku}^{(m)} = \rho$$

$$P_{k,l}(T > m) \leq (1-\rho)$$

$$P(T > m) = \sum_{(k,l)} P_{(k,l)}(T > m)\delta_{ik}\pi_l \leq (1-\rho)$$

Moreover;

$$P(T > n) \leq P(T > [\tfrac{n}{m}]m)$$

We claim that, for $k \geq 1$, $P(T > (k+1)m | T > km) \leq 1 - \rho$. We have, using the total law of probability, the Markov property and the definition of T, that;

$$P(T > (k+1)m | T > km)$$

$$= \sum_{i_{km} \neq j_{km}, i_{km-1} \neq j_{km-1}, \ldots, i_0 \neq j_0} P(T > (k+1)m | W_{km} = (i_{km}, j_{km}), W_{km-1} = (i_{km-1}, j_{km-1}), \ldots, W_0 = (i_0, j_0)) P(W_{km} = (i_{km}, j_{km}), W_{km-1} = (i_{km-1}, j_{km-1}), \ldots, W_0 = (i_0, j_0) | T > km)$$

$$= \sum_{i_{km} \neq j_{km}, i_{km-1} \neq j_{km-1}, \ldots, i_0 \neq j_0} P(T > (k+1)m | W_{km} = (i_{km}, j_{km})) P(W_{km} = (i_{km}, j_{km}), W_{km-1} = (i_{km-1}, j_{km-1}), \ldots, W_0 = (i_0, j_0) | T > km)$$

$$\leq (1-\rho) \sum_{i_{km} \neq j_{km}, i_{km-1} \neq j_{km-1}, \ldots, i_0 \neq j_0} P(W_{km} = (i_{km}, j_{km}), W_{km-1} = (i_{km-1}, j_{km-1}), \ldots, W_0 = (i_0, j_0) | T > km)$$

$= (1-\rho)$

Inductively, we have that;

$P(T > km) = P(T > km, T > (k-1)m)$

$= P(T > km | T > (k-1)m)P(T > (k-1)m)$

$\leq P(T > km | T > (k-1)m)(1-\rho)^{k-1}$

$\leq (1-\rho)^k$

It follows that $|p_{ij}^{(n)} - \pi_j| \leq (1-\rho)^{[\frac{n}{m}]} \leq (1-\rho)^{\frac{n}{m}-1}$, as required.

□

Lemma 0.3. *Let P define a Markov chain with N states, $\{0, 1, \ldots, N-1\}$, where N is odd, such that the transition probabilities of moving from state i to i-2,i+2 (mod N) respectively are $\frac{1}{2}$. Then P is irreducible, aperiodic and π, defined by $\pi_i = \frac{1}{N}$, for $0 \leq i \leq N-1$, defines an invariant distribution. Moreover, we can choose $m = 2N$ and $\rho = \frac{1}{4^N}$ in Lemma 0.2. It follows that;*

$|p_{ij}^{(n)} - \frac{1}{N}| \leq (\frac{4^N-1}{4^N})^{\frac{n}{2N}-1} = \epsilon_n$

Moreover, for any initial probability distribution λ_0, letting $\lambda_j^n = P(X_n = j)$, we have that;

$|\lambda_j^n - \frac{1}{N}| \leq \epsilon_n$, $0 \leq j \leq N-1$ $(*)$

For any initial distribution $\mu_0 = \mu_0^+ - \mu_0^-$, with sums K^+ and K^-, letting $\mu^n = \mu_0 P^n$, and $K = K^+ - K^-$ we have that;

$|\mu_j^n - \frac{K}{N}| \leq (K^+ + K^-)\epsilon_n$, $0 \leq j \leq N-1$

Proof. To prove irreducibility, observe that $i + 2(\frac{N+1}{2}) = i+1$ (mod N), hence, $p_{i,i+1}^{(\frac{N+1}{2})} \geq \frac{1}{2^{\frac{N+1}{2}}}$, for all states $0 \leq i \leq N-1$, $(*)$. To show that all states i, j communicate, it is sufficient, by symmetry, to assume that $i \leq j$. If $i = j$, then we have that $p_{i,i}^{(2)} \geq \frac{1}{4}$. If $j - i$ is even, we have that

$p_{i,j}^{(\frac{j-i}{2})} \geq \frac{1}{2^{\frac{j-i}{2}}}$. If $j-i$ is odd, then $j-(i+1)$ is even. We then have that;

$$p_{i,j}^{\frac{N+1}{2}+\frac{j-(i+1)}{2}} \geq \frac{1}{2^{\frac{N+1}{2}}}\frac{1}{2^{\frac{j-(i+1)}{2}}}$$

using $(*)$. To prove aperiodicity, it is sufficient to show that $p_{ii}^{(n)} > 0$, for sufficiently large n, for any given state i with $0 \leq i \leq N-1$. Observe that $i + 2N = i \pmod N$, hence $p_{ii}^{(N)} \geq \frac{1}{2^N}$. If $n \geq N$, and n is even, then clearly $p_{ii}^{(n)} \geq \frac{1}{2^n}$. If $n \geq N$ and n is odd, then $n-N$ is even, and $p_{ii}^{(n)} \geq \frac{1}{2^N}\frac{1}{2^{n-N}} = \frac{1}{2^n}$. For the invariance claim, we compute;

$$(\pi P)_i = \sum \pi_j P_{ji} = \frac{1}{N}(P_{i-2,i} + P_{i+2,i}) = \frac{1}{N}(\frac{1}{2}+\frac{1}{2}) = \frac{1}{N}$$

To find m and ρ, observe that, by the aperiodicity calculation, that $p_{ii}^{(n)} \geq \frac{1}{2^n}$, for any $0 \leq i \leq N-1$, and $n \geq N$. Observe also that, starting at a given state i, we can cover all the states, by moving in one direction, a total of N steps. It follows that $p_{ij}^{(k)} \geq \frac{1}{2^k}$, for *some* $k \leq N$. Choosing some $1 \leq k_{ij} \leq N$ for each pair of states (i,j), observe that $2N - k_{ij} \geq N$, therefore, for any states (i,j);

$$p_{ij}^{(2N)} \geq \frac{1}{2^{k_{ij}}}\frac{1}{2^{2N-k_{ij}}} = \frac{1}{4^N}$$

We can, therefore, take $m = 2N$ and $\rho = \frac{1}{4^N}$. We then have that;

$$(1-\rho)^{\frac{n}{m}-1} = (\frac{4^N-1}{4^N})^{\frac{n}{2N}-1}$$

and the following claim follows, by Lemma 0.2. The penultimate claim follows by noting that $\lambda_n = \lambda_0 P^n$ and calculating;

$$\lambda_j^n = \lambda_0^0 p_{0j}^{(n)} + \lambda_1^0 p_{1j}^{(n)} + \ldots + \lambda_{N-1}^0 p_{N-1,j}^{(n)}$$

$$= (\lambda_0^0 + \ldots + \lambda_{N-1}^0)(\frac{1}{N}) + \lambda_0^0\epsilon_n^0 + \ldots + \lambda_{N-1}^0\epsilon_n^{N-1}$$

$$= \frac{1}{N} + \epsilon_n'$$

where $\epsilon_n^j \leq \epsilon_n$, for $0 \leq j \leq N-1$ and $\epsilon_n' \leq \epsilon_n$. The final claim follows by observing that;

$$\mu^n = \mu_0 P^n = \mu_0^+ P^n - \mu_0^- P^n = K^+\pi_0^+ P^n - K^-\pi_0^- P^n \ (**)$$

where $\{\pi_0^+,\pi_0^-\}$ are distributions. We then have, using the previous result, and multiplying by an appropriate constant, that;

$|(K^+\pi_0^+P^n)_j-\frac{K^+}{N}|\leq K^+\epsilon_n$

$|(K^-\pi_0^-P^n)_j-\frac{K^-}{N}|\leq K^-\epsilon_n$

for $0\leq j\leq N-1$. Therefore, combining this with $(**)$, we obtain that;

$|\mu_j^n-\frac{K}{N}|=|\mu_j^n-(\frac{K^+-K^-}{N})|\leq (K^++K^-)\epsilon_n$

as required.

□

Lemma 0.4. *Let P define a non standard Markov chain with η states, $\{0,1,\ldots,\eta-1\}$, for η odd infinite, such that the transition probabilities of moving from state i to i-2,i+2 (mod η) respectively are $\frac{1}{2}$. Then, if ϵ is an infinitesimal and*

$n\geq 2\eta(1+\frac{log(\epsilon)}{log(4^\eta-1)-log(4^\eta)})$ $(*)$

we have for any initial probability distribution π_0, that;

$\pi_j^n\simeq\frac{1}{\eta}$ *for* $0\leq j\leq \eta-1$ $(**)$

If $\mu_0=\mu_0^+-\mu_0^-$ is a nonstandard distribution with sums $\{K^+,K^-\}$, possibly infinite, then if $K=K^+-K^-$, and ϵ is an infinitesimal with $(K^++K^-)\epsilon\simeq 0$, and n satisfies $(*)$, we obtain that;

$\mu_j^n\simeq\frac{K}{\eta}$ *for* $0\leq j\leq \eta-1$ $(**)$

Proof. Let $Seq_1=\{f:\mathcal{N}\rightarrow\mathcal{R}\}$ and $Seq_2=\{f:\mathcal{N}^2\rightarrow\mathcal{R}\}$. We let;

$Prob_N=\{f\in Seq_1:(\forall_{m\geq N}f(m)=0)\wedge(\forall_{0\leq m\leq N-1}f(m)\geq 0)\wedge\sum_{0\leq m\leq N-1}f(m)=1\}$.

encode probability vectors of length N. Let $G:\mathcal{N}\rightarrow Seq_2$ be defined by;

$G(N,0,2)=G(N,0,N-2)=\frac{1}{2}$, $(\forall_{m\neq 2,N-2}G(N,0,m)=0$

$(\forall_{0\leq k\leq N-2}\forall_{1\leq m\leq N-1}(G(N,k+1,0)=G(N,k,N-1), G(N,k+1,m)=G(N,k,m-1))$

$\forall_{k\geq N}\forall_{m\geq N}G(N,k,m)=0$

G encodes the transition matrices for the given Markov chain with N states. Let $H:\mathcal{N}^2\rightarrow Seq_2$ be defined by;

$(\forall_{0\leq i,j\leq N-1})H(1,N,i,j)=G(N,i,j)$

$(\forall_{i,j\geq N})H(1,N,i,j)=0$

$(\forall_{0\leq i,j\leq N-1}\forall_{n\geq 2})H(n,N,i,j)=\sum_{0\leq k\leq N-1}H(n-1,N,i,k)G(N,k,j)$

$(\forall_{i,j>N}\forall_{n\geq 2})H(n,N,i,j)=0$

H encodes the powers $G(N)^n$ of the transition matrices. We define maps $L(N,n):Prob_N\rightarrow Prob_N$ by;

$(\forall_{0\leq j\leq N-1})L(N,n)(f)(j)=\sum_{0\leq k\leq N-1}f(k)H(n,N,k,j)$

$(\forall_{j\geq N})L(N,n)(f)(j)=0$

$L(N,n)(f)$ encodes the probability vectors π^n for an initial distribution π_0 represented by f.

By a simple rearrangement, we have that the bound in $|\pi_j^n-\frac{1}{N}|$, from Lemma 0.3, can be formulated in first order logic as;

$\forall N\in\mathcal{N}_{odd}\forall\pi\in Prob_N\forall\epsilon\in\mathcal{R}_{>0}\forall n\in\mathcal{N}(n\geq 2N(1+\frac{log(\epsilon)}{log(4^N-1)-log(4^N)})\rightarrow (|L(n,N)(\pi)(j)-\frac{1}{N}|\leq\epsilon, 0\leq j\leq N-1)$

By transfer, we obtain a corresponding result, quantifying over ${}^*\mathcal{N}$. Taking ϵ to be an infinitesimal and η to be an infinite odd natural number, we obtain the first result. Observe that by construction of G,H,L, the nonstandard Markov chain with η states evolves by the usual nonstandard matrix multiplication by the transition matrix, of the initial probability distribution. The remaining claim is similar and left to the reader.

□

Definition 0.5. *Let $\eta \in {}^{*}\mathcal{N} \setminus \mathcal{N}$, infinite and odd, and let $\nu = \frac{\eta^2}{2}$, $\nu \in {}^{*}\mathcal{Q}_{\geq 0} \setminus \mathcal{Q}$. We let;*

$$\overline{\Omega_\eta} = \{x \in {}^{*}\mathcal{R} : 0 \leq x < 1\},\ \overline{\mathcal{T}_\nu} = \{t \in {}^{*}\mathcal{R}_{\geq 0}\}$$

We let $\mathcal{C}_\eta$ consist of internal unions of the intervals $[\frac{i}{\eta}, \frac{i+1}{\eta})$, for $0 \leq i \leq \eta - 1$, and let $\mathcal{D}_\nu$ consist of internal unions of $[\frac{i}{\nu}, \frac{i+1}{\nu})$, for $i \in {}^{*}\mathcal{Z}_{\geq 0}$.

We define counting measures μ_η and λ_ν on $\mathcal{C}_\eta$ and $\mathcal{D}_\nu$ respectively, by setting $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$, $\lambda_\nu([\frac{i}{\nu}, \frac{i+1}{\nu})) = \frac{1}{\nu}$, for $0 \leq i \leq \eta - 1$, $i \in {}^{*}\mathcal{Z}_{\geq 0}$ respectively.

We let $(\overline{\Omega_\eta}, \mathcal{C}_\eta, \mu_\eta)$ and $(\overline{\mathcal{T}}_\nu, \mathcal{D}_\nu, \lambda_\nu)$ be the resulting measure spaces, in the sense of [1]. We let $(\overline{\Omega_\eta} \times \overline{\mathcal{T}}_\nu, \mathcal{C}_\eta \times \mathcal{D}_\nu, \mu_\eta \times \lambda_\nu)$ denote the corresponding product space.

If $f \in V(\overline{\Omega_\eta} \times \overline{\mathcal{T}}_\nu)$ is measurable, we define;

$$\frac{\partial f}{\partial t}(\frac{i}{\eta}, \frac{j}{\nu}) = \nu(f(\frac{i}{\eta}, \frac{j+1}{\nu}) - f(\frac{i}{\eta}, \frac{j}{\nu})),\ \frac{\partial f}{\partial t}(x, s) = \frac{\partial f}{\partial t}(\frac{[\eta x]}{\eta}, \frac{[\nu s]}{\nu})$$

$$\frac{\partial f}{\partial x}(\frac{i}{\eta}, \frac{j}{\nu}) = \frac{\eta}{2}(f(\frac{i+1}{\eta}, \frac{j}{\nu}) - f(\frac{i-1}{\eta}, \frac{j}{\nu})),\ \frac{\partial f}{\partial x}(y, t) = \frac{\partial f}{\partial x}(\frac{[\eta y]}{\eta}, \frac{[\nu t]}{\nu})$$

where we adopt the usual convention of taking i mod η.

Definition 0.6. *Let $f : \overline{\Omega_\eta} \rightarrow {}^{*}\mathcal{R}$ be measurable with respect to the $*\sigma$-algebra $\mathcal{C}_\eta$, in the sense of [1]. We define $F : \overline{\Omega_\eta} \times \overline{\mathcal{T}_\nu} \rightarrow {}^{*}\mathcal{R}_{\geq 0}$ by;*

$$F(\frac{i}{\eta}, \frac{j}{\nu}) = (\pi_f K^j)(i),\ \text{for } 0 \leq i \leq \eta - 1,\ j \in {}^{*}\mathcal{Z}_{\geq 0}$$

$$F(x, t) = F(\frac{[\eta x]}{\eta}, \frac{[\nu t]}{\nu}),\ (x, t) \in \overline{\Omega_\eta} \times \overline{\mathcal{T}_\nu}$$

where π_f is the nonstandard distribution vector corresponding to f, K is the transition matrix of the above Markov chain with η states, and K^j denotes a nonstandard power.

Theorem 0.7. *Let F be as defined in Definition 0.16, then F is measurable with respect to $\mathcal{C}_\eta \times \mathcal{D}_\nu$, and, moreover F is the unique solution to the nonstandard heat equation;*

$$\frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial x^2} = 0$$

with initial condition f. If f is bounded, then for $\tau \geq \frac{16(4^{\eta})log(\eta)}{\eta}$, we have that $F_{\tau} \simeq C$, where $C = \int_{\overline{\Omega}_{\eta}} f d\mu_{\eta}$.

Proof. The first proposition follows by observing that the defining schema for F is internal and by hyperfinite induction, see Lemma 0.4 for the mechanics of this transfer process. For the second proposition, it is a simple computation, using the definition of the partial derivatives in Definition 0.5, to see that, if F satisfies the nonstandard heat equation, then;

$$F(\tfrac{i}{\eta}, \tfrac{j+1}{\nu}) = \tfrac{\eta^2}{4\nu} F(\tfrac{i+2}{\eta}, \tfrac{j}{\nu}) + (1 - \tfrac{\eta^2}{2\nu}) F(\tfrac{i}{\eta}, \tfrac{j}{\nu}) + \tfrac{\eta^2}{4\nu} F(\tfrac{i-2}{\eta}, \tfrac{j}{\nu}),\ j \in {}^*\mathcal{Z}_{\geq 0}$$

In particular, F is uniquely determined from the initial condition f and taking $\eta^2 = 2\nu$, we obtain that;

$$F(\tfrac{i}{\eta}, \tfrac{j+1}{\nu}) = \tfrac{1}{2} F(\tfrac{i+2}{\eta}, \tfrac{j}{\nu}) + \tfrac{1}{2} F(\tfrac{i-2}{\eta}, \tfrac{j}{\nu}),\ j \in {}^*\mathcal{Z}_{\geq 0}$$

which agrees with the defining schema for F in Definition 0.16. For the last claim, by definition of the nonstandard integral, see [?], and the assumptions on f, we have that $f = f^+ - f^-$, with corresponding sums $\{K^+, K^-, K\}$, where $\frac{(K^+ + K^-)}{\eta^2} \simeq 0$, and $\int_{\overline{\Omega}_{\eta}} f d\mu_{\eta} = \frac{K}{\eta}$. By Lemma 0.4, we have, taking $\epsilon = \frac{1}{\eta^2}$, that for;

$$n \geq 2\eta(1 - \tfrac{log(\eta^2)}{log(4^{\eta-1}) - log(4^{\eta})})$$

$F_{\frac{n}{\nu}} \simeq \int_{\overline{\Omega}_{\eta}} f d\mu_{\eta}$. Then, we compute;

$$2\eta(1 - \tfrac{log(\eta^2)}{log(4^{\eta-1}) - log(4^{\eta})})$$

$$\leq 4\eta(\tfrac{log(\eta)}{log(4^{\eta}) - log(4^{\eta} - 1)}) + 1$$

$$= 4\eta(\tfrac{log(\eta)}{log(1 + \frac{1}{4^{\eta}-1})}) + 1$$

$$\leq 8\eta(4^{\eta} - 1)log(\eta) \text{ as } log(1+x) \geq \tfrac{x}{2}, \text{ for } x \simeq 0$$

It follows that, for $\frac{n}{\nu} \geq \frac{16}{\eta}(4^{\eta} - 1)log(\eta)$, $F_{\frac{n}{\nu}} \simeq \int_{\overline{\Omega}_{\eta}} f d\mu_{\eta}$, therefore, if $\tau \geq \frac{16}{\eta}(4^{\eta})log(\eta)$, $F_{\tau} \simeq \int_{\overline{\Omega}_{\eta}} f d\mu_{\eta}$, as required. □

Lemma 0.8. *Let F satisfy the nonstandard heat equation;*

$\frac{\partial F}{\partial t}-\frac{\partial^2 F}{\partial x^2}=0$, $(*)$

with initial condition $f\in V(\overline{\Omega}_\eta)$, *then* $\{\frac{\partial F}{\partial x},\frac{\partial F^{lsh}}{\partial x},\frac{\partial F^{rsh}}{\partial x},F^{lsh^2},F^{rsh^2}\}$ *satisfy the same equation, with initial conditions* $\{\frac{\partial f}{\partial x},\frac{\partial f^{lsh}}{\partial x},\frac{\partial f^{rsh}}{\partial x},f^{lsh^2},f^{rsh^2}\}$

Proof. If $(*)$ is satisfied for F, then;

$\frac{\partial^2 F}{\partial x\partial t}-\frac{\partial^3 F}{\partial x^3}=0$

As the nonstandard derivatives $\{\frac{\partial}{\partial x},\frac{\partial}{\partial t}\}$ commute;

$\frac{\partial^2 F}{\partial t\partial x}=\frac{\partial^2 F}{\partial x\partial t}=\frac{\partial^3 F}{\partial t\partial x^3}$ $(**)$

hence, $(*)$ is satisfied for $\frac{\partial F}{\partial x}$ with initial condition $\frac{\partial f}{\partial x}$. Applying lsh to $(**)$, and using the fact that lsh commutes with $\{\frac{\partial}{\partial x},\frac{\partial}{\partial t}\}$ we obtain;

$(\frac{\partial^2 F}{\partial t\partial x})^{lsh}=\frac{\partial^3 F}{\partial t\partial x^3}^{lsh}=(\frac{\partial^2 F^{lsh}}{\partial t\partial x})=\frac{\partial^3 F^{lsh}}{\partial t\partial x^3}$

The case for rsh is similar. Clearly, the corresponding initial conditions $\{\frac{\partial f^{lsh}}{\partial x},\frac{\partial f^{rsh}}{\partial x}\}$ are satisfied. A similar argument works for the operators $\{lsh^2,rsh^2\}$

□

Remarks 0.9. *We can give an alternative description of the process given in Theorem 0.7. Namely, we can think of it as the density of a collection of particles , moving independently and at random. For sufficiently large* t*, the density, which we refer to as the equilibrium density, is close to being constant. This idea is made precise in the following.*

Definition 0.10. *We keep the notation of Definition 0.5. We let* $\nu=\frac{\eta^2}{2}$ *but we drop the restriction that* η *is odd. We let;*

$\overline{\Omega}_\kappa=\{(s_i):1\leq i\leq\kappa, s_i=1 \text{ or } -1\}$

so that ${}^*Card(\overline{\Omega}_\kappa)=2^\kappa$. *We let;*

$\omega_i:\overline{\Omega}_\kappa\rightarrow\{1,-1\}$, *for* $1\leq i\leq\kappa$, *be defined by;*

$\omega_i(s)=s_i$

We let;

$\overline{\mathcal{T}_{\nu,\kappa}} = \{t \in \overline{\mathcal{T}_\nu} : 0 \leq [\nu t] \leq \kappa\}$

We let $\chi : \overline{\Omega}_\kappa \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow \overline{\Omega}_\eta$, *be defined by;*

$\chi(s,t) = \frac{1}{\eta}({}^*\sum_{j=1}^{[\nu t]} \omega_j(s))\ mod[0,1),\ 1 \leq [\nu t] \leq \kappa$

$\chi(s,0) = 0$

We let $\overline{\chi} : \overline{\Omega_\eta} \times \overline{\Omega_\kappa} \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow \overline{\Omega_\eta}$ *be defined by;*

$\overline{\chi}(x,s,t) = x + 2\chi(s,t)\ mod[0,1)$

Given an initial condition $f \in V(\overline{\Omega_\eta})$, *with* $f \geq 0$, *we let;*

$N_f : \overline{\Omega_\eta} \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow {}^*\mathcal{R}_{\geq 0}$ *be defined by;*

$N_f(x,t) = {}^*\sum_{0 \leq i \leq \eta-1} \frac{f(\frac{i}{\eta})}{2^\kappa} {}^*Card(\{s \in \overline{\Omega}_\kappa : \overline{\chi}(\frac{i}{\eta}, s, t) = \frac{[\eta x]}{\eta}\})$

Lemma 0.11. *Let* $f \in V(\overline{\Omega}_\eta)$, $f \geq 0$ *be an initial condition, for the heat equation in Theorem 0.7 or the Markov chain in Definition 0.16, then* N_f *as given in Definition 0.10 is exactly the process* F *given by Theorem 0.7.*

Proof. This follows easily by hyperfinite induction. As both N_f and F are measurable on $\overline{\Omega}_\eta \times \overline{\mathcal{T}_{\nu,\kappa}}$, it is sufficient to check the two claims that;

$N_f(x,0) = f(x)$

$N_f(x, \frac{j+1}{\nu}) = \frac{1}{2} N_f(x + \frac{2}{\eta}, \frac{j}{\nu}) + \frac{1}{2} N_f(x - \frac{2}{\eta}, \frac{j}{\nu})$

for $0 \leq j \leq \kappa - 1$, $x \in \overline{\Omega_\eta}$. For the first claim, observe that, if $[\eta x] = i$, then $\overline{\chi}(\frac{i}{\eta}, s, 0) = \frac{[\eta x]}{\eta}$ for all $s \in \overline{\Omega}_\kappa$, and if $[\eta x] \neq i$, then $\overline{\chi}(\frac{i}{\eta}, s, 0) = \frac{[\eta x]}{\eta}$ for no $s \in \overline{\Omega}_\kappa$, by definition of $\overline{\chi}$. Hence, a simple computation of $N_f(x,0)$ gives the result. For the second claim, just observe that, for $0 \leq i \leq \eta - 1$, $0 \leq j \leq \kappa - 1$;

${}^*Card(s \in \overline{\Omega_\kappa} : \overline{\chi}(\frac{i}{\eta}, s, \frac{j+1}{\nu}) = \frac{[\eta x]}{\eta})$

$= \frac{1}{2}{}^*Card(s \in \overline{\Omega_\kappa} : \overline{\chi}(\frac{i}{\eta}, s, \frac{j}{\nu}) = \frac{[\eta x]+2}{\eta}) + \frac{1}{2}{}^*Card(s \in \overline{\Omega_\kappa} : \overline{\chi}(\frac{i}{\eta}, s, \frac{j}{\nu}) = \frac{[\eta x]-2}{\eta})$

The second claim then follows by linearity and the definition of N_f.

□

Definition 0.12. *Let $(\overline{\Omega_\eta}, \mathcal{E}_\eta, \gamma_\eta)$ be a nonstandard $*$-finite measure space. We define a reverse filtration on $\overline{\Omega_\eta}$ to be an internal collection of $*\sigma$-algebras $\mathcal{E}_{\eta,i}$, indexed by $0 \leq i \leq \kappa$, $\kappa \in {}^*\mathcal{N} \setminus \mathcal{N}$, such that;*

(i). $\mathcal{E}_{\eta,0} = \mathcal{E}_\eta$

(ii). $\mathcal{E}_{\eta,i} \subseteq \mathcal{E}_{\eta,j}$, *if* $0 \leq j \leq i \leq \kappa$.

We say that $\overline{F} : \overline{\Omega_\eta} \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow {}^*\mathcal{R}$ is adapted to the filtration if $\overline{F}$ is measurable with respect to $\mathcal{E}_\eta \times \mathcal{D}_\nu$ and $\overline{F}_{\frac{i}{\nu}} : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{R}$ is measurable with respect to $\mathcal{E}_{\eta,i}$, for $0 \leq i \leq \kappa$.

If $f : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{R}$ is measurable with respect to $\mathcal{E}_{\eta,j}$ and $0 \leq j \leq i \leq \kappa$, we define the conditional expectation $E_\eta(f|\mathcal{E}_{\eta,i})$ to be the unique $g : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{R}$ such that g is measurable with respect to $\mathcal{E}_{\eta,i}$ and;

$\int_U g d\gamma_\eta = \int_U f d\gamma_\eta$

for all $U \in \mathcal{E}_{\eta,i}$. We say that $\overline{F} : \overline{\Omega_\eta} \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow {}^*\mathcal{R}$ is a reverse martingale if;

(i). $\overline{F}$ *is adapted to the reverse filtration on* $\overline{\Omega_\eta}$

(ii). $E_\eta(\overline{F}_{\frac{j}{\nu}}|\mathcal{E}_{\eta,i}) = \overline{F}_{\frac{i}{\nu}}$ *for* $0 \leq j \leq i \leq \kappa$

Given $F : \overline{\Omega_\eta} \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow {}^*\mathcal{R}$ measurable with respect to $\mathcal{E}_\eta \times \mathcal{D}_\nu$, we define the cumulative density function; $P : {}^*\mathcal{R} \times [0,1] \rightarrow {}^*\mathcal{R}$ by;

$P(x,t) = \gamma_\eta(F_t \leq x)$

We say that F_1 and F_2 are equivalent in distribution, if their respective cumulative density functions P_1 and P_2 coincide.

Theorem 0.13. *Let F be as in Definition 0.16, without the restriction that η is odd, but keeping $\nu = \frac{\eta^2}{2}$, and let F_κ be its restriction to $\overline{\Omega_\eta} \times \overline{\mathcal{T}_{\nu,\kappa}}$. Then there exists a reverse filtration on $\overline{\Omega_\eta}$ and $\overline{F}_\kappa$ such that $\overline{F}_\kappa$*

is a reverse martingale, and $\overline{F}_{\kappa,\frac{\kappa}{\nu}}=F_{\frac{\kappa}{\nu}}$. *If* G *is the process defined by* $G_t=F_{\frac{\kappa-[\nu t]}{\nu}}$, *for* $0\leq t\leq\frac{\kappa}{\nu}$, $t\in\overline{\mathcal{T}_{\nu,\kappa}}$, *then the processes* G *and* $\overline{F}$ *are equivalent in distribution.*

Proof. We define the reverse filtration, by setting $\mathcal{E}_{\eta,i}$ to be internal unions of the intervals $[\frac{j}{2^{\kappa-i}\eta},\frac{j+1}{2^{\kappa-i}\eta})$ for $0\leq j\leq 2^{\kappa-i}\eta-1$, $0\leq i\leq\kappa$. Clearly, this is an internal collection. It follows that $\mathcal{E}_\eta=\mathcal{E}_{\eta,0}$ consists of internal unions of the intervals $[\frac{j}{2^\kappa\eta},\frac{j+1}{2^\kappa\eta})$ for $0\leq j\leq 2^\kappa\eta-1$, and we define the corresponding measure γ_η by setting $\gamma_\eta([\frac{j}{2^\kappa\eta},\frac{j+1}{2^\kappa\eta}))=\frac{1}{2^\kappa\eta}$. Observe that $\mathcal{E}_{\eta,\kappa}=\mathcal{C}_\eta$, the original $*\sigma$-algebra.

We define bijections $\Phi_i:{}^*\mathcal{N}_{0\leq j\leq\eta-1}\times\overline{\Omega}_{\kappa-i}\rightarrow{}^*\mathcal{N}_{0\leq j\leq 2^{\kappa-i}\eta-1}$, for $0\leq i\leq\kappa$, where $\overline{\Omega}_{\kappa-i}=\{(\omega_k):\omega_k=1\ or-1,1\leq k\leq\kappa-i\}$, by;

$$\Phi_i(j,\omega)=2^{\kappa-i}j+2^{\kappa-i*}\sum_{1\leq k\leq\kappa-i}\frac{\omega_k+1}{2^{k+1}}$$

Define $\overline{F}_\kappa$ by;

$$\overline{F}_\kappa(\frac{r}{2^{\kappa-i}\eta},\frac{i}{\nu})=F_{\frac{i}{\nu}}(\frac{j}{\eta}+\frac{2}{\eta}{}^*\sum_{1\leq k\leq\kappa-i}\omega_k)$$

where $\Phi_i(j,\omega)=r$, for $0\leq r\leq 2^{\kappa-i}\eta-1$, $0\leq i\leq\kappa$.

$$\overline{F}_\kappa(x,t)=\overline{F}_\kappa(\frac{[2^{\kappa-[\nu t]}\eta x]}{2^{\kappa-[\nu t]}\eta},\frac{[\nu t]}{\nu}),\ (x,t)\in\overline{\Omega}_\eta\times\overline{\mathcal{T}_{\nu,\kappa}}$$

It is clear that $\overline{F}_\kappa$ is adapted to the reverse filtration on $\overline{\Omega}_\eta$. Moreover, it is straightforward to see that;

$$\overline{F}_\kappa(\frac{r}{\eta},\frac{\kappa}{\nu})=F_{\frac{\kappa}{\nu}}(\frac{r}{\eta})$$

as $\Phi_\kappa(r)=r$, so $\overline{F}_{\kappa,\frac{\kappa}{\nu}}=F_{\frac{\kappa}{\nu}}$. We claim that $\overline{F}_\kappa$ is a reverse martingale. We have verified condition (i) in Definition 0.12. To verify (ii), by the tower law for conditional expectation, it is sufficient to prove that $E_\eta(\overline{F}_{\kappa,\frac{i}{\nu}}|\mathcal{E}_{i+1})=\overline{F}_{\kappa,\frac{i+1}{\nu}}$, for $0\leq i\leq\kappa-1$. We have that;

$$E_\eta(\overline{F}_{\kappa,\frac{i}{\nu}}|\mathcal{E}_{i+1})(\frac{r}{2^{\kappa-i-1}\eta})$$

$$=2^{\kappa-i-1}\eta\int_{[\frac{r}{2^{\kappa-i-1}\eta},\frac{r+1}{2^{\kappa-i-1}\eta})}E_\eta(\overline{F}_{\kappa,\frac{i}{\nu}}|\mathcal{E}_{i+1})d\gamma_\eta$$

$$=2^{\kappa-i-1}\eta\int_{[\frac{r}{2^{\kappa-i-1}\eta},\frac{r+1}{2^{\kappa-i-1}\eta})}\overline{F}_{\kappa,\frac{i}{\nu}}d\gamma_\eta$$

$= \frac{2^{\kappa-i-1}\eta}{2^{\kappa-i}\eta}(\sum_{m=0}^{1}\overline{F}_{\kappa,\frac{i}{\nu}}(\frac{2r+m}{2^{\kappa-i}\eta}))$

$= \frac{1}{2}(F_{\frac{i}{\nu}}(\frac{j}{\eta}+\frac{2}{\eta}({}^{*}\sum_{1\leq k\leq \kappa-i-1}\omega_k-1))+F_{\frac{i}{\eta}}(\frac{j}{\eta}+\frac{2}{\eta}({}^{*}\sum_{1\leq k\leq \kappa-i-1}\omega_k+1)))$

$= \frac{1}{2}(F_{\frac{i}{\nu}}(x-\frac{2}{\eta})+F_{\frac{i}{\nu}}(x+\frac{2}{\eta}))$

$= F_{\frac{i+1}{\nu}}(x) = \overline{F}_{\kappa,\frac{i+1}{\nu}}(\frac{r}{2^{\kappa-i-1}\eta})$

where $\Phi_{i+1}(j,\omega)=r$, $\omega=(\omega_k)_{1\leq k\leq \kappa-i-1}$ and $x=\frac{j}{\eta}+\frac{2}{\eta}({}^{*}\sum_{1\leq k\leq \kappa-i-1}\omega_k)$, as required.

We claim that;

$2^{\kappa-[\nu t]}{}^{*}Card(\{i: 0\leq i\leq \eta-1, F(\frac{i}{\eta},\frac{[\nu t]}{\nu})=\alpha\})$

$= {}^{*}Card(\{i: 0\leq i\leq 2^{\kappa-[\nu t]}\eta-1, \overline{F}(\frac{i}{2^{\kappa-[\nu t]}\eta-1},\frac{\kappa-[\nu t]}{\nu})=\alpha\})$ $(*)$

We prove that, for $i\in[0,\eta-1]\cap{}^{*}\mathcal{Z}$, then;

${}^{*}Card(\{(j,\omega)\in[0,\eta-1]\cap{}^{*}\mathcal{Z}\times\Omega_{\kappa-[\nu t]} : \frac{j}{\eta}+\frac{2}{\eta}{}^{*}\sum_{1\leq k\leq \kappa-[\nu t]}\omega_k = \frac{i}{\eta}\}) = 2^{\kappa-[\nu t]}$ $(**)$

We can prove $(**)$ by induction. The proof of the base clear when $t=\frac{\kappa-1}{\nu}$ is included in the inductive step. Suppose the claim is true at $\frac{[\nu t]}{\nu}$, and let $i_0\in[0,\eta-1]\cap{}^{*}\mathcal{Z}$, then, we have that;

${}^{*}Card(\{(j,\omega)\in[0,\eta-1]\cap{}^{*}\mathcal{Z}\times\Omega_{\kappa-[\nu t]+1} : \frac{j}{\eta}+\frac{2}{\eta}{}^{*}\sum_{1\leq k\leq \kappa-[\nu t]+1}\omega_k = \frac{i_0}{\eta}\})$

$= {}^{*}Card(\{(j,\omega)\in[0,\eta-1]\cap{}^{*}\mathcal{Z}\times\Omega_{\kappa-[\nu t]+1} : \frac{j}{\eta}+\frac{2}{\eta}{}^{*}\sum_{1\leq k\leq \kappa-[\nu t]}\omega_k = \frac{i_0+2}{\eta}\})$

$+{}^{*}Card(\{(j,\omega)\in[0,\eta-1]\cap{}^{*}\mathcal{Z}\times\Omega_{\kappa-[\nu t]+1} : \frac{j}{\eta}+\frac{2}{\eta}{}^{*}\sum_{1\leq k\leq \kappa-[\nu t]}\omega_k = \frac{i_0-2}{\eta}\})$

$= 2.2^{\kappa-[\nu t]} = 2^{\kappa-[\nu t]+1}$

which proves the claim $(**)$ at $\frac{[\nu t]-1}{\nu}$.

We then have, by the definition of $\overline{F}$, letting $Z_\alpha = \{i : 0 \leq i \leq \eta - 1, F(\frac{i}{\eta}, \frac{[\nu t]}{\nu}) = \alpha\}$, that;

$= {}^*Card(\{i : 0 \leq i \leq 2^{\kappa-[\nu t]}\eta - 1, \overline{F}(\frac{i}{2^{\kappa-[\nu t]}\eta-1}, \frac{\kappa-[\nu t]}{\nu}) = \alpha\})$

$= {}^*\sum_{i \in Z_\alpha} {}^*Card(\{(j,\omega) \in [0, \eta-1] \cap {}^*\mathcal{Z} \times \Omega_{\kappa-[\nu t]}$

$: \frac{j}{\eta} + \frac{2}{\eta} {}^* \sum_{1 \leq k \leq \kappa-[\nu t]} \omega_k = \frac{i}{\eta}\})$

$= {}^*\sum_{i \in Z_\alpha} 2^{\kappa-[\nu t]}$, by $(**)$

$= 2^{\kappa-[\nu t]*}Card(\{i : 0 \leq i \leq \eta - 1, F(\frac{i}{\eta}, \frac{[\nu t]}{\nu}) = \alpha\})$

giving the claim $(*)$. It follows, from $(*)$, that;

$\gamma_\eta(G_s = \alpha)$

$= \gamma_\eta(F_{\frac{\kappa-[s\nu]}{\nu}} = \alpha)$

$= \frac{1}{\eta} {}^*Card(\{i : 0 \leq i \leq \eta - 1, F(\frac{i}{\eta}, \frac{\kappa-[\nu s]}{\nu}) = \alpha\})$

$= \frac{1}{2^{[\nu s]}\eta} {}^*Card(\{i : 0 \leq i \leq 2^{[\nu s]}\eta - 1, \overline{F}(\frac{i}{2^{[\nu s]}\eta-1}, \frac{[\nu s]}{\nu}) = \alpha\})$

$= \gamma_\eta(\overline{F}_{\frac{[\nu s]}{\nu}} = \alpha)$

using the measurability of $G_{\frac{[\nu s]}{\nu}} = F_{\frac{\kappa-[\nu s]}{\nu}}$, and $\overline{F}_{\frac{[\nu s]}{\nu}}$ with respect to the algebras $\mathcal{C}_\eta$ and $\mathcal{E}_{\eta,\nu-[\nu t]}$ respectively. Hence, G and $\overline{F}$ are equivalent in distribution. □

Lemma 0.14. *Let $\overline{F}_\kappa$ be the reverse martingale defined in 0.13, then we have that $\overline{F}_\kappa$ satisfies the nonstandard stochastic equation;*

$d(\overline{F}_\kappa)_t = (\overline{D}_\kappa)_t dW_t$

where;

$d(\overline{F}_\kappa)_t(\frac{j}{\eta}, \omega) = (\overline{F}_\kappa)_{t+\frac{1}{\nu}}(\frac{j}{\eta}, \omega) - (\overline{F}_\kappa)(\frac{j}{\eta}, \omega)$

$W_t(\frac{j}{\eta}, \omega) = \frac{1}{\sqrt{\kappa}} {}^* \sum_{k=1}^{[\nu t]} \omega_k$

for $\omega \in \Omega_\kappa$, $0 \leq t \leq \frac{\kappa-1}{\nu}$

$(\overline{D}_\kappa)_t(\frac{j}{\eta},\omega)=\frac{\sqrt{\kappa}}{\sqrt{2\nu}}(F^{lsh}_{\frac{s-1}{\nu}}+F^{rsh}_{\frac{s-1}{\nu}})'(\frac{j}{\eta}+\frac{2}{\eta}({}^*\sum_{k=1}^{[t\nu]}\omega_k))$

where $s=\kappa-[t\nu]$.

Proof. By the nonstandard martingale representation theorem, see [5], letting $t=\frac{i}{\nu}$, $s=\frac{\kappa-i}{\nu}$, and the result of Theorem 0.13, we have that;

$d(\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega)=C_{\frac{i}{\nu}}(\overline{\omega})\omega_{i+1}(\overline{\omega},\omega)$

where $\omega=\omega^{up}_{i+1}$ or $\omega=\omega^{down}_{i+1}$, $\overline{\omega}\in\Omega_i$.

By the martingale condition, we have;

$\frac{1}{2}((\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega^{up}_{i+1})+(\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega^{down}_{i+1}))=(\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega})$

Rearranging, we obtain;

$(\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega^{up}_{i+1})-(\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega})$

$=(\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega})-(\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega^{down}_{i+1})$

$=-((\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega^{down}_{i+1})-(\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega}))$, $(*)$

Let;

$C_{\frac{i}{\nu}}(\overline{\omega})=((\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega^{up}_{i+1})-((\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega})$

Then;

$C_{\frac{i}{\nu}}(\overline{\omega},\omega^{up}_{i+1})\omega_{i+1}(\overline{\omega},\omega^{up}_{i+1})$

$=C_{\frac{i}{\nu}}(\overline{\omega})\omega_{i+1}(\overline{\omega},\omega^{up}_{i+1})$

$C_{\frac{i}{\nu}}(\overline{\omega})$

$=((\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega^{up}_{i+1})-((\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega})$

$C_{\frac{i}{\nu}}(\overline{\omega},\omega^{down}_{i+1})\omega_{i+1}(\overline{\omega},\omega^{down}_{i+1})$

$=C_{\frac{i}{\nu}}(\overline{\omega})\omega_{i+1}(\overline{\omega},\omega^{down}_{i+1})$

$-C_{\frac{i}{\nu}}(\overline{\omega})$

$= -((\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega_{i+1}^{up}) - ((\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega})$

$= ((\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega_{i+1}^{down}) - (\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega}))$

using $(*)$ in the last step. We compute $C_{\frac{i}{\nu}}$. Using the definition of $\overline{F}_\kappa$ in Theorem 0.13, the definition of C above, and the defining schema for the heat equation in Theorem 0.7, we have;

$C_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega})$

$= ((\overline{F}_\kappa)_{\frac{i+1}{\nu}}(\frac{j}{\eta},\overline{\omega},\omega_{i+1}^{up}) - ((\overline{F}_\kappa)_{\frac{i}{\nu}}(\frac{j}{\eta},\overline{\omega})$

$= F_{\frac{s-1}{\nu}}(a_{i,j}+\frac{2}{\eta}) - F_{\frac{s}{\nu}}(a_{i,j})$

$= F_{\frac{s-1}{\nu}}(a_{i,j}+\frac{2}{\eta}) - \frac{1}{2}(F_{\frac{s-1}{\nu}}(a_{i,j}+\frac{2}{\eta}) + F_{\frac{s-1}{\nu}}(a_{i,j}) - \frac{2}{\eta})$

$= \frac{1}{2}F_{\frac{s-1}{\nu}}(a_{i,j}+\frac{2}{\eta}) - \frac{1}{2}F_{\frac{s-1}{\nu}}(a_{i,j}-\frac{2}{\eta})$

$= \frac{1}{2}F_{\frac{s-1}{\nu}}(a_{i,j}+\frac{2}{\eta}) - \frac{1}{2}F_{\frac{s-1}{\nu}}(a_{i,j}) + \frac{1}{2}F_{\frac{s-1}{\nu}}(a_{i,j}) - \frac{1}{2}F_{\frac{s-1}{\nu}}(a_{i,j}-\frac{2}{\eta})$

$= \frac{1}{\eta}\frac{\eta}{2}[F_{\frac{s-1}{\nu}}(a_{i,j}+\frac{2}{\eta}) - F_{\frac{s-1}{\nu}}(a_{i,j}) + F_{\frac{s-1}{\nu}}(a_{i,j}) - F_{\frac{s-1}{\nu}}(a_{i,j}-\frac{2}{\eta})]$

$= \frac{1}{\eta}(F'_{\frac{s-1}{\nu}}(a_{i,j}+\frac{1}{\eta}) + F'_{\frac{s-1}{\nu}}((a_{i,j}-\frac{1}{\eta})))$

$= \frac{1}{\eta}(F^{lsh}_{\frac{s-1}{\nu}} + F^{rsh}_{\frac{s-1}{\nu}})'(a_{i,j})$

$= \frac{1}{\sqrt{2\nu}}(F^{lsh}_{\frac{s-1}{\nu}} + F^{rsh}_{\frac{s-1}{\nu}})'(a_{i,j})$

where $a_{i,j} = \frac{j}{\eta} + \frac{2}{\eta}(^*\sum_{k=1}^{i}\omega_k)$, $(\text{mod } \eta)$

This gives the result.

□

Lemma 0.15. *The process $\overline{D}_\kappa$ is also a reverse martingale with respect to the filtration on $\overline{\Omega_\eta} \times \overline{T_{\nu,\kappa}}$.*

Proof. Just use the result of Lemma 0.8 and follow the proof of Theorem 0.13 with $\frac{\partial(F^{lsh}+F^{rsh})}{\partial x}$ replacing F.

□

Lemma 0.16. *If η is an infinite prime, $\{4,\eta\}$ are coprime, and $4|\eta+1$, there exists a nontrivial solution f to the differential equation;*

$(f^{lsh}+f^{rsh})'=\frac{\lambda}{2}(f^{lsh^2}+f^{rsh^2})$

when $\lambda\neq 0$, with $f'\neq 0$, for $\lambda=\eta(\frac{1-{}^*cos(\frac{2\pi}{\eta})}{1+{}^*cos(\frac{2\pi}{\eta})})$

with $f\frac{i}{\eta}=w^{\frac{(\eta+1)i}{4}}$, for $0\leq i\leq\eta-1$, and $w={}^*exp(\frac{2\pi i}{\eta})$
and any other such solution is of the form ef with $e\in{}^*\mathcal{C}$

Proof. For $0\leq i\leq\eta-1$, η and 4 coprime, we must have that;

$(f^{lsh}+f^{rsh})'(\frac{i}{\eta})$

$=\frac{\eta}{2}(f^{lsh}(\frac{i+1}{\eta})-f^{lsh}(\frac{i-1}{\eta}))+\frac{\eta}{2}(f^{rsh}(\frac{i+1}{\eta})-f^{rsh}(\frac{i-1}{\eta}))$

$=\frac{\eta}{2}(f(\frac{i+2}{\eta})-f(\frac{i}{\eta})+f(\frac{i}{\eta})-f(\frac{i-2}{\eta}))$

$=\frac{\eta}{2}(f(\frac{i+2}{\eta})-f(\frac{i-2}{\eta}))$, (mod η))

and;

$\frac{\lambda}{2}(f^{lsh^2}+f^{rsh^2})(\frac{i}{\eta})$
$=\frac{\lambda}{2}(f(\frac{i+2}{\eta})+f(\frac{i-2}{\eta}))$ (mod η)

Rearranging gives that;

$(f(\frac{i+2}{\eta})-f(\frac{i-2}{\eta}))$

$=\frac{\lambda}{\eta}(f(\frac{i+2}{\eta})+f(\frac{i-2}{\eta}))$

$(1-\frac{\lambda}{\eta})f(\frac{i+2}{\eta})=(1+\frac{\lambda}{\eta})f(\frac{i-2}{\eta})$

$f(\frac{i+2}{\eta})=wf(\frac{i-2}{\eta})$, where $w=\frac{1+\frac{\lambda}{\eta}}{1-\frac{\lambda}{\eta}}$, (†)

As η and 4 are coprime, we must have that;

$f(\frac{i}{\eta})=w^{\eta}f(\frac{i}{\eta})$, for $0\leq i\leq\eta-1$ $(*)$

As f is non-trivial, we must have that $w^{\eta} = 1$. As η is prime, we must have that $w = {}^*exp(\frac{2\pi i}{\eta})$. Rearranging, we obtain that;

$$\frac{1+\frac{\lambda}{\eta}}{1-\frac{\lambda}{\eta}} = {}^*exp(\frac{2\pi i}{\eta})$$

$$\frac{\lambda}{\eta}(1 + {}^*exp(\frac{2\pi i}{\eta})) = ({}^*exp(\frac{2\pi i}{\eta}) - 1)$$

$$\lambda = \eta\frac{({}^*exp(\frac{2\pi i}{\eta})-1)}{(1+{}^*exp(\frac{2\pi i}{\eta}))}$$

$$\eta\frac{({}^*exp(\frac{2\pi i}{\eta})-1)({}^*exp(\frac{-2\pi i}{\eta})-1)}{|1+{}^*exp(\frac{2\pi i}{\eta})|^2}$$

$$= \eta\frac{1-{}^*cos(\frac{2\pi}{\eta})}{1+{}^*cos(\frac{2\pi}{\eta})}$$

By (†), and using the fact that $4|(\eta + 1)$, we have that;

$f(\frac{i+1}{\eta}) = w^{\frac{(\eta+1)}{4}} f(\frac{i}{\eta})$, for $0 \leq i \leq \eta - 1$, $(**)$

Using $(**)$, we have that;

$f' = 0$ iff $\omega^{\frac{(i+1)(\eta+1)}{4}} = \omega^{\frac{(i-1)(\eta+1)}{4}}$

iff $\omega^{\frac{2(\eta+1)}{4}} = 1$

iff $\omega^{\frac{(\eta+1)}{2}} = 1$

iff $\omega = 1$ which is not the case.

□

Lemma 0.17. *Let $f_\lambda \in V(\overline{\Omega}_\eta)$, be as in Lemma 0.16, and let F_λ be the unique solution to the heat equation;*

$$\frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial x^2} = 0$$

with initial condition f_λ, then;

$$\frac{\partial(F_\lambda^{lsh}+F_\lambda^{rsh})}{\partial x} = \frac{\lambda}{2}(F_\lambda^{lsh^2} + F_\lambda^{rsh^2})$$

Proof. By Lemma 0.8 $\{\frac{\partial F_\lambda}{\partial x}, \frac{\partial F_\lambda^{lsh}}{\partial x}, \frac{\partial F_\lambda^{rsh}}{\partial x}, F_\lambda^{lsh^2}, F_\lambda^{rsh^2}\}$ satisfy the same equation, with initial conditions $\{\frac{\partial f_\lambda}{\partial x}, \frac{\partial f_\lambda^{lsh}}{\partial x}, \frac{\partial f_\lambda^{rsh}}{\partial x}, f_\lambda^{lsh^2}, f_\lambda^{rsh^2}\}$

It follows that;

$\frac{\partial(F_\lambda^{lsh}+F_\lambda^{rsh})}{\partial x}-\frac{\lambda}{2}(F_\lambda^{lsh^2}+F_\lambda^{rsh^2})$

satisfies the heat equation with initial condition;

$\frac{\partial(f_\lambda^{lsh}+f_\lambda^{rsh})}{\partial x}-\frac{\lambda}{2}(f_\lambda^{lsh^2}+f_\lambda^{rsh^2})=0$

By the uniqueness of the solution, given an initial condition, we have that;

$\frac{\partial(F_\lambda^{lsh}+F_\lambda^{rsh})}{\partial x}-\frac{\lambda}{2}(F_\lambda^{lsh^2}+F_\lambda^{rsh^2})=0$

□

Lemma 0.18. *Let $F_\lambda \in V(\overline{\Omega}_\eta \times \overline{\Omega}_{\kappa,\nu})$, be as in Lemma 0.17, and let F_λ be the unique solution to the heat equation;*

with initial condition f_λ, then;

$\frac{\partial(F_\lambda^{lsh}+F_\lambda^{rsh})}{\partial x}(x,t)=\lambda(F_\lambda(x,t+\frac{1}{\nu}))$

Proof. By the defining schema in Lemma 0.7, we have that;

$\frac{1}{2}(F_\lambda^{lsh^2}+F_\lambda^{rsh^2})(x,t)=(F_\lambda(x,t+\frac{1}{\nu}))$

for a solution to the nonstandard heat equation;

$\frac{\partial F}{\partial t}-\frac{\partial^2 F}{\partial x^2}=0$

The result then follows from Lemma 0.17.

□

Lemma 0.19. *Let $\overline{F}_{\lambda,\kappa}$ be the reverse martingale defined in Theorem 0.13, with F_λ as in Lemma 0.18, then we have that $\overline{F}_{\kappa,\lambda}$ satisfies the nonstandard stochastic equation;*

$d(\overline{F}_{\kappa,\lambda})_t=\lambda(\overline{F}_{\kappa,\lambda})_t dW_t$

Proof. Using Lemmas 0.14 and 0.18, we have that;

$(\overline{D}_{\kappa,\lambda})_t(\frac{j}{\eta},\omega)$

$= \frac{\sqrt{\kappa}}{\sqrt{2\nu}}((F_\lambda)^{lsh}_{\frac{s-1}{\nu}} + (F_{lambda})^{rsh}_{\frac{s-1}{\nu}})'(\frac{j}{\eta} + \frac{2}{\eta}({}^*\sum_{k=1}^{[t\nu]}\omega_k))$

$= \frac{\sqrt{\kappa}}{\sqrt{2\nu}}\lambda(F_\lambda)_{\frac{s}{\nu}}(\frac{j}{\eta} + \frac{2}{\eta}({}^*\sum_{k=1}^{[t\nu]}\omega_k))$

$= \frac{\sqrt{\kappa}}{\sqrt{2\nu}}\lambda(\overline{F}_{\kappa,\lambda})_t(\frac{j}{\eta},\omega)$

The result then follows from Lemma 0.14 again. □

Lemma 0.20. *Let $\overline{F}_{\lambda,\kappa}$ be the reverse martingale considered in Lemma 0.19, and let;*

$P_\lambda(t,x) = P(Re[(\overline{F}_{\lambda,\kappa})_t]) \leq x$

$Q_\lambda(t,x) = P(Re[F_{\lambda,\frac{\kappa-t}{\nu}}]) \leq x$, *then;*

for $0 \leq t \leq 1$. *Then;*

$P_\lambda(t,x) = Q_\lambda(t,x) =?$

Proof. Let $\overline{G}_{\kappa,\lambda} = {}^*log(\overline{F}_{\kappa,\lambda})$, then;

$d(\overline{G}_{\kappa,\lambda})_t$

$= (\overline{G}_{\kappa,\lambda})_{t+\frac{1}{\nu}} - (\overline{G}_{\kappa,\lambda})_t$

$= {}^*log(\overline{F}_{\kappa,\lambda})_{t+\frac{1}{\nu}} - {}^*log(\overline{F}_{\kappa,\lambda})_t$

$= {}^*log((\overline{F}_{\kappa,\lambda})_t + d(\overline{F}_{\kappa,\lambda})_t) - {}^*log(\overline{F}_{\kappa,\lambda})_t$

$= {}^*log((\overline{F}_{\kappa,\lambda})_t(1 + \frac{d(\overline{F}_{\kappa,\lambda})_t}{(\overline{F}_{\kappa,\lambda})_t}) - {}^*log(\overline{F}_{\kappa,\lambda})_t$

$= {}^*log(1 + \frac{d(\overline{F}_{\kappa,\lambda})_t}{(\overline{F}_{\kappa,\lambda})_t})$

$= {}^*log(1 + \frac{\lambda(\overline{F}_{\kappa,\lambda})_t dW_t}{(\overline{F}_{\kappa,\lambda})_t})$

$= {}^*log(1 + \lambda dW_t)$

Letting $\kappa = m\nu$, with $m \in {}^*\mathcal{N}$, so that, with $\nu = \frac{\eta^2}{2}$;

$\sqrt{\kappa} = \sqrt{m\nu} = \sqrt{\frac{m\eta^2}{2}} = \frac{\sqrt{m}\eta}{\sqrt{2}}$

We have that;

$|\lambda dW_t| = |\frac{\lambda}{\sqrt{\kappa}}\omega_{\frac{[\nu t]+1}{\nu}}| \leq \frac{2\eta}{\sqrt{\kappa}} = \frac{2\sqrt{2}}{\sqrt{m}} < 1$

if $m > 2\sqrt{2}$. With this condition, we have that;

${}^*log(1+\lambda dW_t) = {}^*\sum_{n\in{}^*\mathcal{N}}(-1)^{n+1}\lambda^n\frac{(dW_t)^n}{n}$

If $m \geq \eta^2 > 2\sqrt{2}$, we have that;

$|{}^*\sum_{n\in{}^*\mathcal{N},n\geq 3}(-1)^{n+1}\lambda^n\frac{(dW_t)^n}{n}|$

$\leq {}^*\sum_{n\in{}^*\mathcal{N},n\geq 3}\frac{|\lambda|^n|dW_t|^n}{n}$

$\leq {}^*\sum_{n\in{}^*\mathcal{N},n\geq 3}\frac{2\eta)^n}{\kappa^{\frac{n}{2}}n}$

$= {}^*\sum_{n\in{}^*\mathcal{N},n\geq 3}\frac{(\sqrt{2}2\eta)^n}{\sqrt{m}\eta^n n}$

$\leq {}^*\sum_{n\in{}^*\mathcal{N},n\geq 3}\frac{(\sqrt{2}2)^n}{\eta^n n}$

$\leq {}^*\sum_{n\in{}^*\mathcal{N},n\geq 3}\frac{(\sqrt{2}2)^n}{\eta^n}$

$\leq {}^*\sum_{n\in{}^*\mathcal{N},n\geq 3}\frac{(\eta)^{\frac{n}{4}}}{\eta^n}$

$= {}^*\sum_{n\in{}^*\mathcal{N},n\geq 3}\frac{1}{\eta^{\frac{3n}{4}}} = \frac{1}{\eta^{\frac{3.3}{4}}}{}^*\sum_{n\geq 0,n\in{}^*\mathcal{Z}}\frac{1}{\eta^{\frac{n}{3}}} \leq \frac{2}{\eta^{\frac{9}{4}}} \simeq 0$

We have;

$(-1)^2\lambda^2\frac{(dW_t)^2}{2} = \frac{\lambda^2}{2\kappa} = \frac{\lambda^2}{2\eta^2\nu} = \frac{\eta^2\mu^2}{2\eta^2\nu} = \frac{\mu^2}{2\nu}$

and;

$-\lambda dW_t = -\frac{\mu\eta\omega_{[t\nu]+1}}{\sqrt{\kappa}} = -\frac{\mu\eta\omega_{[t\nu]+1}}{\eta\sqrt{\nu}} = -\frac{\mu\omega_{[t\nu]+1}}{\sqrt{\nu}}$

where $\mu = \frac{1-{}^*cos(\frac{2\pi}{\eta})}{1+{}^*cos(\frac{2\pi}{\eta})}$

If $0 \leq t \leq 1$, we have;

$(\overline{G}_{\kappa,\lambda})_t - (\overline{G}_{\kappa,\lambda})_0$

$= {}^*\sum_{s=0}^{[t\nu]} d(\overline{G}_{\kappa,\lambda})_t$

$= {}^*\sum_{s=0}^{[t\nu]}(\frac{\mu^2}{2\nu} + \frac{\mu\omega_{[t\nu]+1}}{\sqrt{\nu}}) + \epsilon$

$= {}^*\sum_{s=0}^{[t\nu]}(\frac{\mu^2}{2\nu} + \frac{\mu\omega_{[t\nu]+1}}{\sqrt{\nu}}) + \epsilon$ $(*)$

where;

$|\epsilon(t)| \leq {}^*\sum_{s=0}^{[t\nu]} \frac{2}{\eta^{\frac{9}{4}}} = \frac{[t\nu]+1}{\eta^{\frac{9}{4}}} = \frac{[t\nu]+1}{\eta^{\frac{9}{4}}} \leq \frac{\nu+1}{\eta^{\frac{9}{4}}} = \frac{\frac{\eta^2}{2}+1}{\eta^{\frac{9}{4}}} \leq \frac{\eta^2}{\eta^{\frac{9}{4}}} = \frac{1}{\eta^{\frac{1}{4}}} \simeq 0$

We have;

$= {}^*\sum_{s=0}^{[t\nu]}(\frac{\mu^2}{2\nu} + \frac{\mu\omega_{[t\nu]+1}}{\sqrt{\nu}})$

$= \frac{\mu^2([t\nu]+1)}{2\nu} + \mu\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}} {}^*\sum_{s=0}^{[t\nu]}(\frac{\omega_{[t\nu]+1}}{\sqrt{[t\nu]+1}})$ $(**)$

Using Lemma 0.12 of [4], there exists a finite constant $L \in \mathcal{R}$, such that;

$|P({}^*\sum_{s=0}^{[t\nu]} \frac{\omega_{[t\nu]+1}}{\sqrt{[t\nu]+1}} = \frac{u}{\sqrt{[t\nu]+1}}) - \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-u^2}{2([t\nu]+1)})|$

$\leq L([t\nu]+1)^{\frac{-3}{2}} \simeq 0$

for $-([t\nu]+1) \leq u \leq ([t\nu]+1)$.

Therefore;

$|P(Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}} {}^*\sum_{s=0}^{[t\nu]} \frac{\omega_{[t\nu]+1}}{\sqrt{[t\nu]+1}}$

$= Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}} \frac{u}{\sqrt{[t\nu]+1}}) - \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-u^2}{2([t\nu]+1)})|$

$\leq L([t\nu]+1)^{\frac{-3}{2}} \simeq 0$

for $-([t\nu]+1) \leq u \leq ([t\nu]+1)$.

Similarly, for $Im(\mu)$. By $(*), (**)$, We have that;

We have, for $-([t\nu]+1)\leq u\leq([t\nu]+1)$, that;

$P(Re((\overline{G}_{\kappa,\lambda})_t-(\overline{G}_{\kappa,\lambda})_0=Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}\frac{u}{\sqrt{[t\nu]+1}}+\frac{Re(\mu^2)([t\nu]+1)}{2\nu}+\epsilon(t)))$

$=P({}^*\sum_{s=0}^{[t\nu]}(\frac{Re(\mu^2)}{2\nu}+\frac{Re(\mu)\omega_{[t\nu]+1}}{\sqrt{\nu}})+\epsilon(t)$

$=Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}\frac{u}{\sqrt{[t\nu]+1}}+\frac{Re(\mu^2)([t\nu]+1)}{2\nu}+\epsilon(t))$

$=P(Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}{}^*\sum_{s=0}^{[t\nu]}(\frac{\omega_{[t\nu]+1}}{\sqrt{[t\nu]+1}})=Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}\frac{u}{\sqrt{[t\nu]+1}})$

It follows that, for given $-([t\nu]+1)\leq u_0\leq([t\nu]+1)$, $u_0\in{}^*\mathcal{Z}$, that;

$P(Re((\overline{G}_{\kappa,\lambda})_t-(\overline{G}_{\kappa,\lambda})_0)\leq Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}\frac{u_0}{\sqrt{[t\nu]+1}}+\frac{Re(\mu^2)([t\nu]+1)}{2\nu}+\epsilon(t)))$

$={}^*\sum_{p=-([t\nu]+1)}^{u_0}P(Re((\overline{G}_{\kappa,\lambda})_t-(\overline{G}_{\kappa,\lambda})_0$

$=Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}\frac{p}{\sqrt{[t\nu]+1}}+\frac{Re(\mu^2)([t\nu]+1)}{2\nu}+\epsilon(t)))$

$={}^*\sum_{p=-([t\nu]+1)}^{u_0}P(Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}{}^*\sum_{s=0}^{[t\nu]}(\frac{\omega_{[t\nu]+1}}{\sqrt{[t\nu]+1}})=Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}\frac{p}{\sqrt{[t\nu]+1}})$

$={}^*\sum_{p=-([t\nu]+1)}^{u_0}\frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}}{}^*exp(\frac{-p^2}{2([t\nu]+1)})+{}^*\sum_{p=-([t\nu]+1)}^{u_0}\gamma$

$={}^*\sum_{p=-([t\nu]+1)}^{u_0}\frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}}{}^*exp(\frac{-p^2}{2([t\nu]+1)})+\delta\ (*****)$

where;

$|\gamma|\leq L([t\nu]+1)^{\frac{-3}{2}}$

$|\delta|\leq{}^*\sum_{p=-([t\nu]+1)}^{u_0}L([t\nu]+1)^{\frac{-3}{2}}$

$\leq\frac{2L([t\nu]+1)}{([t\nu]+1)^{\frac{3}{2}}}$

$\leq\frac{2L}{([t\nu]+1)^{\frac{1}{2}}}\simeq 0$, for $t\in\mathcal{R}_{>0}$

Then;

$(\overline{F}_{\kappa,\lambda})_t)={}^*exp(\overline{G}_{\kappa,\lambda})_t)$

$={}^*exp(Re(\overline{G}_{\kappa,\lambda})_t)+iIm(\overline{G}_{\kappa,\lambda})_t)$

$= {}^*exp(Re(\overline{G}_{\kappa,\lambda})_t)^*cos(Im(\overline{G}_{\kappa,\lambda})_t + i^*sin(Im(\overline{G}_{\kappa,\lambda}))_t)$

$= (\overline{H_{\kappa,\lambda}})_t((\overline{K_{\kappa,\lambda}})_t + i(\overline{L_{\kappa,\lambda}})_t)$

so that;

${}^*log(\overline{H_{\kappa,\lambda}})_t = Re(\overline{G}_{\kappa,\lambda})_t$

Then letting;

${}^*log(x) - Re(\overline{G}_{\kappa,\lambda})_0 = Re(\mu)\frac{\sqrt{[t\nu]+1}}{\sqrt{\nu}}\frac{u_0}{\sqrt{[t\nu]+1}} + \frac{Re(\mu^2)([t\nu]+1)}{2\nu} + \epsilon(t)$

with the equilibrium condition, see Lemma 0.22;

$Re(\overline{G}_{\kappa,\lambda})_0(y) = Re(\overline{F}_{\lambda,\frac{\kappa}{\nu}}) = \theta(z)$

$|\theta(z)| \leq 2\eta\frac{(4^\eta-1)}{4^\eta}^{4(4^\eta-1)^*log(\eta)}$, $z \in \overline{\Omega}_\eta$

$v_0 = [w_0]$

$w_0 = ({}^*log(x) - Re(\overline{G}_{\kappa,\lambda})_0 - \frac{Re(\mu^2)([t\nu]+1)}{2\nu} - \epsilon(t))\frac{\sqrt{\nu}}{Re(\mu)\sqrt{[t\nu]+1}}\sqrt{[t\nu]+1}$

$= ({}^*log(x) - \theta(z) - \frac{Re(\mu^2)([t\nu]+1)}{2\nu} - \epsilon(t))\frac{\sqrt{\nu}}{Re(\mu)\sqrt{[t\nu]+1}}\sqrt{[t\nu]+1}$ (note error in taking integer part)

$v_1 = [w_1]$

$w_1 = ({}^*log(x-\rho) - Re(\overline{G}_{\kappa,\lambda})_0 - \frac{Re(\mu^2)([t\nu]+1)}{2\nu} - \epsilon(t))\frac{\sqrt{\nu}}{Re(\mu)\sqrt{[t\nu]+1}}\sqrt{[t\nu]+1}$

$w_1 = ({}^*log(x-\rho) - \theta(z) - \frac{Re(\mu^2)([t\nu]+1)}{2\nu} - \epsilon(t))\frac{\sqrt{\nu}}{Re(\mu)\sqrt{[t\nu]+1}}\sqrt{[t\nu]+1}$

It follows that, as ${}^*log$ is increasing on ${}^*R_{>0}$, using $(*****)$ for the last line;

$P((\overline{H_{\kappa,\lambda}})_t \leq x)$

$= P({}^*log(\overline{H_{\kappa,\lambda}})_t \leq {}^*log(x))$

$P(Re(\overline{G}_{\kappa,\lambda})_t \leq {}^*log(x))$

$P((Re(\overline{G}_{\kappa,\lambda})_t - Re(\overline{G}_{\kappa,\lambda})_0) \leq {}^*log(x) - Re(\overline{G}_{\kappa,\lambda})_0)$

$= {}^*\sum_{p=-([t\nu]+1)}^{v_0} \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-p^2}{2([t\nu]+1)}) + \delta(v_0)$

$\simeq ({}^*\sum_{p=-([t\nu]+1)}^{v_0} \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-p^2}{2([t\nu]+1)})$

$P((\overline{H_{\kappa,\lambda}})_t \leq x - \rho)$

$= ({}^*\sum_{p=-([t\nu]+1)}^{v_1} \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-p^2}{2([t\nu]+1)}) + \delta(v_1)$

$\simeq ({}^*\sum_{p=-([t\nu]+1)}^{v_1} \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-p^2}{2([t\nu]+1)})$, $(****)$

where $max(|\delta(v_0)|, |\delta(v_1)|) \leq \frac{2L}{([t\nu]+1)^{\frac{1}{2}}}$, see estimate on $|\delta|$ above.
Let ρ be a sufficiently small infinitesimal, then;

$P((\overline{H_{\kappa,\lambda}})_t = x)$

$= P((\overline{H_{\kappa,\lambda}})_t \leq x) - P((\overline{H_{\kappa,\lambda}})_t \leq x - \frac{1}{\rho})$

$= \frac{\rho}{\rho}(P((\overline{H_{\kappa,\lambda}})_t \leq x) - P((\overline{H_{\kappa,\lambda}})_t \leq x - \rho))$

$= \frac{\rho}{\rho}({}^*\sum_{p=-([t\nu]+1)}^{v_0} \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-p^2}{2(\sqrt{[t\nu]+1})})$

$+\delta(v_0) - ({}^*\sum_{p=-([t\nu]+1)}^{v_1} \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-p^2}{2([t\nu]+1)}) - \delta(v_1))$

$= (g \circ f)^{D_\rho}(x) + \delta''$, where;

$|\delta''| = |\delta(v_0) - \delta(v_1)|$

$\leq |\delta(v_0)| + |\delta(v_1)| \leq \frac{4L}{([t\nu]+1)^{\frac{1}{2}}}$

$f(x) = {}^*log(x) - Re(\overline{G}_{\kappa,\lambda})_0 - \frac{Re(\mu^2)([t\nu]+1)}{2\nu} - \epsilon(t))\frac{\sqrt{\nu}}{Re(\mu)\sqrt{[t\nu]+1}}\sqrt{[t\nu]+1}$

$= {}^*log(x) - \theta(z) - \frac{Re(\mu^2)([t\nu]+1)}{2\nu} - \epsilon(t))\frac{\sqrt{\nu}}{Re(\mu)\sqrt{[t\nu]+1}}\sqrt{[t\nu]+1}$

$f^{D_\rho}(x) \simeq {}^*log^{D_\rho}(x)$

$g(y) = \rho^* \sum_{p=-([t\nu]+1)}^{y} \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-p^2}{2([t\nu]+1)})$

Letting $y = \frac{p}{\sqrt{[t\nu]+1}}$, we have that $g^{D_\rho}(y) = \frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}} {}^*exp(\frac{-y^2}{2})$

Noting that $\frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}}$ is finite and the fact that $exp(-\frac{y^2}{2})$ is continuous, using Lemma 0.27, we have that;

$$P((\overline{H_{\kappa,\lambda}})_t = x)$$

$$\simeq (\frac{\sqrt{2}}{\sqrt{\pi([t\nu]+1)}}{}^*exp(-\frac{[({}^*log(x) - Re(\overline{G}_{\kappa,\lambda})_0 - \frac{Re(\mu^2)([t\nu]+1)}{2\nu} - \epsilon(t))\frac{\sqrt{\nu}}{Re(\mu)\sqrt{[t\nu]+1}}\sqrt{[t\nu]+1}\quad]^2}{2}){}^*log^{D_\rho}(x))$$

$$= {}^*exp(-\frac{[({}^*log(x)-c)d]^2}{2}){}^*log^{D_\rho}(x))$$

$$= x^{cd^2}\frac{{}^*exp(-c^2d^2){}^*exp(-d^2\,{}^*log^2(x))}{2}{}^*log^{D_\rho}(x)$$

where;

$$c = Re(\overline{G}_{\kappa,\lambda})_0 - \frac{Re(\mu^2)([t\nu]+1)}{2\nu} - \epsilon(t)$$

$$d = \frac{\sqrt{\nu}}{Re(\mu)\sqrt{[t\nu]+1}}\sqrt{[t\nu]+1}$$

.... For the last part, using Theorem 0.7, by the definition of $\overline{F}_{\lambda,\kappa}$, letting $\overline{Re(F)}_{\lambda,\kappa}$ be the reverse martingale corresponding to the real part $Re(F_\lambda)$ of the process F_λ,we have that;

$$P_\lambda(t,x) = P(Re[(\overline{F}_{\lambda,\kappa})_t]) \geq x$$

$$= P((\overline{Re(F)}_{\lambda,\kappa})_t]) \geq x)$$

$$= P(Re(F_\lambda)_{\frac{\kappa-t}{\nu}} \geq x)$$

□

Lemma 0.21. *We have that, for ρ infinitesimal, for $x \in {}^*\mathcal{R}$, $x > 0$, ${}^\circ x \neq 0$;*

$${}^*log^{D_\rho}(x) \simeq {}^*(\frac{1}{x})$$

and for $x \in {}^*(-1,1)$, $|y| < \frac{\pi}{2}$, ;

$${}^*sin^{-1D_\rho}(x) \simeq {}^*(\frac{1}{\sqrt{1-x^2}})$$

and for $x \in {}^*(-1,1)$, $0 \leq y \leq \pi$, ${}^\circ x \notin \{-1,1\}$;

${}^{*}cos^{-1}{}^{D_{\rho}}(x)\simeq{}^{*}(\frac{1}{-\sqrt{1-x^{2}}})$, ${}^{\circ}x\notin\{-1,1\}$

Proof. ${}^{*}log^{D_{\rho}}(x)$

$=\frac{1}{\rho}({}^{*}log(x+\rho)-{}^{*}log(x))$

$=\frac{1}{\rho}({}^{*}log(x)+{}^{*}log(1+\frac{\rho}{x})-{}^{*}log(x))$

$=\frac{1}{\rho}({}^{*}log(1+\frac{\rho}{x}))$

$=\frac{1}{\rho}({}^{*}\sum_{n\geq 1}(-1)^{n+1}(\frac{\rho}{x})^{n})$, as $|\frac{\rho}{x}|<1$

$={}^{*}(\frac{1}{x})+\frac{1}{\rho}({}^{*}\sum_{n\geq 2}(-1)^{n+1}(\frac{\rho}{x})^{n})$

Observe that;

$|\frac{1}{\rho}({}^{*}\sum_{n\geq 2}(-1)^{n+1}(\frac{\rho}{x})^{n})|$

$\leq\frac{1}{\rho}({}^{*}\sum_{n\geq 2}(\frac{\rho}{x})^{n})$

$=\frac{1}{\rho}(\frac{\rho}{x})^{2}({}^{*}\sum_{n\geq 0}(\frac{\rho}{x})^{n})$

$=(\frac{\rho}{x^{2}})({}^{*}\sum_{n\geq 0}(\frac{\rho}{x})^{n})$

$=(\frac{\rho}{x^{2}})(\frac{1}{1-\frac{\rho}{x}})$

$\leq(\frac{2\rho}{x^{2}})$, as $\frac{\rho}{x}\simeq 0$

$\simeq 0$

For the last two results, considering the appropriate branch, one can use the result, that the functions $\{sin^{-1},cos^{-1}\}$ are smooth and bounded on $(-1+\epsilon,1-\epsilon)$ for $\epsilon>0$, and ${}^{*}cos{-}1^{D_{\rho}}(x)\simeq{}^{*}((cos^{-1})')$, for $x\in{}^{*}(-1+\epsilon,1-\epsilon)$, see [4].

□

Lemma 0.22. *Let f_{λ} be as in Lemma 0.7, with corresponding solution F_{λ} to the nonstandard heat equation, then $Re(f_{\lambda})$ and $Im(f_{\lambda})$ are bounded and for $\kappa\geq 8\eta(4^{\eta}-1){}^{*}log(\eta)$, we have that, with $\nu=\frac{\eta^{2}}{2}$,*

$t=\frac{\kappa}{\nu}$, $t\geq\frac{16(4^{\eta}-1)^{*}log(\eta)}{\eta}$;

$Re(F_{\lambda,t})\simeq 0$, $Im(F_{\lambda,t})\simeq 0$

Moreover, for $\kappa\geq 8\eta(4^{\eta}-1)^{*}log(\eta)$, $t\geq\frac{16(4^{\eta}-1)^{*}log(\eta)}{\eta}$;

$|Re(F_{\lambda,\frac{\kappa}{\nu}})|=|Re(F_{\lambda,t})|\leq 2\eta(\frac{(4^{\eta}-1)}{4^{\eta}})^{\frac{\kappa}{2\eta}-1}\leq 2\eta(\frac{(4^{\eta}-1)}{4^{\eta}})^{4(4^{\eta}-1)^{*}log(\eta)-1}$

$|Im(F_{\lambda,\frac{\kappa}{\nu}})|=|Re(F_{\lambda,t})|\leq 2\eta(\frac{(4^{\eta}-1)}{4^{\eta}})^{\frac{\kappa}{2\eta}-1}\leq 2\eta(\frac{(4^{\eta}-1)}{4^{\eta}})^{(4^{\eta}-1)^{*}log(\eta)-1}$

Proof. By Lemma 0.7, we have that $f_{\lambda}(\frac{i}{\eta})=w^{3i}$, where $w={}^{*}exp_{\eta}(\frac{2\pi i}{\eta})$. We have that $|f_{\lambda}(\frac{i}{\eta})|=|w^{3i}|=1$, in particular $Re(f_{\lambda})$ and $Im(f_{\lambda})$ are bounded. It follows, by Theorem 0.16, that, for $\kappa\geq 8\eta(4^{\eta}-1)^{*}log(\eta)$;

$Re(F_{\lambda})\simeq C_1=\int_{\overline{\Omega}_{\eta}}Re(f_{\lambda})d\mu_{\eta}, Im(F_{\lambda})\simeq C_2=\int_{\overline{\Omega}_{\eta}}Im(f_{\lambda})d\mu_{\eta}$

Moreover;

$\int_{\overline{\Omega}_{\eta}}f_{\lambda}d\mu_{\eta}=\frac{1}{\eta}{}^{*}\sum_{i=0}^{\eta-1}w^{3i}=\frac{1}{\eta}\frac{w^{3\eta}-1}{w^{3}-1}=0$

so that;

$\int_{\overline{\Omega}_{\eta}}Re(f_{\lambda})d\mu_{\eta}=\int_{\overline{\Omega}_{\eta}}Im(f_{\lambda})d\mu_{\eta}=0$

Letting $t=\frac{\kappa}{\nu}$, $\nu=\frac{\eta^{2}}{2}$, with $t\geq\frac{16(4^{\eta}-1)^{*}log(\eta)}{\eta}$, gives the first result.

By Lemma 0.3, we have, taking $Re(F_{\lambda})$ as the initial distribution, that;

$|Re((F_{\lambda}))|\leq(K^{+}+K^{-})(\frac{(4^{\eta}-1)}{4^{\eta}})^{\frac{\kappa}{2\eta}-1}$

$|Im((F_{\lambda}))|\leq(L^{+}+L^{-})(\frac{(4^{\eta}-1)}{4^{\eta}})^{\frac{\kappa}{2\eta}-1}$

where;

$K^{+}=K^{-}={}^{*}\sum_{j\in K_{\eta}}{}^{*}\cos(\frac{6\pi j}{\eta})$

$L^{+}=L^{-}={}^{*}\sum_{j\in L_{\eta}}{}^{*}\sin(\frac{6\pi j}{\eta})$

$K_{\eta}=\{j:0\leq j\leq\eta-1,{}^{*}\cos(\frac{6\pi j}{\eta})>0\}$

$L_\eta = \{j : 0 \leq j \leq \eta - 1, {}^*\sin(\frac{6\pi j}{\eta}) > 0\}$

Clearly, $max(K^+, L^+) \leq \eta$ which gives the result, for the appropriate t, and substituting the lower bound on κ.

□

Lemma 0.23. *Let ρ be an infinitesimal, and let $f : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with f^D bounded, and ${}^*g : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with $g : \mathcal{R} \rightarrow \mathcal{R}$, g second differentiable on $\mathcal{R}$. Then, for $x \in {}^*\mathcal{R}$;*

$({}^*g \circ f)^D(x) \simeq ((g')^* \circ f)f^D(x)$

Proof. By the definition of the operator D, as $h^D(x) = \rho(h(x + \rho) - h(x))$, we have that;

$({}^*g \circ f)^D(x)$

$= \rho(({}^*g \circ f)(x + \rho) - ({}^*g \circ f)(x))$

$= \frac{({}^*g \circ f)(x+\rho) - ({}^*g \circ f)(x)}{f(x+\rho) - f(x)}(\rho(f(x + \rho) - f(x)))$

$= \frac{{}^*g(f(x+\delta)) - {}^*g(f(x))}{\delta}(\rho(f(x + \rho)) - f(x)))$

$= \frac{{}^*g(f(x+\delta)) - {}^*g(f(x))}{\delta}f^D(x)$

letting $\delta = f(x + \rho) - f(x)$. As f^D is bounded, we have that $\delta \simeq 0$. As g is second differentiable, using Taylor's Theorem, we have that;

${}^*g(y + \delta) = {}^*g(y) + \delta^*(g')(y) + \frac{\delta^2}{2}{}^*(g'')(c)$

where $c \in [y, y + \delta]$, $y \in {}^*\mathcal{R}$. Therefore;

$|\frac{g(y+\delta) - g(y)}{\delta} - ({}^*g)'(y)| \leq \frac{\delta}{2}|{}^*(g'')(c)| \simeq 0$

It follows that, as f^D is bounded, letting $y = f(x)$, that;

$\frac{{}^*g(f(x+\delta)) - {}^*g(f(x))}{\delta}f^D(x) \simeq ({}^*g)'(f(x))f^D(x)$

□

Lemma 0.24. *Let ρ be an infinitesimal, and let $f \in V(\overline{\mathcal{R}}_\rho)$, with $f^{D'}$ bounded, and ${}^*g : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with $g : \mathcal{R} \rightarrow \mathcal{R}$, g second differentiable* *on $\mathcal{R}$. Then, for $x \in V(\overline{\Omega}_\eta)$;*

$$({}^*g \circ f)^{D'}(x) \simeq ((g')^* \circ f) f^{D'}(x)$$

Proof. By the definition of the operator D', as $h^{D'} = \rho(h(x) - h(x - \frac{1}{\rho}))$, using the fact that $({}^*g \circ f) \in V(\overline{\Omega}_\eta)$, we have that;

$$({}^*g \circ f)^{D'}(x)$$

$$= \rho(({}^*g \circ f)(x) - ({}^*g \circ f)(x - \tfrac{1}{\rho}))$$

$$= \frac{({}^*g \circ f)(x) - ({}^*g \circ f)(x - \frac{1}{\rho})}{f(x) - f(x - \frac{1}{\rho})}(\rho(f(x) - f(x - \tfrac{1}{\rho})))$$

$$= \frac{{}^*g(f(x)) - {}^*g(f(x) - \delta)}{\delta}(\rho(f(x)) - f(x - \tfrac{1}{\rho})))$$

$$= \frac{{}^*g(f(x)) - {}^*g(f(x) - \delta)}{\delta} f^{D'}(x)$$

letting $\delta = f(x) - f(x - \frac{1}{\rho})$. As $f^{D'}$ is bounded, we have that $\delta \simeq 0$. As g is second differentiable, using Taylor's Theorem, we have that;

$${}^*g(y - \delta) = {}^*g(y) - \delta^*(g')(y) + \frac{\delta^2}{2}{}^*(g'')(c)$$

where $c \in [y, y - \delta]$, $y \in {}^*\mathcal{R}$. Therefore;

$$\left|\frac{g(y) - g(y - \delta)}{\delta} - ({}^*g)'(y)\right| \leq \frac{\delta}{2}|{}^*(g'')(c)| \simeq 0$$

It follows that, as f^D is bounded, letting $y = f(x)$, that;

$$\frac{{}^*g(f(x)) - {}^*g(f(x) - \delta)}{\delta} f^{D'}(x) \simeq ({}^*g)'(f(x)) f^D(x)$$

□

Lemma 0.25. *Let ρ be an infinitesimal, and let $f : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with f^{D_ρ} bounded, and ${}^*g : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with $g^{D_\rho} = q^*h$, h continuous, $q \in {}^*\mathcal{R}$ Then, for $x \in {}^*\mathcal{R}$;*

$$(g \circ f)^{D_\rho}(x) \simeq ({}^*h \circ f) f^{D_\rho}(x)$$

Proof. Suppose that $f^{D_\rho}(x) = \frac{f(x+\rho) - f(x)}{\rho} = M$. By re-scaling f, (Case $f(x) = 0$) we can suppose that $M > 1$, $M \in \mathcal{N}$. Then, let $\delta = f(x + \rho) - f(x) = M\rho > \rho$. We claim that $g^{D_\delta}(x) \simeq q^*h(x)$, $(*)$.

As $\rho < \delta$, we have that;

$g^{D_\delta}(x)$

$= \frac{1}{\delta}(g(x+\delta) - g(x)) = \frac{1}{\delta}\int_x^{x+\delta} g_{D_\rho} d\mu_\rho(x)$

$= \frac{1}{\delta}\int_x^{x+\delta} q^* h d\mu_\rho(x)$

$= \frac{q}{M\rho}\rho^* \sum_{j=0}^{M-1} {}^* h(x+j\rho)$

$= \frac{q}{M}{}^* \sum_{j=0}^{M-1} {}^* h(x+j\rho)$

$\simeq q^* h(x)$, as h is continuous, $(*)$

By the definition of the operator D_ρ, as $f^{D_\rho} = \rho(f(x) - f(x+\rho))$, we have that;

$(g \circ f)^{D_\rho}(x)$

$= \rho((g \circ f)(x+\rho) - (g \circ f)(x))$

$= \frac{(g\circ f)(x+\rho)-(g\circ f)(x)}{f(x+\rho)-f(x)}(\rho(f(x+\rho) - f(x)))$

$= \frac{g(f(x+\delta))-g(f(x))}{\delta}(\rho(f(x)) - f(x-\rho)))$

$= \frac{g(f(x+\delta))-g(f(x))}{\delta} f^{D_\rho}(x)$

letting $\delta = f(x+\rho) - f(x)$. It follows that, as f^{D_ρ} is bounded, letting $y = f(x)$, and using $(*)$, that;

$\frac{g(f(x+\delta))-g(f(x))}{\delta} f^{D_\rho}(x) \simeq ({}^*h)(f(x)) f^{D_\rho}(x)$

□

Lemma 0.26. *Let ρ be an infinitesimal, and let $f : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with $f^{D'_\rho}$ bounded, and ${}^*g : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with $g^{D'_\rho} = {}^*h$, h continuous. Then, for $x \in {}^*\mathcal{R}$;*

$(g \circ f)^{D'_\rho}(x) \simeq ({}^*h \circ f) f^{D'_\rho}(x)$

Proof. Suppose that $f^{D'_\rho}(x) = \frac{f(x)-f(x-\rho)}{\rho} = M$. By considering $\alpha f + \beta$, $\{\alpha, \beta\} \subset \mathcal{R}$, we can suppose that $M > 1$, $M \in \mathcal{N}$. Then, let

$\delta = f(x) - f(x-\rho) = M\rho > \rho$. We claim that $g^{D'_\delta}(x) \simeq {}^*h(x)$, $(*)$. As $\rho < \delta$, we have that;

$g^{D'_\delta}(x)$

$= \frac{1}{\delta}(g(x) - g(x-\delta)) = \frac{1}{\delta}\int_{x-\delta}^{x} g_{D'_\rho} d\mu_\rho(x)$

$= \frac{1}{\delta}\int_{x-\delta}^{x} {}^*h d\mu_\rho(x)$

$= \frac{1}{M\rho}\rho^* \sum_{j=0}^{M-1} {}^*h(x-\delta+j\rho)$

$= \frac{1}{M}{}^* \sum_{j=0}^{M-1} {}^*h(x-\delta+j\rho)$

$\simeq {}^*h(x)$, as h is continuous, $(*)$

By the definition of the operator D'_ρ, as $f^{D'_\rho} = \rho(f(x) - f(x-\rho))$, we have that;

$(g \circ f)^{D'_\rho}(x)$

$= \rho((g \circ f)(x) - (g \circ f)(x-\rho))$

$= \frac{(g\circ f)(x) - (g\circ f)(x-\rho)}{f(x)-f(x-\rho)}(\rho(f(x) - f(x-\rho)))$

$= \frac{g(f(x)) - g(f(x-\delta))}{\delta}(\rho(f(x)) - f(x-\rho)))$

$= \frac{g(f(x)) - g(f(x-\delta))}{\delta} f^{D'_\rho}(x)$

letting $\delta = f(x) - f(x-\rho)$. It follows that, as $f^{D'_\rho}$ is bounded, letting $y = f(x)$, and using $(*)$, that;

$\frac{g(f(x-\delta)) - g(f(x))}{\delta} f^{D'_\rho}(x) \simeq ({}^*h)(f(x)) f^{D'_\rho}(x)$

□

Lemma 0.27. *Let ρ be an infinitesimal, and let $f : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with f^{D_ρ} bounded, and ${}^*g : {}^*\mathcal{R} \rightarrow {}^*\mathcal{R}$, with $g^{D_\rho} = q^*h$, h continuous, $q \in {}^*\mathcal{R}$ Then, for $x \in {}^*\mathcal{R}$;*

$(g \circ f)^{D_\rho}(x) \simeq (q^*h \circ f) f^{D_\rho}(x)$

Proof. Suppose that $f^{D_\rho}(x)=\frac{f(x+\rho)-f(x)}{\rho}=M$. By re-scaling f, (Case $f(x)=0$) we can suppose that $M>1$, $M\in\mathcal{N}$. Then, let $\delta=f(x+\rho)-f(x)=M\rho>\rho$. We claim that $g^{D_\delta}(x)\simeq q^*h(x)$, $(*)$. As $\rho<\delta$, we have that;

$g^{D_\delta}(x)$

$=\frac{1}{\delta}(g(x+\delta)-g(x))=\frac{1}{\delta}\int_x^{x+\delta}g_{D_\rho}d\mu_\rho(x)$

$=\frac{1}{\delta}\int_x^{x+\delta}q^*hd\mu_\rho(x)$

$=\frac{q}{M\rho}\rho^*\sum_{j=0}^{M-1}{}^*h(x+j\rho)$

$=\frac{q}{M}{}^*\sum_{j=0}^{M-1}{}^*h(x+j\rho)$

$\simeq q^*h(x)$, as h is continuous, and q is finite, $(*)$

By the definition of the operator D_ρ, as $f^{D_\rho}=\rho(f(x)-f(x+\rho))$, we have that;

$(g\circ f)^{D_\rho}(x)$

$=\rho((g\circ f)(x+\rho)-(g\circ f)(x))$

$=\frac{(g\circ f)(x+\rho)-(g\circ f)(x)}{f(x+\rho)-f(x)}(\rho(f(x+\rho)-f(x)))$

$=\frac{g(f(x+\delta))-g(f(x))}{\delta}(\rho(f(x))-f(x-\rho)))$

$=\frac{g(f(x+\delta))-g(f(x))}{\delta}f^{D_\rho}(x)$

letting $\delta=f(x+\rho)-f(x)$. It follows that, as f^{D_ρ} is bounded, letting $y=f(x)$, and using $(*)$, that;

$\frac{g(f(x+\delta))-g(f(x))}{\delta}f^{D_\rho}(x)\simeq(q^*h)(f(x))f^{D_\rho}(x)$

□

Lemma 0.28. *Let the density ρ satisfy the diffusion equation;*

$\frac{\partial\rho}{\partial t}-\frac{\partial^2\rho}{\partial^2 x}=0$, $(*)$

then so does the current $J = \frac{\partial \rho}{\partial x}$*, and, moreover, the pair* $\{\rho, J\}$ *satify the continuity equation;*

$\frac{\partial \rho}{\partial t} - \frac{\partial J}{\partial x} = 0$

Moreover, if $Q(t) = \int_{\overline{\Omega_\eta}} \rho(x,t) d\mu_\eta(x)$*, then* $Q(t) = Q(0)$ *is constant, that is the amount is conserved.*

Proof. If;

$\frac{\partial \rho}{\partial t} - \frac{\partial^2 \rho}{\partial^2 x} = 0$

then, letting $J = \frac{\partial \rho}{\partial x}$, we have that;

$\frac{\partial \rho}{\partial t} - \frac{\partial J}{\partial x} = 0$

Applying the operator $\frac{\partial}{\partial x}$ to $(*)$, and, using the fact that the derivatives $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\}$ commute, we obtain;

$\frac{\partial \rho}{\partial t \partial x} - \frac{\partial^3 \rho}{\partial^3 x} = 0$

Hence;

$\frac{\partial J}{\partial t} - \frac{\partial^2 J}{\partial^2 x} = 0$

giving the first two parts of the claim. For the last part, if;

$Q(t) = \int_{\overline{\Omega_\eta}} \rho(x,t) d\mu_\eta(x)$

then, differentiating under the integral sign, using the second part, and the argument of Lemma 0.45(i) ,we have, for $t \in \overline{T}_\nu$;

$\frac{\partial Q}{\partial s}$

$= \int_{\overline{\Omega_\eta}} \frac{\partial \rho}{\partial s}(x,t) d\mu_\eta(x)$

$= \int_{\overline{\Omega_\eta}} \frac{\partial J}{\partial x}(y,t) d\mu_\eta(y) = 0$

Using the definition of $\frac{\partial}{\partial t}$ in 0.5, we obtain that $Q(t) = Q(0)$, for $t \in \overline{T}_\nu$, as required.

□

Remarks 0.29. *As observed in Lemma 0.28, if* $f_\lambda \in V(\overline{\Omega}_\eta)$ *satisfies the differential equation;*

$(f^{lsh} + f^{rsh})' = \frac{\lambda}{2}(f^{lsh^2} + f^{rsh^2})$

with corresponding F_λ *satisfying the heat equation;*

$\frac{\partial F_\lambda}{\partial t} - \frac{\partial^2 F_\lambda}{\partial x^2}$

then, so does the nonstandard derivative $(f_\lambda)'$*, with corresponding* $\frac{\partial F_\lambda}{\partial x}$. *By Lemma 0.16,* $(f_\lambda)' = e f_\lambda$*, where* $e = \frac{\eta\lambda}{2}(1 - \lambda^{\eta-2})$. *Interpreting* F_λ *as a particle or charge density, see Remark 0.9, we can consider* $\frac{\partial F_\lambda}{\partial x}$ *as a flow of charge or current, linked to particle velocity. In the case without collisions, the distribution function* R_λ *might be linked to the Boltzmann distribution for molecular speeds in the theory of thermodynamcs. More specifically, in the terminology of Lemma 0.28, if we can compute the distribution functions* $\{R_\lambda, S_\lambda\}$ *for the processes* $\{\rho_\lambda, J_\lambda\}$*, with initial conditions* $\{f_\lambda, \frac{\partial f_\lambda}{\partial x}\}$*, then, we can, using the same method, hope to compute the joint distribution function* T_λ *for the vector process* $\overline{V_\lambda} = (\rho_\lambda, J_\lambda)$*, satisfying the vector equation;*

$\frac{\partial \overline{V_\lambda}}{\partial t} - \frac{\partial^2 \overline{V_\lambda}}{\partial x^2} = 0$

with initial condition $(f_\lambda, \frac{\partial f_\lambda}{\partial x})$

We can then obtain the distribution function U_λ*, for the* 1*-dimensional process;*

$v_\lambda = \frac{J_\lambda}{\rho_\lambda}$

as $f_\lambda \neq 0$ *implies that* $\rho_\lambda^t \neq 0$*, for all* $t \in \overline{T}_\nu$*, using the defining schema for the diffusion in Theorem 0.7 (check this?).*

We should then be able to obtain the distribution of the individual particle velocities, modelling a particle as $\epsilon\delta_p$ *for a fixed infinitesimal* ϵ*, and* δ_p*, with* $|\delta_p| = 1$*, supported at a point* $p \in \overline{V_\eta}$. *More specifically, we introduce path spaces* $\overline{\Omega}_{p,\eta}$ *for each* $p \in \overline{V_\eta}$*, with* ${}^*Card(\overline{\Omega}_{p,\eta}) = [\kappa_p]$*, where* $N_p = [\frac{f_\lambda}{\epsilon}]$ *and* $\kappa_p = {}^*log_2(N_p)$. *We then restrict to the time interval* $\overline{T}_\kappa$*, where* $\kappa = max_{p \in \overline{V_\eta}}[\kappa_p]$*, with path spaces* $\overline{\Omega'}_{p,\eta}$*, for an augmenting sequence of zeroes of length* $\kappa - \kappa_p$. *We can then define a* *diffusion* G_{f_λ} *on* $\overline{V_\eta}\times\overline{T}_\kappa$ *by;*

$G_{f_\lambda}(x,t)$

$= {}^*\sum_{p\in\overline{V_\eta}}\epsilon^* Card(\{\omega\in\overline{\Omega}_{p,\eta}:\chi^t_{p,\eta}(\omega)=x\})$

where $\chi^t_{p,\eta}(\omega)=p+\frac{1}{\eta}({}^*\sum_{i=1}^{j}\omega_j)$ *(mod 1)*

with the convention that $\omega_j=0$, *for* $\kappa-\kappa_p\leq j\leq\kappa$ *and* $t=\frac{j}{\nu}$.

If ϵ *is small enough, we can take* $min_{p\in\overline{V_\eta}}[\kappa_p]$ *to be at least the condition for equilibrium given by* $\tau=\frac{\kappa}{\nu}$ *in Theorem 0.7. In this case, we can check that* $G_{f_\lambda}\simeq\rho_\lambda$, $(**)$. *We can then define the velocities of the particles* $\{v_{p,\omega,t}:p\in\overline{V_\eta},\omega\in\overline{\Omega}_{p,\eta},t\in\overline{T_\nu}\}$, *by;*

$v_{p,\omega,t}=\frac{\nu}{j}\chi^t_{p,\eta}(\omega)$

and use the formula with $(**)$*;*

$v_\lambda(x,t)\simeq {}^*\sum_{(p,\omega):\chi^t_{p,\eta}(\omega)=x}v_{p,\omega,t}$

to compute the distribution function W_λ *for the individual velocities.*

Definition 0.30. *Let* $c\in{}^*\mathcal{N}\setminus\mathcal{N}$, *infinite and odd,* $\nu\in{}^*\mathcal{Q}_{\geq 0}\setminus\mathcal{Q}$. *We let;*

$\overline{\Omega_{\eta c}}=\{x\in{}^*\mathcal{R}:0\leq x<1\}$, $\overline{\mathcal{T}_{\nu c}}=\{t\in{}^*\mathcal{R}_{\geq 0}\}$

We let $\mathcal{C}_{\eta c}$ *consist of internal unions of the intervals* $[\frac{i}{\eta c},\frac{i+1}{\eta c})$, *for* $0\leq i\leq\eta c-1$, *and let* $\mathcal{D}_\nu$ *consist of internal unions of* $[\frac{i}{\nu},\frac{i+1}{\nu})$, *for* $i\in{}^*\mathcal{Z}_{\geq 0}$.

We define counting measures $\mu_{\eta c}$ *and* $\lambda_{\nu,c}$ *on* $\mathcal{C}_{\eta c}$ *and* $\mathcal{D}_{\nu,c}$ *respectively, by setting* $\mu_{\eta c}([\frac{i}{\eta c},\frac{i+1}{\eta c}))=\frac{1}{\eta c}$, $\lambda_\nu([\frac{i}{\nu},\frac{i+1}{\nu}))=\frac{1}{\nu}$, *for* $0\leq i\leq\eta c-1$, $i\in{}^*\mathcal{Z}_{\geq 0}$ *respectively.*

We let $(\overline{\Omega_{\eta c}},\mathcal{C}_{\eta c},\mu_{\eta c})$ *and* $(\overline{\mathcal{T}}_{\nu,c},\mathcal{D}_{\nu,c},\lambda_{\nu,c})$ *be the resulting measure spaces, in the sense of* [1]. *We let* $(\overline{\Omega_{\eta c}}\times\overline{\mathcal{T}}_{\nu,c},\mathcal{C}_{\eta c}\times\mathcal{D}_{\nu,c},\mu_{\eta c}\times\lambda_{\nu,c})$ *denote the corresponding product space.*

If $f\in V(\overline{\Omega_{\eta c}}\times\overline{\mathcal{T}}_{\nu,c})$ is measurable, we define;

$\frac{\partial f}{\partial t}(\frac{i}{\eta c},\frac{j}{\nu})=\nu(f(\frac{i}{\eta},\frac{j+1}{\nu})-f(\frac{i}{\eta c},\frac{j}{\nu}))$, $\frac{\partial f}{\partial t}(x,s)=\frac{\partial f}{\partial t}(\frac{[\eta cx]}{\eta c},\frac{[\nu s]}{\nu})$

$\frac{\partial f}{\partial x}(\frac{i}{\eta},\frac{j}{\nu})=\frac{\eta c}{2}(f(\frac{i+1}{\eta c},\frac{j}{\nu})-f(\frac{i-1}{\eta c},\frac{j}{\nu}))$, $\frac{\partial f}{\partial x}(y,t)=\frac{\partial f}{\partial x}(\frac{[\eta cy]}{\eta c},\frac{[\nu t]}{\nu})$

where we adopt the usual convention of taking i mod ηc.

Theorem 0.31. *Let F satisfy the nonstandard rescaled heat equation, with the choice of $\nu=\frac{\eta^2}{2c^2}$*
;

$\frac{\partial F}{\partial t}-\frac{1}{c^4}\frac{\partial^2 F}{\partial x^2}=0$

Then we have that;

$F(\frac{i}{\eta c},\frac{j+1}{\nu})=\frac{1}{2}F(\frac{i+2}{\eta c},\frac{j}{\nu})+\frac{1}{2}F(\frac{i-2}{\eta c},\frac{j}{\nu})$, $j\in{}^*\mathcal{Z}_{\geq 0}, 0\leq j\leq \eta c-1$, *mod* (ηc)

Proof. We have, as above, that;

$\nu(F(\frac{i}{\eta c},\frac{j+1}{\nu})-F(\frac{i}{\eta c},\frac{j}{\nu}))$

$=\frac{1}{c^4}\frac{\eta^2c^2}{4}(F(\frac{i+2}{\eta c},\frac{j}{\nu})-2F(\frac{i}{\eta},\frac{j}{\nu})+F(\frac{i-2}{\eta},\frac{j}{\nu}))$

$F(\frac{i}{\eta c},\frac{j+1}{\nu})=\frac{1}{\nu c^4}\frac{\eta^2c^2}{4}F(\frac{i+2}{\eta c},\frac{j}{\nu})$

$+(1-\frac{2}{\nu c^4}\frac{\eta^2c^2}{4})F(\frac{i}{\eta c},\frac{j}{\nu})$

$+\frac{1}{\nu c^4}\frac{\eta^2c^2}{4}F(\frac{i-2}{\eta c},\frac{j}{\nu})$

$F(\frac{i}{\eta c},\frac{j+1}{\nu})=\frac{1}{\nu c^2}\frac{\eta^2}{4}F(\frac{i+2}{\eta c},\frac{j}{\nu})$

$+(1-\frac{2}{\nu c^2}\frac{\eta^2}{4})F(\frac{i}{\eta c},\frac{j}{\nu})$

$+\frac{1}{\nu c^2}\frac{\eta^2}{4}F(\frac{i-2}{\eta c},\frac{j}{\nu})$

$F(\frac{i}{\eta c},\frac{j+1}{\nu})=\frac{1}{2}F(\frac{i+2}{\eta c},\frac{j}{\nu})$

$+\frac{1}{2}F(\frac{i-2}{\eta c},\frac{j}{\nu})$

□

Lemma 0.32. *If c is odd then Lemma 0.4 holds, replacing η with ηc. Similarly, Definition 0.6, Theorem 0.7 hold, replacing η with ηc, and F being the unique solution to the nonstandard heat equation;*

$$\frac{\partial F}{\partial t}-\frac{1}{c^4}\frac{\partial^2 F}{\partial x^2}=0$$

with initial condition f, with $\tau_{new}\geq\frac{kappa}{\nu}$, $\nu=\frac{\eta^2}{2c^2}$, $\kappa=8\eta c(4^{\eta c}-1)^* log(\eta c$. Lemma 0.8 holds. Definition 0.10 is the same replacing η with ηc. Lemma 0.11 is the same, replacing η with ηc, and using the rescaled heat equation, with the new choice of ν. Definition 0.12 and Theorem 0.13 are the same, replacing η with ηc. Lemma 0.14 is the same with the modification that;

$$(\overline{D_\kappa,\lambda,c})_t(\tfrac{j}{\eta c})=\frac{\sqrt{\kappa}}{\sqrt{2\nu}c^2}$$

Lemma 0.15 is the same. Lemma 0.16 is still true, replacing η with ηc. Similarly for Lemma 0.17, with $F_{\lambda,c}$ solving the rescaled heat equation, and the initial condition $f_{\lambda,c}$ from Lemma 0.16. Lemma 0.18 holds for $F_{\lambda,c}$. The same equation from Lemma 0.19 holds with the corresponding $\overline{F}_{\kappa,\lambda,c}$. The computation of the distribution of $\overline{F}_{\kappa,\lambda,c}$ is similar.

Proof. The proof follows by noting that with the choice of $\nu=\frac{\eta^2}{2c^2}$, the diffusion for the rescaled heat equation is the same as in Theorem 0.7. For Lemma 0.14, substitute η for ηc in the proof, then using the relation $\nu=\frac{\eta^2}{2c^2}$, we obtain, $\frac{1}{\eta c}=\frac{1}{c\sqrt{2\nu}c}=\frac{1}{\sqrt{2\nu}c^2}$. The new scalar doesn't effect the result of Lemma 0.15. In Lemma 0.19, the new constant doesn't matter. □

Definition 0.33. *We let $\overline{\Omega_{\eta,c}}=\{x\in{}^*\mathcal{R}:0\leq x<c^2+\frac{(1-c)}{\eta}\}$, $\overline{\mathcal{T}_{\nu c}}=\{t\in{}^*\mathcal{R}_{\geq 0}\}$*

We let $\mathcal{C}_{\eta,c}$ consist of internal unions of the intervals $[\frac{i}{\eta},\frac{i+1}{\eta})$, for $0\leq i\leq(\eta c-1)c$, and let $\mathcal{D}_\nu$ consist of internal unions of $[\frac{i}{\nu},\frac{i+1}{\nu})$, for $i\in{}^*\mathcal{Z}_{\geq 0}$.

We define counting measures $\mu_{\eta,c}$ and $\lambda_{\nu,c}$ on $\mathcal{C}_{\eta,c}$ and $\mathcal{D}_{\nu,c}$ respectively, by setting $\mu_{\eta,c}([\frac{i}{\eta},\frac{i+1}{\eta}))=\frac{1}{\eta}$, $\lambda_\nu([\frac{i}{\nu},\frac{i+1}{\nu}))=\frac{1}{\nu}$, for $0\leq i\leq(\eta c-1)c$, $i\in{}^*\mathcal{Z}_{\geq 0}$ respectively.

If c is odd, $g\in V(\overline{\Omega_\eta})$, we define the contraction $g^{c^2}\in V(\overline{\Omega_{\eta c}})$, by;

$g^{c^2}(\frac{i}{\eta c}) = g_{ext}(\frac{ic}{\eta})$, $0 \leq i \leq \eta c - 1$

where $g_{ext} \in V(\overline{\Omega_{\eta,c}})$ *is the periodic extension, defined by;*

$g_{ext}(\frac{j}{\eta}) = g(\frac{j(mod \eta)}{\eta})$, $0 \leq j \leq (\eta c - 1)c$

We let $\mathcal{N}_\eta = {}^*\mathcal{N} \cap [0, \eta - 1]$, *and if* $g \in V(\mathcal{N}_\eta)$, c *and* η *are coprime, we define the extension* $g_{c^2} \in V(\mathcal{N}_{\eta c})$, *by letting, for* $n \in \mathcal{N}_{\eta c}$

$g_{c^2}(n) = g(m)$, *where* $m \in \mathcal{N}_\eta$ *is unique with the property that* $mc^2 = n$ *(mod* ηc*)*

Note this is a good definition, as if $0 \leq m_1 < m_2 \leq \eta - 1$, *with;*

$m_1 c^2 = m_2 c^2$ *(mod (*ηc*))*

then;

$m_1 c = m_2 c$ *(mod (*η*))*

so that as $\eta | m_1 c - m_2 c$ *and* $\{c, \eta\}$ *are coprime,* $\eta | m_1 - m_2$, *which cannot be the case, as,* $0 < (m_2 - m_1) < \eta$.

Definition 0.34. *If* $f \in V(\overline{\Omega}_\eta)$, *and* $m \in \mathcal{N}_\eta$, *we define;*

$\mathcal{F}_\eta(f)(m) = \int_{\overline{\Omega}_\eta} f(x) exp_\eta(-2\pi i m x) d\mu_\eta(x)$

By checking that $\{exp_\eta(2\pi i m x : m \in \mathcal{N}_\eta\}$ *forms an orthonormal basis for* $V(\overline{\Omega}_\eta)$, *see for example the calculation in* [**?**], *we have the inversion theorem;*

$f(x) = {}^* \sum_{m \in \mathcal{N}_\eta} \mathcal{F}_\eta(f)(m) exp_\eta(2\pi i m x)$

Lemma 0.35. *Let* $f_\lambda \in V(\overline{\Omega}_\eta)$, *be as in Lemma 0.16,* $m \in \mathcal{N}_\eta$, *then;*

$\mathcal{F}_\eta(f_\lambda)(m) = 0$, *if* $m \neq \frac{(\eta+1)}{4}$

$\mathcal{F}_\eta(f_\lambda)(m) = 1$, *if* $m = \frac{(\eta+1)}{4}$ $(*)$

Let $m \in \mathcal{N}_{\eta c}$, *then;*

$\mathcal{F}_{\eta c}((f_\lambda)^{c^2})(m) = 0,\ if\ m \neq \kappa c$

$\mathcal{F}_{\eta c}((f_\lambda)^{c^2})(m) = 1,\ if\ m = \kappa c$

where $c = \beta\eta + \delta,\ 0 \leq \delta \leq \eta - 1$

and $\frac{(\eta+1)}{4}\delta = \epsilon\eta + \kappa,\ 0 \leq \kappa \leq \eta - 1,\ (**)$

Finally;

$(\mathcal{F}_\eta(f_\lambda))_{c^2} = \mathcal{F}_{\eta c}((f_\lambda)^{c^2}),\ (***).$

Proof. We have, for $m \in \mathcal{N}_\eta$, $m \neq \frac{\eta+1}{4}$, that;

$\mathcal{F}_\eta(f_\lambda)(m)$

$= \int_{\overline{\Omega}_\eta} f_\lambda(x) exp_\eta(-2\pi i m x) d\mu_\eta(x)$

$= \frac{1}{\eta}{}^* \sum_{j=0}^{\eta-1} f_\lambda(\frac{j}{\eta}) exp_\eta(-2\pi i \frac{mj}{\eta})$

$= \frac{1}{\eta}{}^* \sum_{j=0}^{\eta-1} \omega^{\frac{(\eta+1)j}{4}}{}^* exp(-2\pi i \frac{mj}{\eta})$

$= \frac{1}{\eta}{}^* \sum_{j=0}^{\eta-1}{}^* exp(2\pi i \frac{(\eta+1)j}{4\eta})^* exp(-2\pi i \frac{mj}{\eta})$

$= \frac{1}{\eta}{}^* \sum_{j=0}^{\eta-1}{}^* exp(\pi i \frac{\frac{\eta+1}{2\eta} j - 2mj}{\eta})$

$= \frac{1}{\eta}{}^* \sum_{j=0}^{\eta-1} \xi_m^j$

$= \frac{1}{\eta} \frac{\xi_m^\eta - 1}{\xi_m - 1}$

where $\xi_m = {}^* exp(\frac{\pi i (\frac{\eta+1}{2\eta} - 2m)}{\eta})$

$= \frac{1}{\eta} \frac{{}^* exp(\pi i (\frac{\eta+1}{2\eta} - 2m)) - 1}{\xi_m - 1}$

$= \frac{1}{\eta} \frac{{}^* exp(-2\pi i m) - 1}{\xi_m - 1}$, as $\frac{\eta+1}{2\eta}$ is even

$= \frac{1}{\eta} \frac{(1)^m - 1}{\xi_m - 1} = 0$

We have, for $m \in \mathcal{Z}_\eta$, $m = \frac{\eta+1}{4}$, that;

$\mathcal{F}_\eta(f_\lambda)(m)$

$= \frac{1}{\eta} {}^* \sum_{j=0}^{\eta-1} f_\lambda(\frac{j}{\eta}) exp_\eta(-2\pi i \frac{(\eta+1)j}{4\eta})$

$= \frac{1}{\eta} {}^* \sum_{j=0}^{\eta-1} \omega^{\frac{(\eta+1)j}{4}} {}^* exp(-2\pi i \frac{(\eta+1)j}{4\eta})$

$= \frac{1}{\eta} {}^* \sum_{j=0}^{\eta-1} {}^* exp(2\pi i \frac{(\eta+1)j}{4\eta}) {}^* exp(-2\pi i \frac{(\eta+1)j}{4\eta})$

$= \frac{1}{\eta} {}^* \sum_{j=0}^{\eta-1} 1$

$= \frac{\eta}{\eta} = 1$, as required.

We have that;

$(f_\lambda)^{c^2}(\frac{j}{\eta c})$

$= (f_{\lambda,ext})(\frac{jc}{\eta})$

$= \omega^{\frac{cj(\eta+1)}{4}}$

$= {}^* exp(2\pi i \frac{cj(\eta+1)}{4\eta})$

$= {}^* exp(2\pi i \frac{c(\eta+1)}{4\eta} \eta c \frac{j}{\eta c})$

$= {}^* exp(2\pi i \frac{c(\eta+1)}{4\eta} \eta c x)$, $(x = \frac{j}{\eta c}, 0 \leq j \leq \eta c - 1)$

We need to find $m_0 \in \mathcal{N}_{\eta c}$, such that;

$= {}^* exp(2\pi i \frac{c(\eta+1)}{4\eta} \eta c x) = {}^* exp(2\pi i m_0 x)$

We find $m_0 \in [0, \eta c - 1] \cap {}^*\mathcal{N}$ such that;

$m_0 = \frac{(\eta+1)c^2}{4}$, mod ηc

Letting $\alpha = \frac{(\eta+1)}{4}$, we have that;

$\alpha c^2 = s(\eta c) + t$, $0 \leq t \leq \eta c - 1$

if $\alpha c = s\eta + t'$, $0 \leq t' \leq \eta - 1$

Let $c = \beta\eta + \delta$, with $0 \leq \delta \leq \eta - 1$

then, we require that;

$\alpha(\beta\eta+\delta)=s\eta+t'$

iff $\alpha\beta\eta+\alpha\delta=s\eta+t'$

Letting $\alpha\delta=\epsilon\eta+\kappa$, with $0\leq\kappa\leq\eta-1$

we require that;

$\alpha\beta\eta+\epsilon\eta+\kappa=s\eta+t'$

iff $(\alpha\beta+\epsilon)\eta+\kappa=s\eta+t'$

So, noting that s and t are unique, and taking $s=(\alpha\beta+\epsilon)$, $t'=\kappa$, $t=t'c=\kappa c$;

we have that $m_0=\kappa c$, $\alpha c^2=s(\eta c)+m_0$. Then, for $x\in\overline{\Omega}_{\eta c}$, we have;

${}^*exp(2\pi i\frac{c(\eta+1)}{4\eta}\eta c\frac{[x\eta c]}{\eta c})$

$={}^*exp(2\pi i(s(\eta c)+m_0)\frac{[x\eta c]}{\eta c})$

$={}^*exp(2\pi i(\frac{s(\eta c)[x\eta c]}{\eta c})){}^*exp(2\pi im_0\frac{[x\eta c]}{\eta c})$

$={}^*exp(2\pi im_0\frac{[x\eta c]}{\eta c})$

so that $exp_{\eta c}(2\pi im_0x)=exp_{\eta c}(2\pi i\frac{c(\eta+1)}{4\eta}\eta cx)$

Using the fact that $\{exp_{\eta c}(2\pi imx):m\in\mathcal{N}_{\eta c}\}$ forms an orthonormal basis, with respect to $<,>_{\eta c}$, we obtain the second result, $(**)$.

Then, using $(*)$ and the definition of an extension in Definition 0.33, we have that;

$(\mathcal{F}_\eta(f_\lambda))_{c^2}(m)$

$=(\mathcal{F}_\eta(f_\lambda)(n)$

where $n\in\mathcal{N}_\eta$ is unique with the property that $c^2n=m\ (\mathrm{mod}\ \eta c)$

$= 1$, for $m_0 = \frac{(\eta+1)}{4}c^2$, (mod ηc)

$= 0$, for $m \neq \frac{(\eta+1)}{4}c^2$, (mod ηc)

which agrees with the result for $\mathcal{F}_{\eta c}((f_{\lambda,ext})^{c^2})(m)$, by $(**)$, taking $m_0 = \kappa c$, $\frac{(\eta+1)}{4}c^2 = s(\eta c) + m_0$

$\square$

Lemma 0.36. *If $f \in V(\overline{\Omega_\eta})$, then;*

$\mathcal{F}_{\eta c}(f^{c^2}) = ((\mathcal{F}_\eta(f)))_{c^2}$

Remarks 0.37. *This generalises the proof of the result above.*

Proof. Let $m \in \mathcal{N}_{\eta c}$, then we have that;

$\mathcal{F}_{\eta c}(f^{c^2})(m)$

$= \int_{\overline{\Omega_{\eta c}}} f^{c^2}(x) exp_{\eta c}(-2\pi i m x) d\mu_{\eta c}(x)$

$= \frac{1}{\eta c} {}^*\sum_{j=0}^{\eta c-1} f^{c^2}(\frac{j}{\eta c}) exp_{\eta c}(-2\pi i m \frac{j}{\eta c})$

$= \frac{1}{\eta c} {}^*\sum_{j=0}^{\eta c-1} f_{ext}(\frac{jc}{\eta}) exp_{\eta c}(-2\pi i m \frac{j}{\eta c})$

Using the inversion theorem, we have that;

$f(x) = {}^*\sum_{k=0}^{\eta-1} \mathcal{F}_\eta(f)(k) exp_\eta(2\pi i k x)$

so that;

$\frac{1}{\eta c} {}^*\sum_{j=0}^{\eta c-1} f_{ext}(\frac{jc}{\eta}) exp_{\eta c}(-2\pi i m \frac{j}{\eta c})$

$= \frac{1}{\eta c} {}^*\sum_{j=0}^{\eta c-1} [{}^*\sum_{k=0}^{\eta-1} \mathcal{F}_\eta(f)(k) exp_\eta(2\pi i k x)]_{ext}(\frac{jc}{\eta}) exp_{\eta c}(-2\pi i m \frac{j}{\eta c})$

$\frac{1}{\eta c} {}^*\sum_{j=0}^{\eta c-1} {}^*\sum_{k=0}^{\eta-1} \mathcal{F}_\eta(f)(k) [exp_\eta(2\pi i k x)]_{ext}(\frac{jc}{\eta}) exp_{\eta c}(-2\pi i m \frac{j}{\eta c})$

We have that;

$[exp_\eta(2\pi i k x)]_{ext}(\frac{jc}{\eta}) = {}^*\exp(2\pi i k \frac{jc}{\eta})$

as ${}^{*}\exp(2\pi ikx)$ has period 1, see Definition 0.35. Then;

$\frac{1}{\eta c}{}^{*}\sum_{j=0}^{\eta c-1}{}^{*}\sum_{k=0}^{\eta-1}\mathcal{F}_{\eta}(f)(k)[exp_{\eta}(2\pi ikx)]_{ext}(\frac{jc}{\eta})exp_{\eta c}(-2\pi im\frac{j}{\eta c})$

$=\frac{1}{\eta c}{}^{*}\sum_{j=0}^{\eta c-1}{}^{*}\sum_{k=0}^{\eta-1}\mathcal{F}_{\eta}(f)(k){}^{*}\exp(2\pi ik\frac{jc}{\eta}){}^{*}\exp(-2\pi im\frac{j}{\eta c})$

$=\frac{1}{\eta c}{}^{*}\sum_{k=0}^{\eta-1}\mathcal{F}_{\eta}(f)(k){}^{*}\sum_{j=0}^{\eta c-1}{}^{*}\exp(-2\pi ij(\frac{-kc}{\eta}+\frac{m}{\eta c}))$

$=\frac{1}{\eta c}{}^{*}\sum_{k=0}^{\eta-1}\mathcal{F}_{\eta}(f)(k)\frac{\xi_{k,m}^{\eta c}-1}{\xi_{k,m}-1}$, $(*)$

(if $\frac{-kc}{\eta}+\frac{m}{\eta c}\notin{}^{*}\mathcal{Z}$), where $\xi_{k,m}={}^{*}\exp(-2\pi i(\frac{-kc}{\eta}+\frac{m}{\eta c})))$

$=0$, as $\xi_{k,m}^{\eta c}=1$

We have that;

$\frac{-kc}{\eta}+\frac{m}{\eta c}\in{}^{*}\mathcal{Z}$

iff $\frac{-kc}{\eta}+\frac{m}{\eta c}=n$, some $n\in{}^{*}\mathcal{Z}$

iff $-kc^{2}+m=n\eta c$

iff $kc^{2}=m$, mod ηc, $(**)$

In this case, we have that;

$\frac{1}{\eta c}{}^{*}\sum_{j=0}^{\eta c-1}{}^{*}\exp(-2\pi ij(\frac{-kc}{\eta}+\frac{m}{\eta c}))$

$=\frac{1}{\eta c}{}^{*}\sum_{j=0}^{\eta c-1}1$

$=1$

Considering the expression $(*)$, and using $(**)$, we then obtain, that;

$\mathcal{F}_{\eta c}(f^{c^{2}})(m)=\mathcal{F}_{\eta}(f)(k)$

where $0\leq k\leq\eta-1$ is unique such that $kc^{2}\equiv m$, (mod ηc), and $0\leq m\leq\eta c-1$, see the note in Definition 0.33

The result then follows from the definition of an extension in Definition 0.33.

□

Lemma 0.38. *Let $f \in V(\overline{\Omega}_\eta)$, then;*

$\mathcal{F}_\eta(f'')(m) = \psi_\eta^2(m)\mathcal{F}_\eta(f)(m)$

for $m \in \mathcal{N}_\eta$, where;

$\psi_\eta(m) = \frac{\eta}{2}(exp_\eta(2\pi i\frac{m}{\eta}) - exp_\eta(-2\pi i\frac{m}{\eta}))$

Proof. We have, using the definition of the derivative in Lemma 0.43 and (iii) of Lemma 0.45, that;

$\mathcal{F}_\eta(f')(m)$

$= \int_{\overline{\Omega}_\eta} f'(x)exp_\eta(-2\pi imx)d\mu_\eta(x)$

$= -\int \overline{\Omega}_\eta f(x)exp'_\eta(-2\pi imx)d\mu_\eta(x)$

$= -\int \overline{\Omega}_\eta f(x)\frac{\eta}{2}(exp_\eta(-2\pi im(x+\frac{1}{\eta}) - exp_\eta(-2\pi im(x-\frac{1}{\eta})))d\mu_\eta(x)$

$= -\frac{\eta}{2}\int \overline{\Omega}_\eta f(x)exp_\eta(-2\pi imx)(exp_\eta(-2\pi i(\frac{m}{\eta}) - exp_\eta(2\pi i\frac{m}{\eta})))d\mu_\eta(x)$

$\psi_\eta(m)\int \overline{\Omega}_\eta f(x)exp_\eta(-2\pi imx)d\mu_\eta(x)$

$= \psi_\eta(m)\mathcal{F}_\eta(f)(m)$

where $\psi_\eta(m) = \frac{\eta}{2}(exp_\eta(2\pi i\frac{m}{\eta}) - exp_\eta(-2\pi i\frac{m}{\eta}))$

It follows that;

$\mathcal{F}_\eta(f'')(m)$

$= \psi_\eta(m)\mathcal{F}_\eta(f')(m)$

$= \psi_\eta^2(m)\mathcal{F}_\eta(f)(m)$

as required.

□

Lemma 0.39. *Let $g \in V(\overline{\Omega}_\eta)$, $f \in V(\overline{\Omega}_{\eta c})$, with corresponding G, F satisfying the nonstandard heat equations;*

$\frac{\partial G}{\partial t}-\frac{\partial^2 G}{\partial x^2}=0$, $(*)$

$\frac{\partial F}{\partial t}-\frac{1}{c^4}\frac{\partial^2 F}{\partial x^2}=0$, $(**)$

Let ψ_η be as in Lemma 0.38, with corresponding $\psi_{\eta c}$, then;

$(\mathcal{F}_\eta(G))(m,t)=[1+\frac{\psi_\eta^2(m)}{\nu}]^{[t\nu]}\mathcal{F}_\eta(g)(m)$

for $m\in\mathcal{N}_\eta$

$(\mathcal{F}_{\eta c}(F))(m,t)=[1+\frac{\psi_{\eta c}^2(m)}{\nu c^4}]^{[t\nu]}\mathcal{F}_{\eta c}(f)(m)$

for $m\in\mathcal{N}_{\eta c}$

Proof. Applying $\{\mathcal{F}_\eta,\mathcal{F}_{\eta c}\}$ to the equations $(*),(**)$, using the facts that $\{\mathcal{F}_\eta,\mathcal{F}_{\eta c}\}$ commute with $\frac{\partial}{\partial t}$, and Lemma 0.38, we obtain, for $m\in\mathcal{N}_\eta$, $m\in\mathcal{N}_{\eta c}$ respectively;

$\frac{\partial\mathcal{F}_\eta(G)}{\partial t}(m)-\mathcal{F}_\eta(\frac{\partial^2 G}{\partial x^2})(m)$

$=\frac{\partial\mathcal{F}_\eta(G)}{\partial t}(m)-\psi_\eta^2(m)\mathcal{F}_\eta(g)(m)=0$

$\frac{\partial\mathcal{F}_{\eta c}(F)}{\partial t}(m)-\frac{1}{c^4}\mathcal{F}_{\eta c}(\frac{\partial^2 F}{\partial x^2})(m)$

$=\frac{\partial\mathcal{F}_{\eta c}(F)}{\partial t}(m)-\frac{\psi_{\eta c}^2}{c^4}(m)\mathcal{F}_{\eta c}(f)(m)=0$

Using the definitions of the nonstandard derivative $\frac{\partial}{\partial t}$ in Definition 0.5 and Definition 0.29, we obtain, for $t\in\mathcal{T}_\nu$, $m\in\mathcal{N}_\eta$, $m\in\mathcal{N}_{\eta c}$ respectively, that;

$\mathcal{F}_\eta(G)(m,t+\frac{1}{\nu})=(1+\frac{\psi_\eta^2(m)}{\nu})\mathcal{F}_\eta(G)(m,t)$

$\mathcal{F}_{\eta c}(F)(m,t+\frac{1}{\nu})=(1+\frac{\psi_{\eta c}^2(m)}{\nu c^4})\mathcal{F}_{\eta c}(F)(m,t)$

and iterating, using the initial conditions $\mathcal{F}_\eta(G)(m,0)=\mathcal{F}_\eta(g)(m)$, $\mathcal{F}_{\eta c}(F)(m,0)=\mathcal{F}_{\eta c}(f)(m)$, we obtain;

$\mathcal{F}_\eta(G)(m,t)$

$=(1+\frac{\psi_\eta^2(m)}{\nu})^{[\nu t]}\mathcal{F}_\eta(G)(m,0)$

$= (1+\frac{\psi_\eta^2(m)}{\nu})^{[\nu t]}\mathcal{F}_\eta(g)(m)$

and;

$\mathcal{F}_{\eta c}(F)(m,t)$

$= (1+\frac{\psi_{\eta c}^2(m)}{\nu c^4})^{[\nu t]}\mathcal{F}_{\eta c}(F)(m,0)$

$= (1+\frac{\psi_{\eta c}^2(m)}{\nu c^4})^{[\nu t]}\mathcal{F}_{\eta c}(f)(m)$

□

Lemma 0.40. *If $g \in C^\infty([0,1])$, there exists a unique $G \in C^\infty(T)$, with $G_0 = g$, such that G satisfies the heat equation;*

$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2}$ $(*)$

on T^0.

Proof. Suppose, first, there exists such a solution G, then, applying $\mathcal{F}$ to $(*)$, we must have that;

$\mathcal{F}(\frac{\partial G}{\partial t} - \frac{\partial^2 G}{\partial x^2})(m,t) = 0$ $(t > 0, m \in \mathcal{Z})$

Differentiating under the integral sign, we have that;

$\mathcal{F}(\frac{\partial G}{\partial t}) = \frac{\partial \mathcal{F}(G)}{\partial t}(m,t)$, for $t > 0, m \in \mathcal{Z}$

Integrating by parts and using the fact that $G_t \in C^\infty([0,1])$, for $t > 0$, we have that;

$\mathcal{F}\frac{\partial^2 G}{\partial x^2} = -4\pi^2 m^2 \mathcal{F}(G)(m,t)$, for $t > 0, m \in \mathcal{Z}$

We thus obtain the sequence of ordinary differential equations, indexed by $m \in \mathcal{Z}$;

$\frac{\partial \mathcal{F}(G)}{\partial t} + 4\pi^2 m^2 \mathcal{F}(G)(m,t) = 0$ $(t > 0)$

As $G \in C(T)$, $G_t \to G_0$ pointwise , as $t \to 0$, and, using the Dominated Convergence Theorem, $\mathcal{F}(G)(m,t) \to \mathcal{F}(G)(m,0)$, as $t \to 0$, for each $m \in \mathcal{Z}$. By Picard's and Peano's Theorem, see [3], Chapter 4, this system of equations has a unique continuous solution, given by;

$\mathcal{F}(G)(m,t) = e^{-4\pi^2 m^2 t}\mathcal{F}(g)(m)\ (t \geq 0)$

As $G_t \in C^\infty([0,1])$, its Fourier series converges absolutely to G_t and, in particular, G_t is determined by its Fourier coefficients, for $t > 0$. It follows that G is a unique solution.

If $g \in C^\infty([0,1])$, its Fourier series converges absolutely to g, hence, the series;

$\sum_{m\in\mathcal{Z}} e^{-4\pi^2 m^2 t}\mathcal{F}(g)(m)e^{2\pi imx}$

are absolutely convergent for $t > 0$. It follows that G defined by;

$G(x,t) = \sum_{m\in\mathcal{Z}} e^{-4\pi^2 m^2 t}\mathcal{F}(g)(m)e^{2\pi imx}$

is a solution of the required form. □

We introduce more notation.

Definition 0.41. *If $\eta \in {}^*\mathcal{N}\setminus\mathcal{N}$, we let $\overline{\mathcal{V}_\eta} = {}^*\bigcup_{0\leq i\leq \eta-1}[-1+\frac{i}{\eta}, -1+\frac{i+1}{\eta})$, so that $\overline{\mathcal{V}_\eta} = {}^*[-1,1)$. We let $\mathcal{D}_\eta$ denote the associated $*$-finite algebra, generated by the intervals $[-1+\frac{i}{\eta}, -1+\frac{i+1}{\eta})$, for $0 \leq i \leq \eta-1$, and μ_η the associated counting measure defined by $\mu_\eta([-1+\frac{i}{\eta}, -1+\frac{i+1}{\eta})) = \frac{1}{\eta}$. We let $(\overline{\mathcal{V}_\eta}, L(\mathcal{D}_\eta), L(\mu_\eta))$ denote the associated Loeb space, see [1]. If $\nu \in {}^*\mathcal{N}\setminus\mathcal{N}$, we let $\overline{\mathcal{T}_\nu} = {}^*\bigcup_{0\leq i\leq \nu^2-1}[\frac{i}{\nu}, \frac{i+1}{\nu})$, so that $\overline{\mathcal{T}_\nu} = [0,\nu) \subset {}^*\mathcal{R}_{\geq 0}$. We let $\mathcal{C}_\eta$ denote the associated $*$-finite algebra, generated by the intervals $[\frac{i}{\nu}, \frac{i+1}{\nu})$, for $0 \leq i \leq \nu^2-1$, and λ_ν the associated counting measure defined by $\lambda_\nu([\frac{i}{\nu}, \frac{i+1}{\nu})) = \frac{1}{\nu}$. We let $(\overline{\mathcal{T}_\nu}, L(\mathcal{C}_\nu), L(\lambda_\nu))$ denote the associated Loeb space.*

We let $([0,1], \mathfrak{D}, \mu)$ denote the interval $[0,1]$, with the completion $\mathfrak{D}$ of the Borel field, and μ the restriction of Lebesgue measure. We let $(\mathcal{R}_{\geq 0}\cup\{+\infty\}, \mathfrak{C}, \lambda)$ denote the extended real half line, with the completion $\mathfrak{C}$ of the extended Borel field, and λ the extension of Lebesgue measure, with $\lambda(+\infty) = \infty$, see [3], Chapter 6.

We let $(\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, \mathcal{D}_\eta\times\mathcal{C}_\nu, \mu_\eta\times\lambda_\nu)$ be the associated product space and $(\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta\times\mathcal{C}_\eta), L(\mu_\eta\times\lambda_\nu))$ be the corresponding Loeb space. $(\overline{\mathcal{V}_\eta}\times$

$\overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta) \times L(\mathcal{C}_\nu), L(\mu_\eta) \times L(\lambda_\nu))$ is the complete product of the Loeb spaces $(\overline{\mathcal{V}_\eta}, L(\mathcal{D}_\eta), L(\mu_\eta))$ and $(\overline{\mathcal{T}_\nu}, L(\mathcal{C}_\nu), L(\lambda_\nu))$. Similarly, $([0,1] \times (\mathcal{R}_{\geq 0} \cup \{+\infty\}, \mathfrak{D} \times \mathfrak{C}, \mu \times \lambda)$ is the complete product of $([0,1], \mathfrak{D}, \mu)$ and $(\mathcal{R}_{\geq 0} \cup \{+\infty\}, \mathfrak{C}, \lambda)$.

We let $({}^*\mathcal{R}, {}^*\mathfrak{E})$ denote the hyperreals, with the transfer of the Borel field $\mathfrak{E}$ on $\mathcal{R}$. A function $f : (\overline{\mathcal{V}_\eta}, \mathcal{D}_\eta) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{E})$ is measurable, if $f^{-1} : {}^*\mathfrak{E} \rightarrow \mathcal{D}_\eta$. The same definition holds for $\mathcal{T}_\nu$. Similarly, $f : (\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, \mathcal{D}_\eta \times \mathcal{C}_\nu) \rightarrow ({}^*\mathcal{R}, {}^*\mathfrak{E})$ is measurable, if $f^{-1} : {}^*\mathfrak{E} \rightarrow \mathcal{D}_\eta \times \mathfrak{C}_\nu$. Observe that this is equivalent to the definition given in [1]. We will abbreviate this notation to $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{R}$, $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{R}$ or $f : \overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{R}$ is measurable, $(*)$. The same applies to $({}^*\mathcal{C}, {}^*\mathfrak{E})$, the hyper complex numbers, with the transfer of the Borel field $\mathfrak{E}$, generated by the complex topology. Observe that $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$, $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ $f : \overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ is measurable, in this sense, iff $Re(f)$ and $Im(f)$ are measurable in the sense of $(*)$.

We let $\overline{\mathcal{S}_{\eta,\nu}} = \overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}$ and;

$V(\overline{\mathcal{V}_\eta}) = \{f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C},\ f \text{ measurable } d(\mu_\eta)\}$

and, similarly, we define $V(\overline{\mathcal{T}_\nu})$. Let;

$V(\overline{\mathcal{S}_{\eta,\nu}}) = \{f : \overline{\mathcal{S}_{\eta,\nu}} \rightarrow {}^*\mathcal{C},\ f \text{ measurable } d(\mu_\eta \times \lambda_\nu)\}$

Lemma 0.42. *The identity;*

$i : (\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta \times \mathcal{C}_\nu), L(\mu_\eta \times \lambda_\nu))$

$\rightarrow (\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta) \times L(\mathcal{C}_\nu), L(\mu_\eta) \times L(\lambda_\nu))$

and the standard part mapping;

$st : (\overline{\mathcal{V}_\eta} \times \overline{\mathcal{T}_\nu}, L(\mathcal{D}_\eta) \times L(\mathcal{C}_\nu), L(\mu_\eta) \times L(\lambda_\nu)) \rightarrow [-\pi, \pi] \times \mathcal{R}_{\geq 0} \cup \{+\infty\}$

are measurable and measure preserving.

Proof. The proof is similar to work in [3], Chapter 6, using Caratheodory's Extension Theorem and Theorem 22 of [**?**].

□

Definition 0.43. *Discrete Partial Derivatives*

Let $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ *be measurable. We define the discrete derivative* f' *to be the unique measurable function satisfying;*

$f'(-1+\frac{i}{\eta}) = \frac{\eta}{2}(f(-1+\frac{i+1}{\eta}) - f(-1+\frac{i-1}{\eta}))$;

for $i \in {}^*\mathcal{N}_{1\leq i\leq \eta-2}$.

$f'(1-\frac{1}{\eta}) = \frac{\eta}{2}(f(-1) - f(1-\frac{2}{\eta}))$

$f'(-1) = \frac{\eta}{2}(f(-1+\frac{1}{\eta}) - f(1-\frac{1}{\eta}))$

Let $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ *be measurable. We define the discrete derivative* f' *to be the unique measurable function satisfying;*

$f'(\frac{i}{\nu}) = \nu(f(\frac{i+1}{\nu}) - f(\frac{i}{\nu}))$;

for $i \in {}^*\mathcal{N}_{0\leq i\leq \nu^2-2}$.

$f'(\frac{\nu-1}{\nu}) = 0$;

If $f : \overline{\mathcal{V}_\eta} \rightarrow {}^*\mathcal{C}$ *is measurable, then we define the shift (left, right);*

$f^{lsh}(-1+\frac{j}{\eta}) = f(-1+\frac{j+1}{\eta})$ *for* $0 \leq j \leq \eta - 2$

$f^{lsh}(\eta - \frac{1}{\eta}) = f(-1)$

$f^{rsh}(-1+\frac{j}{\eta}) = f(-1+\frac{j-1}{\eta})$ *for* $1 \leq j \leq \eta - 1$

$f^{rsh}(-1) = f(1-\frac{1}{\eta})$

If $f : \overline{\mathcal{T}_\nu} \rightarrow {}^*\mathcal{C}$ *is measurable, then we define the shift (left, right);*

$f^{lsh}(\frac{j}{\nu}) = f(\frac{j+1}{\nu})$ *for* $0 \leq j \leq \nu^2 - 2$

$f^{lsh}(\nu - \frac{1}{\nu}) = f(0)$

$f^{rsh}(\frac{j}{\nu}) = f(\frac{j-1}{\nu})$ *for* $1 \leq j \leq \nu^2 - 1$

$f^{rsh}(0) = f(\nu - \frac{1}{\nu})$

If $f:\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}\rightarrow{}^*\mathcal{C}$ is measurable. Then we define $\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial t}\}$ to be the unique measurable functions satisfying;

$\frac{\partial f}{\partial x}(-1+\frac{i}{\eta},t)=\frac{\eta}{2}(f(-1+\frac{i+1}{\eta},t)-f(-1+\frac{i-1}{\eta},t))$;

for $i\in{}^*\mathcal{N}_{1\leq i\leq\eta-1}, t\in\overline{\mathcal{T}_\nu}$

$\frac{\partial f}{\partial x}(1-\frac{1}{\eta},t)=\frac{\eta}{2}(f(-1,t)-f(1-\frac{2}{\eta},t))$

$\frac{\partial f}{\partial x}(-1,t)=\frac{\eta}{2}(f(-1+\frac{1}{\eta},t)-f(1-\frac{1}{\eta},t))$

$\frac{\partial f}{\partial t}(x,\frac{j}{\nu})=\nu(f(x,\frac{j+1}{\nu})-f(x,\frac{j}{\nu}))$;

for $j\in{}^*\mathcal{N}_{0\leq j\leq\nu^2-2}, x\in\overline{\mathcal{H}_\eta}$

$\frac{\partial f}{\partial t}(x,\nu-\frac{1}{\nu})=0$

We define $\{f^{lsh_x},f^{lsh_t},f^{rsh_x},f^{rsh_t}\}$ *by;*

$f^{lsh_x}(x_0,t_0)=(f_{t_0})^{lsh}(x_0)$

$f^{lsh_t}(x_0,t_0)=(f_{x_0})^{lsh}(t_0)$

$f^{rsh_x}(x_0,t_0)=(f_{t_0})^{rsh}(x_0)$

$f^{rsh_t}(x_0,t_0)=(f_{x_0})^{rsh}(t_0)$

where, if $(x_0,t_0)\in\overline{\mathcal{V}_\eta}\times\overline{\mathcal{T}_\nu}$;

$f_{t_0}(x_0)=f_{x_0}(t_0)=f(\pi\frac{[\frac{\eta x_0}{\pi}]}{\eta},\frac{[\nu t_0]}{\nu})$

Lemma 0.44. *If f is measurable, then so are;*

$$\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial t},\frac{\partial^2 f}{\partial x^2},f_x,f_t,f^{lsh_x},f^{lsh_t},f^{rsh_x},f^{rsh_t},f^{lsh_x^2},f^{lsh_t^2},f^{rsh_x^2},f^{rsh_t^2}\}$$

Proof. This follows immediately, by transfer, from the corresponding result for the discrete derivatives and shifts of discrete functions $f:\mathcal{H}_n\times\mathcal{T}_m\rightarrow\mathcal{C}$, where $n,m\in\mathcal{N}$, see [3], Chapter 6. □

Lemma 0.45. *Let $g,h:\overline{\mathcal{V}_\eta}\rightarrow{}^*\mathcal{C}$ be measurable. Then;*

(i). $\int_{\overline{\mathcal{V}_\eta}} g'(y)d\mu_\eta(y) = 0$

(ii). $(gh)' = g'h^{lsh} + g^{rsh}h'$

(iii). $\int_{\overline{\mathcal{V}_\eta}}(g'h)(y)d\mu_\eta(y) = -\int_{\overline{\mathcal{V}_\eta}} gh'd\mu_\eta(y)$

(iv). $\int_{\overline{\mathcal{V}_\eta}} g(y)d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}} g^{lsh}(y)d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}} g^{rsh}(y)d\mu_\eta(y)$

(v). $(g')^{rsh} = (g^{rsh})'$, $(g')^{lsh} = (g^{lsh})'$

(vi). $\int_{\overline{\mathcal{V}_\eta}}(g''h)(y)d\mu_\eta(y) = \int_{\overline{\mathcal{V}_\eta}}(gh'')(y)d\mu_\eta(y)$

Proof. In the first part, for (i), we have, using Definition 0.43, that;

$\int_{\overline{\mathcal{V}_\eta}} g'(y)d\mu_\eta(y)$

$= \frac{1}{\eta}[^*\sum_{1\leq j\leq \eta-2}\frac{\eta}{2}[g(-1+(\frac{j+1}{\eta})) - g(-1+(\frac{j-1}{\eta}))]$

$+\frac{\eta}{2}[g(-1+\frac{1}{\eta}) - g(1-\frac{1}{\eta})] + \frac{\eta}{2}[g(-1) - g(\pi-\frac{2}{\eta})]] = 0$

For (ii), we calculate;

$(gh)'(-1+\frac{j}{\eta}) =$

$= \frac{\eta}{2}(gh(-1+\frac{j+1}{\eta}) - gh(-1+\frac{j-1}{\eta}))$

$= \frac{\eta}{2}(gh(-1+\frac{j+1}{\eta}) - g(-1+\frac{j-1}{\eta})h(-1+\frac{j+1}{\eta})$

$+g(-1+\frac{j-1}{\eta})h(-1+\frac{j+1}{\eta}) - gh(-1+\frac{j-1}{\eta}))$

$= g'(-1+\frac{j}{\eta})h(-1+\frac{j+1}{\eta}) + g(-1+\frac{j-1}{\eta})h'(-1+\frac{j}{\eta})$

$= (g'h^{lsh} + g^{rsh}h')(-1+\frac{j}{\eta})$

Combining $(i),(ii)$, we have;

$0 = \int_{\overline{\mathcal{V}_\eta}}(gh)'(x)d\mu_\eta(x)$

$= \int_{\overline{\mathcal{V}_\eta}}(g'h^{lsh} + g^{rsh}h')(x)d\mu_\eta(x)$

and, rearranging, that;

$\int_{\overline{\mathcal{V}_\eta}}(g'h^{lsh})d\mu_\eta = -\int_{\overline{\mathcal{V}_\eta}}(g^{rsh}h')d\mu_\eta$

For (iv), we have that;

$\int_{\overline{\mathcal{V}_\eta}} g^{rsh}(y)d\mu_\eta(y)$

$= \frac{1}{\eta}({}^*\sum_{0\leq j\leq \eta-1} g^{rsh}(-1+\frac{j}{\eta}))$

$= \frac{1}{\eta}({}^*\sum_{1\leq j\leq \eta-2} g(-1+\frac{j-1}{\eta}) + g(1-\frac{1}{\eta}))$

$= \frac{1}{\eta}({}^*\sum_{0\leq j\leq \eta-1} g(-1+\frac{j}{\eta})$

$= \int_{\overline{\mathcal{V}_\eta}} g(y)d\mu_\eta(y)$

A similar calculation holds with g^{lsh}. For (v), we have for $2\leq j\leq \eta-2$;

$(g')^{rsh}(-1+\pi\frac{j}{\eta})$

$= g'(-1+\frac{j-1}{\eta})$

$= \frac{\eta}{2}(g(-1+\frac{j}{\eta}) - g(-1+\frac{j-2}{\eta}))$

$(g^{rsh})'(-1+\frac{j}{\eta})$

$= \frac{\eta}{2}(g^{rsh}(-1+\frac{j+1}{\eta}) - g^{rsh}(-1+\frac{j-1}{\eta}))$

$= \frac{\eta}{2}(g(-1+\frac{j}{\eta}) - g(-1+\pi\frac{j-2}{\eta}))$

Similar calculations hold for the remaining j to give that $(g')^{rsh} = (g^{rsh})'$, and the calculation $(g')^{lsh} = (g^{lsh})'$ is also similar.

It follows that;

$\int_{\overline{\mathcal{V}_\eta}}(g'h)d\mu_\eta$

$= \int_{\overline{\mathcal{V}_\eta}}(g'(h^{rsh})^{lsh})d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(g^{rsh}(h^{rsh})')d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(g^{rsh}(h')^{rsh})d\mu_\eta$

$= -\int_{\overline{\mathcal{V}_\eta}}(gh'))d\mu_\eta$

which gives (iii), using $(iv),(v)$. The calculation (vi) is then immediate from (iii).

□

Lemma 0.46. *Similar results to Lemma 0.45 hold for* $\{lsh_x, rsh_x, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\}$. *Namely, if* $g,h:\overline{\mathcal{S}_{\eta,\nu}}\rightarrow {}^*\mathcal{C}$ *are measurable. Then;*

(i). $\int_{\overline{\mathcal{S}_{\eta,\nu}}}\frac{\partial g}{\partial x}d(\mu_\eta\times\lambda_\nu)=0$

(ii). $\frac{\partial gh}{\partial x}=\frac{\partial g}{\partial x}h^{lsh_x}+g^{rsh_x}\frac{\partial h}{\partial x}$

(iii). $\int_{\overline{\mathcal{S}_{\eta,\nu}}}\frac{\partial g}{\partial x}hd(\mu_\eta\times\lambda_\nu)=-\int_{\overline{\mathcal{S}_{\eta,\nu}}}g\frac{\partial h}{\partial x}d(\mu_\eta\times\lambda_\nu)$

(iv). $\int_{\overline{\mathcal{S}_{\eta,\nu}}}gd(\mu_\eta\times\lambda_\nu)=\int_{\overline{\mathcal{S}_{\eta,\nu}}}g^{lsh_x}d(\mu_\eta\times\lambda_\nu)=\int_{\overline{\mathcal{S}_{\eta,\nu}}}g^{rsh_x}d(\mu_\eta\times\lambda_\nu)$

(v). $(\frac{\partial g}{\partial x})^{lsh_x}=\frac{\partial(g^{lsh_x})}{\partial x}$, *and, similarly, with* rsh_x *replacing* lsh_x.

(vi). $\int_{\overline{\mathcal{S}_{\eta,\nu}}}(\frac{\partial^2 g}{\partial x^2}h)d(\mu_\eta\times\lambda_\nu)=\int_{\overline{\mathcal{S}_{\eta,\nu}}}(g\frac{\partial^2 h}{\partial x^2})d(\mu_\eta\times\lambda_\nu)$ $(*)$

Proof. For (i), using (i) from the argument in Lemma 0.45, we have;

$\int_{\overline{\mathcal{S}_{\eta,\nu}}}\frac{\partial g}{\partial x}d(\mu_\eta\times\lambda_\nu)$

$=\int_{\overline{\mathcal{V}_\eta}}(\int_{\overline{\mathcal{T}_\nu}}(\frac{\partial g}{\partial x})_t d\mu_\eta)d\lambda_\nu(t)$

$=\int_{\overline{\mathcal{V}_\eta}}(\int_{\overline{\mathcal{T}_\nu}}(\frac{\partial g_t}{\partial x})d\mu_\eta)d\lambda_\nu(t)$

$=\int_{\overline{\mathcal{T}_\nu}}0d\lambda_\nu(t)=0$

The proofs of $(ii),(iii),(iv)$ are similar to Lemma 0.45, relying on the result of (i). (v) follows easily from Definitions 0.43 and (vi) follows, repeating the result of (iii), and applying (v).

□

References

[1] Conversion from Nonstandard to Standard Measure Spaces and Applications in Probability Theory, Peter Loeb, Transactions of the American Mathematical Society, (1975).

[2] Markov Chains, J. R. Norris, Cambridge Series in Statistical and Probabilistic Mathematics, (1997).

[3] Advances in Nonstandard Analysis, Tristram de Piro, (2017).

[4] Nonstandard Methods for Solving the Heat Equation, Tristram de Piro, (2017).

[5] A Simple Proof of the Martingale Representation Theorem using Nonstandard Analysis, Tristram de Piro, (2015).

[6] A Simple Proof of the Uniform Convergence of Fourier Series using Nonstandard Analysis, Tristram de Piro, (2013).
[**?**] Nonstandard Methods for Solving Schrodinger's Equation, Tristram de Piro, in progress, (2019).

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