AN NONSTANDARD VERSION OF DIRICHLET'S THEOREM

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Abstract. We prove a nonstandard version of Dirichlet's Theorem on the pointwise convergence of Fourier Series, a uniform convergence result was shown in [2]. We apply this result in [].

Definition 0.1. *We let $f:(-1.1)\rightarrow\mathcal{R}$ be analytic, and suppose that $f(-1)$ and $f(1)$ are defined, with upper and lower limits existing,$f^{-}=lim_{x\rightarrow -1}f(x)$ and $f^{+}=lim_{x\rightarrow 1}f(x)$. We let $C_f=\frac{f^{+}+f^{-}}{2}$. We let $V(\overline{V}_\eta)={}^{*}[-1,1)$, with the $*$-topology generated by $\{[\frac{i}{\eta},\frac{i+1}{\eta}),-\eta\leq i\leq\eta-1\}$, with $\eta\in{}^{*}\mathcal{N}$ prime. If $g\in V(\overline{V}_\eta)$, we use the forward derivative;*

$g^{D}(x)=\eta(g(x+\frac{1}{\eta})-g(x))$, *for* $x\in V(\overline{V}_\eta)\setminus[\frac{\eta-1}{\eta},1)$

$g^{D}(\frac{\eta-1}{\eta})=\eta(g(-1)-g(\frac{\eta-1}{\eta}))$

We use the notation in [3].

Lemma 0.2. *With f as in the above definition, we have that f_η^{D} is finitely bounded on $[\frac{1}{\eta},\frac{\eta-1}{\eta})$*

Proof. For $x\in(-1,1)$ and $\epsilon>0$, with $x+\epsilon\in(-1,1)$, we have, using Taylor's Theorem, that;

$f(x+\epsilon)=f(x)+\epsilon f'(x)+\frac{\epsilon^{2}f''(c)}{2}$

and $c\in(x,x+\epsilon)$. It follows that;

$|\frac{f(x+\epsilon)-f(x)}{\epsilon}-f'(x)|\leq|\epsilon f''(c)|\leq D\epsilon$

where $D\in\mathcal{R}$as f'' is bounded on $(-1,1)$. Taking ϵ to be the infinitesimal $\frac{1}{\eta}$, and transferring this result, we obtain that;

$f^{D}(x)\simeq(f')_\eta(x)$

for $x\in[\frac{1}{\eta},\frac{\eta-1}{\eta})$. As $(f')_\eta$ is bounded by assumption, we obtain the result.

□

Definition 0.3. *For $g\in V(\overline{V}_\eta)$, and $k\in\mathcal{Z}_\eta$, with $|k|\leq\eta-1$ we define the nonstandard partial Fourier sum, by;*

$$S(k,g)=\tfrac{1}{2}{}^*\textstyle\sum_{|m|\leq k}\mathcal{F}_\eta(f)(m)exp_\eta(\pi ixm)$$

and the Dirichlet kernel by;

$$D_k(x)={}^*\textstyle\sum_{|m|\leq k}exp_\eta(\pi ixm)$$

Lemma 0.4. *We have that;*

$$D_k(x)=\frac{sin_\eta(\pi x(k+\frac{1}{2}))}{sin_\eta(\frac{\pi x}{2})},\ for\ x\not in[0,\tfrac{1}{\eta})$$

$$D_k(0)=2k+1$$

Proof. We have, when $x\neq 0$;

$$D_k(x)={}^*\textstyle\sum_{|m|\leq k}exp_\eta(\pi ixm)$$

$$=exp_\eta(-\pi ixk)({}^*sum_{m=0}^{2k}exp_\eta(\pi ixm))$$

$$=exp_\eta(-\pi ixk)(\tfrac{exp_\eta(\pi ix)^{2k+1}-1}{exp_\eta(\pi ix)-1})$$

$$=\frac{exp_\eta(\frac{\pi ix}{2})(exp_\eta(-\pi ix(k+\frac{1}{2}))-exp_\eta(\pi ix(k+\frac{1}{2})))}{exp_\eta(\frac{\pi ix}{2})(exp_\eta(-\frac{\pi ix}{2}))-exp_\eta(\frac{\pi ix}{2})}$$

$$=\frac{-2isin_\eta(\pi x(k+\frac{1}{2}))}{-2isin_\eta(\frac{\pi x}{2})}$$

$$=\frac{sin_\eta(\pi x(k+\frac{1}{2}))}{sin_\eta(\frac{\pi x}{2})}$$

The calculation when $x=0$ is clear.

□

Lemma 0.5. *For $g\in V(\overline{V}_\eta)$;*

$$S(k,g)\simeq\frac{1}{2}\int_{[0,1)_\eta}D_k(v)(g(x+v)+g(x-v))d\mu_\eta(v)$$

Proof. Using the definitions above, the nonstandard definition of the convolution, and the symmetry $D_k(v)=D_k(-v)$, for $v\in[-1,1)_\eta\setminus[-1,\frac{1-\eta}{\eta})$, and Lemma 0.4, we for $k \in {}^*\mathcal{Z}$, with $|k| \leq \eta - 1$, that;

$S(k,g) = \frac{1}{2}{}^*\sum_{|m|\leq k}\mathcal{F}_\eta(f)(m)exp_\eta(\pi ixm)$

$= \frac{1}{2}{}^*\sum_{|m|\leq k}(\int_{[-1,1)_\eta} g(x)exp_\eta(-\pi ixm)d\mu_\eta(x)))exp_\eta(\pi ixm)$

$= \frac{1}{2}\int_{[-1,1)_\eta} g(y)({}^*\sum_{|m|\leq k}exp_\eta(\pi ixm)exp_\eta(-\pi imy))d\mu)\eta(y)$

$= \frac{1}{2}\int_{[-1,1)_\eta} D_k(x-y)g(y)d\mu_\eta(y)$

$= \frac{1}{2}(D_k * g)$

$= \frac{1}{2}(g * D_k)$

$= \frac{1}{2}\int_{[-1,1)_\eta} g(x-v)D_k(v)d\mu_\eta(v)$

$= \frac{1}{2\eta}g(x+1)D_k(-1) + \frac{1}{2}\int_{[\frac{1-\eta}{\eta},1)_\eta} g(x-v)D_k(v)d\mu_\eta(v)$

$= \frac{1}{2\eta}g(x+1)D_k(-1) + \frac{1}{2}\int_{[\frac{1-\eta}{\eta},1)_\eta} g(x+v)D_k(-v)d\mu_\eta(v)$

$= \frac{1}{2\eta}g(x+1)D_k(-1) + \frac{1}{2}\int_{[\frac{1-\eta}{\eta},1)_\eta} g(x+v)D_k(v)d\mu_\eta(v)$

$= \frac{1}{2\eta}g(x+1)D_k(-1) - \frac{1}{2\eta}g(x-1)D_k(-1) + \frac{1}{2}\int_{[-1,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v)$

$= \frac{(-1)^k(g(x+1)-g(x-1))}{2\eta} + \frac{1}{2}\int_{[-1,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v)$

$\simeq \frac{1}{2}\int_{[-1,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v)$

as g is bounded. It follows that;

$S(k,g) \simeq \frac{1}{2}(\int_{[-1,0)_\eta} g(x+v)D_k(v)d\mu_\eta(v) + \int_{[0,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v))$

$= \frac{D_k(-1)g(x-1)}{2\eta} + \frac{1}{2}(\int_{[\frac{1-\eta}{\eta},0)_\eta} g(x+v)D_k(v)d\mu_\eta(v) + \int_{[0,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v))$

$= \frac{D_k(-1)g(x-1)}{2\eta} + \frac{1}{2}(\int_{(0,\frac{\eta-1}{\eta}]_\eta} g(x-v)D_k(-v)d\mu_\eta(v) + \int_{[0,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v))$

$= \frac{D_k(-1)g(x-1)}{2\eta} + \frac{1}{2}(\int_{(0,1)_\eta} g(x-v)D_k(v)d\mu_\eta(v) + \int_{[0,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v))$

$= \frac{D_k(-1)g(x-1)}{2\eta} - \frac{D_k(0)g(x)}{2\eta} + \frac{1}{2}(\int_{[0,1)_\eta} g(x-v)D_k(v)d\mu_\eta(v) + \int_{[0,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v))$

$= \frac{(-1)^k g(x-1)}{2\eta} - \frac{(2k+1)g(x)}{2\eta} + \frac{1}{2}(\int_{[0,1)_\eta} g(x-v)D_k(v)d\mu_\eta(v) + \int_{[0,1)_\eta} g(x+v)D_k(v)d\mu_\eta(v))$

$= \frac{(-1)^k g(x-1)-(2k+1)g(x)}{2\eta} + \frac{1}{2}(\int_{[0,1)_\eta}(g(x+v)+g(x-v))D_k(v)d\mu_\eta(v))$

$\simeq \frac{1}{2}(\int_{[0,1)_\eta}(g(x+v)+g(x-v))D_k(v)d\mu_\eta(v))$

□

Lemma 0.6. *If f as above is analytic on $(-1,1)$, then for $r \in \mathcal{R}_{>0}$, there exists $N_r \in {}^*\mathcal{N}$, with $N_r \leq \eta - 1$, such that;*

$\int_{[r,1)_\eta} f(x)D_k(x)d\mu_\eta(x) \simeq 0$, *for $k \leq N_r$, k infinite.*

Proof. We have that;

$\int_{[r,1)_\eta} f(x)D_k(x)d\mu_\eta(x)$

$= \int_{[r,1)_\eta} \frac{f(x)sin(\pi(k+\frac{1}{2})x)}{sin\frac{\pi x}{2}} d\mu_\eta(x)$

A simple calculation shows that $\frac{1}{sin(\frac{\pi x}{2})}$ is analytic on $(r,1)$ for $0 < r < 1$, hence $\frac{f}{sin(\frac{\pi x}{2})}$ is analytic on $(r,1)$. Using the alternation method and Lemma 6 from the paper [4], we obtain the result. □

Definition 0.7. *We define the Fourier kernel by;*

$F_k(x) = \frac{2sin_\eta(\pi x(k+\frac{1}{2}))}{\pi x_\eta}$, *for $x \in [-1,1)_\eta \setminus [0,\frac{1}{\eta})$*

$F_k(0) = 2k+1$

Lemma 0.8. *For $g \in V([0,1)_\eta)$, with g analytic, there exists $N \in {}^*\mathcal{N}$, infinite, $|N| \leq \eta - 1$, such that for all $k \in {}^*\mathcal{N}$, with $0 \leq k \leq N$, we have that;*

$\int_{[0,1)_\eta} g(x)F_k(x)d\mu_\eta(x) \simeq \int_{[0,1)_\eta} g(x)D_k(x)d\mu_\eta(x)$

Proof. We have that $\frac{1}{sin_\eta(\frac{\pi x}{2})} - \frac{2}{\pi x}$ is analytic on $(0,1)$, hence, so is $h = \frac{g}{sin_\eta(\frac{\pi x}{2})} - \frac{2g}{\pi x}$. Moreover;

$\int_{[0,1)_\eta} g(x)D_k(x)d\mu_\eta(x)$

$= \int_{[0,1)_\eta} g(x)(D_k - F_k + F_k)d\mu_\eta(x)$

$= \int_{[0,1)_\eta} g(x)(D_k - F_k)d\mu_\eta(x)$

$+ \int_{[0,1)_\eta} g(x)F_k d\mu_\eta(x)$

$= \int_{[0,1)_\eta} h(x)sin_\eta(\pi(k+\frac{1}{2})x)d\mu_\eta(x)$

$+ \int_{[0,1)_\eta} g(x)F_k d\mu_\eta(x)$

$\simeq \int_{[0,1)_\eta} g(x)F_k d\mu_\eta(x)$

using the argument of Lemma 0.6.

□

Lemma 0.9. *For $g \in V(\overline{V}_\eta)$ analytic on $[-1,1)\eta$ and $x \in [-1,1)_\eta$, there exists an infinite N_x, as in Lemma 0.8, such that, for infinite $|k| \leq N_x$*

$S(k,g) \simeq \frac{1}{2}\int_{[0,1)_\eta} F_k(v)(g(x+v)+g(x-v))d\mu_\eta(v)$

Proof. Using Lemma 0.5, we have that;

$S(k,g) \simeq \frac{1}{2}\int_{[0,1)_\eta} D_k(v)(g(x+v)+g(x-v))d\mu_\eta(v)$

where $(g(x+v)+g(x-v))$ is the transfer of an analytic function on $[0,1)_\eta$. Using Lemma 0.8, there exists $N_x \in {}^*\mathcal{N}$, such that;

$\frac{1}{2}\int_{[0,1)_\eta} D_k(v)(g(x+v)+g(x-v))d\mu_\eta(v)$

$\simeq \frac{1}{2}\int_{[0,1)_\eta} F_k(v)(g(x+v)+g(x-v))d\mu_\eta(v)$

for all k infinite, with $|k| \leq N_x$. □

Lemma 0.10. *For infinite k, with $|k| \leq N_x$, we have that*

$\frac{1}{2}\int_{[0,1)_\eta} F_k(v)(g(x+v)+g(x-v))d\mu_\eta(v)$

$= \frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^+}{v}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x-v)-g_x^-)}{v}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^+}{v}d\mu_\eta(v)$

Proof. Using the definition of $F_k(v)$ and rearranging, we have that;

$\frac{1}{2}\int_{[0,1)_\eta} F_k(v)(g(x+v)+g(x-v))d\mu_\eta(v)$

$=\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^++g_x^++g(x-v)-g_x^-+g_x^-)}{v}d\mu_\eta(v)$

$=\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^+}{v}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x-v)-g_x^-)}{v}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^+}{v}d\mu_\eta(v)$

as required. □

Lemma 0.11. *Suppose that $g_x^+ = g_x^-$, then there exists an infinite $N \in {}^*\mathcal{N}$, such that for infinite k, with $|k| \leq N$;*

$\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^+}{v}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^-}{v}d\mu_\eta(v)$

$\simeq \frac{(g_x^++g_x^-)}{2}$

Proof. Using the alternation argument, we take $N \leq \frac{\eta}{2}$ we have;

$\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^+}{v}d\mu_\eta(v)$

$=\frac{2}{\pi}\frac{g_x^+}{2}\int_{[0,1)_\eta} \frac{sin_\eta(\pi(k+\frac{1}{2})v)}{v}d\mu_\eta(v)$

$=\frac{2}{\pi}\frac{\pi g_x^+}{4}\int_{[0,1)_\eta} \frac{sin_\eta(\pi(k+\frac{1}{2})v)}{\frac{\pi v}{2}}d\mu_\eta(v)$

$\simeq \frac{2}{\pi}\frac{\pi g_x^+}{4}\int_{[0,1)_\eta} \frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

Similarly;

$\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^-}{v}d\mu_\eta(v)$

$\simeq \frac{2}{\pi}\frac{\pi g_x^-}{4}\int_{[0,1)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

It follows that;

$\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^+}{v}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{1}{2}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{g_x^-}{v}d\mu_\eta(v)$

$\simeq \frac{2}{\pi}\frac{\pi g_x^+}{4}\int_{[0,1)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{\pi g_x^-}{4}\int_{[0,1)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

$= \frac{2}{\pi}\frac{\pi g_x^+}{4}\int_{[0,1)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{\pi g_x^-}{4}\int_{(\frac{-(\eta-1)}{\eta},0]_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})-v)}{sin_\eta(\frac{\pi-v}{2})}d\mu_\eta(v)$

$= \frac{2}{\pi}\frac{\pi g_x^+}{4}\int_{[0,1)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{\pi g_x^-}{4}\int_{[-1,0)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

$= \frac{2}{\pi}\frac{\pi g_x^+}{4}\int_{[0,1)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

$+\frac{2}{\pi}\frac{\pi g_x^-}{4}\int_{[-1,0)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)+\frac{2}{\pi}\frac{\pi g_x^-}{4}\frac{2k+1}{\eta}-\frac{(-1)^k}{\eta}$

$\simeq \frac{2}{\pi}\frac{\pi g_x^+}{4}\int_{[-1,1)_\eta}\frac{sin_\eta(\pi(k+\frac{1}{2})v)}{sin_\eta(\frac{\pi v}{2})}d\mu_\eta(v)$

$= \frac{2}{\pi}\frac{2\pi g_x^+}{4}$

$= \frac{2}{\pi}\frac{\pi g_x^+}{2}$

$= g_x^+ = \frac{g_x^+ + g_x^-}{2}$

as $\int[-1,1)_\eta F_k(x)d\mu_\eta(x)=2$, for all k.

by the choice of N.

□

Lemma 0.12. *For infinite k, with $k\leq min(N,N_x)$, and $g_x^+=g_x^-$, we have that;*

$S_k(g) \simeq g_x^+$

$+\frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)$

$+\frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x-v)-g_x^-)}{v}d\mu_\eta(v)$

Proof. This follows by combining the previous Lemmas 0.9,0.10 and 0.11. □

Lemma 0.13. *If g is analytic on $(-1,1)$, with g right differentiable at -1, and g left differentiable at 1, and $g_1^- = g_{-1}^+$, then, there exists M_x infinite, $|M_x| \leq \eta - 1$, such that for all $0 \leq \frac{M_x}{2} \leq k \leq M_x$, if $x \in [-1,1)_\eta$, with ${}^\circ x \subset \{-1,1\}$;*

$\frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)$

$\simeq \frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x-v)-g_x^-)}{v}d\mu_\eta(v)$

$\simeq 0$

Proof. Suppose that $x = 1$, then for $v \in [0,1)_\eta$. with $v \in \mathcal{V}_0$, we have, as g is analytic, using Taylor's theorem, that $\frac{g(x+v)-g_x^+}{v} = \frac{g(x+v)-g_x^-}{v} \simeq g^{',+}(-1)$, and, for $v \in [0,1)_\eta$, with $v \in \mathcal{V}_1$, we have that $\frac{g(x+v)-g_x^+}{v} \simeq -g^{',-}(1)$, both of which are bounded. For $v_0 \in \mathcal{R}_{>0}$, we have that, $\frac{(g(x+v)-g_x^+)}{v}$ is analytic on $[0,1)_\eta \setminus ([0,v_0) \cup [1-v_0,1))_\eta$. It follows that, for infinite κ ;

$\frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)$

$\simeq \frac{1}{\pi}\int_{[0,1)_\eta \setminus ([0,\frac{1}{\kappa}) \cup [1-\frac{1}{\kappa},1))} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)$

In particular, for a given $\epsilon_r = \frac{1}{r}$, with $r \in \mathcal{N}$, using underflow, uniformly in k, as $|sin_\eta(\pi(k+\frac{1}{2})v) \leq 1|$, there exists $n_r \in \mathcal{N}$, for which;

$|\frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)$

$-\frac{1}{\pi}\int_{[0,1)_\eta \setminus ([0,\frac{1}{n_r}) \cup [1-\frac{1}{n_r},1))} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)| < \epsilon_r$

Then, using the alternation argument above, we can find an infinite N_r, such that for $0 \leq \frac{N_r}{2} \leq k \leq N_r$;

$\frac{1}{\pi}\int_{[\frac{1}{n_r},1-\frac{1}{n_r})_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v) \simeq 0$

Extending the sequence $\{N_r : r \in \mathcal{N}\}$ by countable comprehension, we can for infinite $r' \in {}^*\mathcal{N}$, find an infinite $N_{r'}$ such that;

(i). For $0 \leq \frac{N_{r'}}{2} \leq k \leq N_{r'}$;

$\frac{1}{\pi}\int_{[\frac{1}{n_{r'}},1-\frac{1}{n_{r'}})_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v) \simeq 0$

(ii). $|\frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)$

$-\frac{1}{\pi}\int_{[0,1)_\eta\setminus([0,\frac{1}{n_{r'}})\cup[1-\frac{1}{n_{r'}},1))} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v)| < \epsilon_{r'} \simeq 0$

It follows that;

$\frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x+v)-g_x^+)}{v}d\mu_\eta(v) \simeq 0$

A similar argument produces $N_{r''}$ to show that;

$\frac{1}{\pi}\int_{[0,1)_\eta} sin_\eta(\pi(k+\frac{1}{2})v)\frac{(g(x-v)-g_x^-)}{v}d\mu_\eta(v) \simeq 0$

We can then take $M_x = min(N_{r'}, N_{r''})$. The proofs for $x = -1$, and $^\circ x \subset \{-1, 1\}$ are similar and left to the reader.

□

Lemma 0.14. *For g analytic on $(0,1)$, g finite, left and right differentiable at $\{1,-1\}$, with the limits $g_x^+ = g_x^-$ and $^\circ x \subset \{-1,1\}$ there exists $P_x \in {}^*\mathcal{N}$, with P_x infinite, such that for all $0 \leq k \leq P_x$, k infinite;*

$S_k(g) \simeq g_x^+ = g_x^- = g_x$

If g is continuous at 1, then $S_k(g) \simeq g(x)$.

Proof. The proof is clear by the previous Lemmas 0.12 and 0.13,

□

Definition 0.15. *We define the standard sawtooth function on $[-1,1]$ by;*

$S(x) = x$

and the Fourier series $S(S)$ by;

$lim_{k\rightarrow\infty}S_m(S)$

where $S_k(S)=\frac{1}{2}\sum_{|m|\leq k}\mathcal{F}(S)(k)e^{\pi ikx}$

and the nonstandard sawtooth function on $[-1,1)_\eta$ *by;*

$S(x)=x_\eta$

Lemma 0.16. *For $m\in\mathcal{Z}$, we have that;*

$\mathcal{F}(S)(m)=\frac{2i(-1)^{|m|}}{\pi m}$

and $S(S)=\frac{2}{\pi}\sum_{m>0}\frac{(-1)^{m+1}sin(\pi mx)}{m}$

Proof. We compute, for $m\neq 0$;

$\mathcal{F}(S)(m)=\int_{-1}^{1}S(x)e^{-\pi imx}dx$

$=\int_{-1}^{1}xe^{-\pi imx}dx$

$=[\frac{xe^{-\pi imx}}{-\pi im}]_{-1}^{1}+\frac{1}{\pi im}\int_{-1}^{1}e^{-\pi imx}dx$

$=\frac{e^{-\pi im}+e^{\pi im}}{-\pi im}+\frac{1}{\pi im}[e^{\frac{-\pi imx}{-\pi im}}]_{-1}^{1}$

$=\frac{2cos(\pi m)}{-\pi mi}+\frac{1}{\pi^2m^2}(e^{\pi im}-e\pi im)$

$=\frac{2(-1)^m}{-\pi mi}$

$=\frac{2i(-1)^m}{\pi m}$

We have $\mathcal{F}(S)(0)=\int_{-1}^{1}xdx=0$

Then;

$S_k(S)=\frac{1}{2}\sum_{|m|\leq k,m\neq 0}\frac{2i(-1)^m}{\pi m}e^{\pi imx}$

$=\frac{i}{\pi}(\sum_{|m|\leq k,m\neq 0}\frac{cos(\pi mx)(-1)^m}{m}+\frac{isin(\pi mx)(-1)^m}{m})$

$= \frac{2}{\pi}\sum_{0<m\leq k}\frac{sin(\pi mx)(-1)^{m+1}}{m}$

so that;

$S(S) = \frac{2}{\pi}\sum_{m>0}\frac{(-1)^{m+1}sin(\pi mx)}{m}$

□

Lemma 0.17. *Let f be analytic on $(-1,1)$, such that the left and right derivatives exist at -1 and 1 respectively. Then there exists g analytic on $(-1,1)$ and constants λ,μ,ν such that;*

$g = \lambda f + \mu S + \nu$

with $g_1^+ = g_{-1}^-$.
A similar result holds for f_η, replacing g by g_η and S by S_η.

Proof. We have that $S_1^+ = 1$ and $S_{-1}^- = -1$. Suppose that $f_1^+ + f_1^- \neq 0$ and $f_1^+ - f_1^- \neq 0$, $(*)$, then we solve the equations;

$1 = \lambda f_1^+ + \mu S_1^+ = \lambda f_1^+ + \mu = \lambda f_{-1}^- + \mu S_{-1}^- = \lambda f_{-1}^- - \mu$
to obtain that;

$\lambda = \frac{2}{f_1^+ + f_1^-} \neq 0$

$\mu = \frac{f_1^- - f_1^+}{f_1^+ + f_1^-} \neq 0$

Setting $g = \lambda f + \mu S$ gives the result. For the general case, if $f_1^+ = f_{-1}^-$, we can take $g = f$. Otherwise, we can find a constant c, such that $(*)$ is satisfied for $f+c$. Applying the previous calculation, we can find g such that $g = \lambda(f+c)+\mu S$. Taking $\nu = \lambda c+\mu$ gives the result. The final claim is clear, taking the measurable versions.

□

Lemma 0.18. *For $\{f,g\} \subset V([-1,1)_\eta)$, we have that;*

$\int_{[-1,1)_\eta} fg^D d\mu_\eta = -\int_{[-1,1)_\eta} f^D g^{lsh} d\mu_\eta$

Proof. We have $(fg)^D = f^D g^{lsh} + fg^D$, so that;

$0 = \int_{[-1,\frac{\eta-1}{\eta}]}(fg)^D d\mu_\eta$

$= \int_{[-1,\frac{\eta-1}{\eta}]} f^D g^{lsh} + f g^D d\mu_\eta$

$= \int_{[-1,\frac{\eta-1}{\eta}]} f^D g^{lsh} d\mu_\eta + \int_{[-1,\frac{\eta-2}{\eta}]} f g^D d\mu_\eta$

Rearranging, we obtain that;

$\int_{[-1,\frac{\eta-1}{\eta}]} f g^D d\mu_\eta = -\int_{[-1,\frac{\eta-1}{\eta}]} f^D g^{lsh} d\mu_\eta$

□

Lemma 0.19. *For $m \in \overline{\mathcal{Z}_\eta}$, we have that;*

$\mathcal{F}_\eta(S_\eta)(m) = \frac{2(-1)^m}{\theta_\eta(m)}$

where $\theta_\eta(m) = \eta(exp_\eta(\frac{-\pi m}{\eta}) - 1)$

Moreover, for $m \in \overline{\mathcal{Z}_\eta}$ finite;

${}^\circ\mathcal{F}_\eta(S_\eta)(m) = \mathcal{F}(S)(m)$

Finally, for $x \in [-1,1)_\eta$, with ${}^\circ x \in \{-1,1\}$ there exists an infinite T_x, such that for $k \leq T_x$, $S_k(S_\eta)(x) \simeq 0$
,

Proof. Let $f(x) = x_\eta$, $h(x) = exp_\eta(-\pi i x m)$, $h^D(x) = \theta_\eta(m) exp_\eta(-\pi i m x)$, so if $g(x) = \frac{h(x)}{\theta_\eta(m)} = \frac{exp_\eta(-\pi i m x)}{\theta_\eta(m)}$, then $g^D(x) = exp_\eta(-\pi i x m)$ and $g^{lsh}(x) = \frac{\phi_\eta(m) exp_\eta(-\pi i x m)}{\theta_\eta(m)}$, where $\phi_\eta(m) = exp_\eta(-\frac{\pi i m}{\eta})$. We have that $f^D(x) = x_\eta^D = 1 + r\delta_{\frac{\eta-1}{\eta}}$, where $r = x_\eta^D(\frac{\eta-1}{\eta}) - 1 = \eta(x_\eta(-1) - x_\eta) - 1 = \eta(-1 - \frac{\eta-1}{\eta}) - 1 = -2\eta$, so that $x_\eta^D = 1 - 2\eta\delta_{\frac{\eta-1}{\eta}}$.

Using the previous lemma, we obtain that;

$\mathcal{F}_\eta(S_\eta)(m)$

$= \int_{[-1,1)_\eta} x_\eta exp_\eta(-\pi i m x) d\mu_\eta$

$\int_{[-1,1)_\eta} f g^D d\mu_\eta$

$= -\int_{[-1,1)_\eta} f^D g^{lsh} d\mu_\eta$

$= -\int_{[-1,1)_\eta} (1 - 2\eta\delta_{\frac{\eta-1}{\eta}})(\frac{\phi_\eta(m) exp_\eta(-\pi i x m)}{\theta_\eta(m)}) d\mu_\eta$

$= -\frac{\phi_\eta(m)}{\theta_\eta(m)} \int_{[-1,1)_\eta} exp_\eta(-\pi ixm) d\mu_\eta(x) - 2\eta \frac{\phi_\eta(m)}{\theta_\eta(m)} \int_{[-1,1)_\eta} \delta_{\frac{\eta-1}{\eta}} exp_\eta(-\pi ixm)) d\mu_\eta$

$= -2\eta \frac{1}{\eta} \frac{\phi_\eta(m)}{\theta_\eta(m)} exp_\eta(-\pi i \frac{\eta-1}{\eta} m)$

$= -2 \frac{\phi_\eta(m)}{\theta_\eta(m)} \phi_\eta(-m) exp_\eta(-\pi im)$

$= \frac{2(-1)^m}{\theta_\eta(m)}$

as required. For the final part, either use the fact that for m finite $x_\eta exp_\eta(-\pi ixm)$ is S-continuous and S-integrable, or check directly, for finite m that;

$\theta_\eta(m) = exp_\eta(-\pi ixm)^D(0) \simeq (exp(-\pi ixm)')_\eta(0) = -\pi im$

as $exp(-\pi ixm)$ is analytic.

For x as in the statement of the lemma, we have already checked that, for finite m, ${}^\circ\mathcal{F}_\eta(S_\eta)(m) = \mathcal{F}(S)(m)$, in particular, for finite k;

$S_k(f)(x) \simeq S_k(S)({}^\circ x) = 0$

$|S_k(f)(x)| < \frac{1}{k}$

so that, by overflow there exists an infinite T_x, such that for $k \leq T_x$, k infinite, $|S_k(S_\eta(x)| < \frac{1}{k} \simeq 0$

$\square$

Lemma 0.20. *For h analytic on $(0,1)$, h finite, left and right differentiable at $\{1,-1\}$, with the limits h_x^+ and h_x^- and ${}^\circ x \subset \{-1,1\}$ there exists $Q_x \in {}^*\mathcal{N}$, with Q_x infinite, such that for all $0 \leq k \leq Q_x$, k infinite;*

$S_k(h) \simeq \frac{g_x^+ + g_x^-}{2}$

Proof. Consider the case when ${}^\circ x = 1$. If $h_x^+ = h_x^-$, the result is proved in Lemma 0.14. Otherwise, using Lemma 0.17, we can find λ, μ, ν such that $g = \lambda h + \mu S_\eta + \nu$, with $g_x^+ = g_x^-$. By Lemma 0.14, for k infinite, with $|k| \leq Q_x = min(T_x, P_x)$, we obtain;

$S_k(\lambda h + \mu S_\eta + \nu)(x)$

$= \lambda S_k(h)(x) + \mu S_k(S_\eta)(x) + \nu$

$= \lambda S_k(h)(x) + \nu$

$= S_k(g)(x) \simeq g_x^+$.

Rearranging, with $\lambda \neq 0$, $S_k(h)(x+) = \frac{g_x(+)-\nu}{\lambda}$. In the generic case, when $\nu = 0$, we obtain that;

$S_k(h)(x) = \frac{g_x(+)}{\lambda}$

$= \frac{g_x(f_x^+ + f_x^-)}{2}$

$= \frac{(\lambda f_x^+ + \mu)(f_x^+ + f_x^-)}{2}$

$= \frac{2f_x^+ + (f_x^- - f_x^+)}{f_x^+ + f_x^-} \frac{f_x^+ + f_x^-}{2}$

$= \frac{f_x^+ + f_x^-}{2}$

The case, when $\nu \neq 0$ is similar, and left to the reader.

□

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