ANTIDERIVATIVES OF INVERSE FUNCTIONS

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ABSTRACT. We find a formula for the antiderivative of an inverse function, by considering areas. We apply it in the computation of typical integrals which might arise from probability.

Lemma 0.1. *Let $f : [a, b] \rightarrow \mathcal{R}$ be increasing and continuous, with inverse f^{-1}, then;*

(i). $(f^{-1})' = \frac{1}{(f' \circ f^{-1})}$

(ii). Supposing $g' = f$, then if;

$s(x) = xf^{-1}(x) - g(f^{-1}(x)) + g(a)$

we have that $s' = f^{-1}$.

Proof. For (i), as $(f \circ f^{-1})(x) = x$, differentiating both sides, and using the chain rule, we obtain;

$(f \circ f^{-1})'(x)$

$= (f' \circ f^{-1})(f^{-1})'(x) = 1$

Rearranging gives (i)

For (ii), assume that $a = 0$, with $a \leq x$, for $x \in dom(f)$, with $f(0) = x_0$, and $g' = f$, then let;

$h(x) = \int_{x_0}^{x} f^{-1}(y)dy$

then, by the fundamental theorem of calculus, we have that $h'(x) = f^{-1}(x)$, for $x \in ran(f)$. Moreover, as;

$\int_{x_0}^{x} f^{-1}(y)dy + \int_{0}^{f^{-1}(x)} f(y)dy = xf^{-1}(x)$

we have, rearranging, that;

$h(x) = xf^{-1}(x) - \int_0^{f^{-1}(x)} f(y)dy$

$= xf^{-1}(x) - r(x)$

where $r(x) = F(f^{-1}(x))$, $F' = f$, by the Fundamental theorem of calculus, so that $F = g + c$, $r(x) = g(f^{-1}(x)) + c$, and;

$h(x) = xf^{-1}(x) - g(f^{-1}(x)) - c.$

As $f^{-1}(x_0) = 0$, and $h(x_0) = 0$, we have, substituting, that;

$0 = -g(0) - c$, so that $c = -g(0)$, and;

$h(x) = xf^{-1}(x) - g(f^{-1}(x)) + g(0)$ $(*)$

Now assuming $a \neq 0$, with $a \leq x$ for $x \in dom(f)$, and consider $f_a(x) = f(x+a)$, $g_a(x) = g(x+a)$, with $dom(f_a) = [0, b-a]$, f_a increasing, and $g_a' = f_a$. With h_a defined as above, $h_a^{'} = f_a^{-1}$, and, by $(*)$;

$h_a(x) = xf_a^{-1}(x) - g_a f_a^{-1}(x) + g_a(0)$

$= xf_a^{-1}(x) - g(f_a^{-1}(x) + a) + g(a)$

$= x(f^{-1}(x) - a) - g(f^{-1}(x)) + g(a)$

as $f_a^{-1}(x) = f^{-1}(x) - a$

As $h_a^{'} = f^{-1} - a$

$(h_a + ax)' = h_a^{'} + a = f^{-1}$

Letting $h(x) = h_a(x) + ax$, we have that;

$h(x) = xf^{-1}(x) - ax - g(f^{-1}(x)) + g(a) + ax$

$= xf^{-1}(x) - g(f^{-1}(x)) + g(a)$

□

Lemma 0.2. *Let $f : [a, b] \to \mathcal{R}$ be decreasing and continuous, with inverse f^{-1}, then;*

Supposing $g' = f$, then if;

$s(x) = xf^{-1}(x) - g(f^{-1}(x)) + g(a)$

we have that $s' = f^{-1}$ again.

Proof. We have that $-f$ is increasing and continuous, with inverse $(-f)^{-1}(x) = f^{-1}(-x)$

Supposing $g' = f$, then $(-g)' = -f$, and applying the previous result, we have, with $h(x) = x(-f)^{-1}(x) - (-g \circ (-f)^{-1})(x) - g(a)$, that $h'(-f)^{-1}$. Then;

$h(x) = xf^{-1}(-x) + g \circ (-f)^{-1}(x) - g(a)$

$= xf^{-1}(-x) + g(f^{-1}(-x)) - g(a)$

with $h'(x) = f^{-1}(-x)$

But then;

$(-h(-x))' = - - h'(-x) = f^{-1}(x)$

and $-h(-x) = xf^{-1}(x) - g \circ f^{-1}(x) + g(a)$ as before.

□

Lemma 0.3. *We have, for $0 \leq x \leq 1$, that;*

$g(x) = \int_0^x cos^{-1}(y)dy == xcos^{-1}(x) - sin(cos^{-1}(x))$

In particularly, for $0 \leq x \leq 1$;

$g'(x) = cos^{-1}(x)$

and $\int_0^1 cos^{-1}(y)dy = cos^{-1}(1) - sin(cos^{-1}(1))$

Proof. The result follows from the previous lemama. The last result is the Fundamental Theorem of Calculus.

□

Lemma 0.4. *Let* $J_n = \int_0^1 x^{2n-1} sin(x)dx$, *for* $n \geq 1$

Then:

$J_n = cos(1) - sin(1) + cos(1)\sum_{1\leq r\leq n-2} \frac{(2n-1)!}{(2n-2r+1)!}$

$-sin(1)\sum_{1\leq r\leq n-2} \frac{(2n-1)!(2n-2r-1)}{(2n-2r+1)!} - (2n-1)!(cos(1)+sin(1))$

Proof. Using integration by parts, we have, for $n \geq 2$ that;

$J_n = [-x^{2n-1}cos(x)]_0^1 + (2n-1)\int_0^1 x^{2n-2}cos(x)dx$

$= -cos(1) + (2n-1)([x^{2n-2}sin(x)]_0^1 - (2n-2)\int_0^1 x^{2n-3}sin(x)dx)$

$= -cos(1) + (2n-1)(sin(1) - (2n-2)\int_0^1 x^{2n-3}sin(x)dx)$

$= -cos(1) + (2n-1)(sin(1) - (2n-2)J_{n-1})$

$= -cos(1) + (2n-1)sin(1) - (2n-1)(2n-2)J_{n-1}$

It follows that;

$J_n = d_n - \sum_{1\leq r\leq n-2}(2n-1)\ldots 2(n-r)d_{n-r} - (2n-1)\ldots 2J_1$

$= d_n - \sum_{1\leq r\leq n-2} \frac{(2n-1)!}{(2n-2r+1)!} d_{n-r} - (2n-1)!J_1$

where $d_{n-r} = -cos(1) + (2(n-r)-1)sin(1)$, for $0 \leq r \leq n-2$

Here;

$J_1 = \int_0^1 x sin(x)dx$

$= [-xcos(x)]_0^1 + \int_0^1 cos(x)dx$

$= cos(1) + [sin(x)]_0^1$

$= cos(1) + sin(1)$

We have that;

$\sum_{1\leq r\leq n-2}\frac{(2n-1)!}{(2n-2r+1)!}d_{n-r}$

$= \sum_{1\leq r\leq n-2}\frac{(2n-1)!}{(2n-2r+1)!}(-cos(1)+(2(n-r)-1)sin(1))$

$= -cos(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!}{(2n-2r+1)!} + sin(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!(2n-2r-1)}{(2n-2r+1)!}$

So that;

$J_n = cos(1) - sin(1) + cos(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!}{(2n-2r+1)!}$

$-sin(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!(2n-2r-1)}{(2n-2r+1)!} - (2n-1)!(cos(1)+sin(1))$

□

Lemma 0.5. $\int_0^1 cos(x)cos^{-1}(x)dx$

$= cos^{-1}(1) - sin(cos^{-1}(1)) + \sum_{n=1}^{\infty}[\frac{cos-1(1)-sin(1)}{2n+1} + \frac{2n}{2n+1}(cos(1)$

$-sin(1) + cos(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!}{(2n-2r+1)!}$

$-sin(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!(2n-2r-1)}{(2n-2r+1)!} - (2n-1)!(cos(1)+sin(1)))]$

Proof. We have that;

$cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^{2n}x^{2n}}{n!}$ on $[0,1]$, so that, using the Dominated Convergence Theorem;

$\int_0^1 cos(x)cos^{-1}(x)dx$

$= \int_0^1(\sum_{n=0}^{\infty}\frac{(-1)^{2n}x^{2n}}{n!})cos^{-1}(x)dx$

$= \sum_{n=0}^{\infty}(-1)^{2n}\int_0^1 x^{2n}cos^{-1}(x)dx$

Let $I_n = \int_0^1 x^{2n}cos^{-1}(x)dx$, for $n\geq 1$

$J_n = \int_0^1 x^{2n-1}sin(x)dx$, for $n\geq 1$

Then, using integration by parts and the fact that;

$g'(x) = cos^{-1}(x)$ on $[0,1]$, where $g(x) = xcos^{-1}(x) + 1 - sin(x)$

We have;

$I_n = [x^{2n}(xcos^{-1}(x)+1-sin(x))]_0^1 - \int_0^1 2nx^{2n-1}(xcos^{-1}(x)+1-sin(x))dx$

$= cos{-1}(1)+1-sin(1)-2n(\int_0^1 x^{2n}(cos^{-1}(x)dx+\int_0^1 x^{2n-1}dx-\int_0^1 x^{2n-1}sin(x))dx$

$= cos{-1}(1)+1-sin(1)-2n(I_n+[\frac{x^{2n}}{2n}]_0^1-J_n)dx$

$= cos{-1}(1)+1-sin(1)-2nI_n-\frac{2n}{2n}+2nJ_n$

$= cos{-1}(1)-sin(1)-2nI_n+2nJ_ndx$

so that;

$(2n+1)I_n = cos{-1}(1)-sin(1)+2nJ_n$

and $I_n = \frac{cos{-1}(1)-sin(1)}{2n+1}+\frac{2n}{2n+1}J_n$

Applying the previous lemma, we have that;

$I_n = \frac{cos{-1}(1)-sin(1)}{2n+1}+\frac{2n}{2n+1}(cos(1)-sin(1)+cos(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!}{(2n-2r+1)!}$

$-sin(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!(2n-2r-1)}{(2n-2r+1)!}-(2n-1)!(cos(1)+sin(1)))$

Therefore,;

$\int_0^1 cos(x)cos^{-1}(x)dx$

$= \int_0^1 cos^{-1}(x)dx+\sum_{n=1}^{\infty} I_ndx$

$= cos^{-1}(1)-sin(cos^{-1}(1))+\sum_{n=1}^{\infty}[\frac{cos{-1}(1)-sin(1)}{2n+1}$

$+\frac{2n}{2n+1}(cos(1)-sin(1)+cos(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!}{(2n-2r+1)!}$

$-sin(1)\sum_{1\leq r\leq n-2}\frac{(2n-1)!(2n-2r-1)}{(2n-2r+1)!}-(2n-1)!(cos(1)+sin(1)))]$

□

References

[1] Further Pure Mathematics, Bostock, Chandler and Rourke, OUP, (1982).

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