A NONSTANDARD SOLUTION TO THE WAVE EQUATION

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Abstract. We follow d'Alembert's method for solving the standard wave equation, in the nonstandard case, and compare the solutions.

Definition 0.1. *Given $u \in V(\overline{\mathcal{R}_\eta} \times \overline{\mathcal{T}_\nu})$, we let;*

$u_t(y,s) = \nu(u(y, s+\frac{1}{\nu}) - u(y,s))$, *for* $(y,s) \in \overline{\mathcal{R}_\eta} \times (\overline{\mathcal{T}_\nu} \setminus \frac{\nu^2-1}{\nu})$

$u_x(y,s) = \frac{\sqrt{\eta}}{2}(u(y+\frac{1}{\sqrt{\eta}}, s) - u(y-\frac{1}{\sqrt{\eta}}, s))$, *for* $(y,s) \in \overline{\mathcal{R}_\eta} \times \overline{\mathcal{T}_\nu}$

We define the nonstandard unconvoluted wave equation, on $\overline{\mathcal{R}_\eta} \times (\overline{\mathcal{T}_\nu} \setminus \frac{\nu^2-1}{\nu}, \frac{\nu^2-2}{\nu})$ *by;*

$u_{tt} - u_{xx} = 0$

Lemma 0.2. *Given initial conditions $\{f,g\} \subset V(\overline{\mathcal{R}_\eta})$ there exists a unique $u \in V(\overline{\mathcal{R}_\eta} \times \overline{\mathcal{T}_\nu})$ solving the nonstandard unconvoluted wave equation, such that $u^0 = f$, and $u^0_t = g$, and;*

$u(x, t+\frac{2}{\nu}) = 2u(x, t+\frac{1}{\nu}) + \frac{\eta}{4\nu^2}u(x+\frac{2}{\sqrt{\eta}}, t) - (1+\frac{\eta}{2\nu^2})u(x,t)$

$+\frac{\eta}{4\nu^2}u(x-\frac{2}{\sqrt{\eta}}, t)$, $(*)$

for $(x,t) \in \overline{\mathcal{R}_\eta} \times (\overline{\mathcal{T}_\nu} \setminus \frac{\nu^2-1}{\nu}, \frac{\nu^2-2}{\nu})$

Moreover, if the standard initial conditions $\{u^0, u^0_t\}$ are given with $\frac{\partial^i(u^0)}{\partial x^i} \leq D$ and $\frac{\partial^j(u^0_t)}{\partial x^j} \leq D$, uniformly, for finite $(i,j) \in \mathcal{N}^2$, then for $(t_0, x_0) \in \overline{\mathcal{T}_\nu} \times \overline{\mathcal{R}_\eta}$, with t_0 finite, and the choice $\eta \leq 4\nu$, $u_{txx}|_{t_0,x_0}$ and $u_{xx}|_{t_0,x_0}$ are bounded, for the nonstandard equation generated by initial conditions $\{v, v_t\}$, with $v^0 = (u^0)_\eta$, $v^0_t = (u^0_t)_\eta$.

Proof. Using the definition of the derivatives, the equation in Definition 0.1, and rearranging, we obtain the defining schema for u given in

$(*)$. We are free to choose the values for the first two time steps, by setting;

$$u(x,0)=f(x)$$

$$u_t(x,0)=\nu(u(x,\tfrac{1}{\nu})-u(x,0))=g(x)$$

First, observe, using Taylors' Theorem, that $v^0_{x^n}$ and $v^0_{tx^n}$ are bounded. Suppose inductively, that there exists a constant $C_i\in\mathcal{R}$, for $\frac{i}{\nu}$ finite, with;

$$max(\{u_{\frac{j}{\nu},x^n},u_{\frac{k}{\nu}},x^nt:0\leq j\leq i,0\leq k\leq i-1\})\leq C_i$$

for n even, where v_{x^n} denotes the n'th derivative of v with respect to x.

We have that;

$$v(x,\tfrac{i+1}{\nu})=2v(x,\tfrac{i}{\nu})+\tfrac{\eta}{4\nu^2}v(x+\tfrac{2}{\sqrt{\eta}},\tfrac{i-1}{\nu})-(1+\tfrac{\eta}{2\nu^2})v(x,\tfrac{i-1}{\nu})$$

$$+\tfrac{\eta}{4\nu^2}v(x-\tfrac{2}{\sqrt{\eta}},\tfrac{i-1}{\nu})$$

and taking the n'th even derivative with respect to x;

$$v_{x^n}(x,\tfrac{i+1}{\nu})=2v_{x^n}(x,\tfrac{i}{\nu})+\tfrac{\eta}{4\nu^2}v_{x^n}(x+\tfrac{2}{\sqrt{\eta}},\tfrac{i-1}{\nu})-(1+\tfrac{\eta}{2\nu^2})v_{x^n}(x,\tfrac{i-1}{\nu})$$

$$+\tfrac{\eta}{4\nu^2}v_{x^n}(x-\tfrac{2}{\sqrt{\eta}},\tfrac{i-1}{\nu})$$

Abbreviating notation;

$v_{i,tx^n}=\nu(v_{i+1,x^n}-v_{i,x^n})=\nu(2v_{i,x^n}-v_{i-1,x^n}+\frac{\eta}{4\nu^2}(v^{lsh^2}_{i-1,x^n}-2v_{i-1,x^n}+v^{rsh^2}_{i-1,x^n})-v_{i,x^n})$

$$\nu(v_{i,x^n}-v_{i-1,x^n})+\tfrac{\eta}{4\nu}(v^{lsh^2}_{i-1,x^n}-2v_{i-1,x^n}+v^{rsh^2}_{i-1,x^n}))$$

$$=v_{i-1,tx^n}+\tfrac{1}{4\nu}v_{i-1,x^{n+2}}$$

$$|v_{i,tx^n}|\leq|v_{i-1,tx^n}|+\tfrac{1}{4\nu}|v_{i-1,x^{n+2}}|$$

$$\leq C+\tfrac{1}{4\nu}C=C(1+\tfrac{1}{4\nu})$$

$v_{i+1,x^n} = 2v_{i,x^n} + \frac{\eta}{4\nu^2}(v^{lsh^2}_{i-1,x^n} - (1+\frac{\eta}{2\nu^2})v_{i-1,x^n} + \frac{\eta}{4\nu^2}v^{rsh^2}_{i-1,x^n})$

$= v_{i,x^n} - v_{i-1,x^n} + v_{i,x^n} + \frac{\eta}{4\nu^2}(v^{lsh^2}_{i-1,x^n} - 2v_{i-1,x^n} + v^{rsh^2}_{i-1,x^n})$

$= \frac{v_{i-1,tx^n}}{\nu} + v_{i,x^n} + \frac{\eta}{4\nu^2}(v_{i-1,x^{n+2}})$

$|v_{i+1,x^n}| \leq \frac{C}{\nu} + C(1+\frac{\eta}{4\nu^2})$

Iterating, we obtain that;

for finite t;

$|v_{t,tx^n}| \leq C(1+\frac{1}{4\nu})^{[\nu t]}$

$= C(1+\frac{1}{4\nu})^{\nu\frac{[\nu t]}{\nu}}$

$\simeq Ce^{4\frac{[\nu t]}{\nu}} \simeq Ce^{4t}$

$|v_{t,x^n}| \leq C(1+\frac{1}{\nu}+\frac{\eta}{4\nu^2})^{[\nu t]}$

$= C(1+(\frac{1+\frac{\eta}{4\nu}}{\nu})^{[\nu t]})$

$\leq C(1+(\frac{2}{\nu})^{[\nu t]})$

$\simeq Ce^{2t}$

with $\eta \leq 4\nu$, as required. A similar argument works for the odd derivatives $\{u_{x^n}, u_{tx^n}\}$, with $n \in \mathcal{N}$ odd.

□

Lemma 0.3. *Suppose that u satisfies the nonstandard equation in Lemma 0.2, with the extra assumption that for finite $(i,j) \in \mathcal{N}^2$, $\frac{\partial^i(u^0)}{\partial x^i}$ and $\frac{\partial^j(u^0_t)}{\partial x^j}$ are bounded, for finite $(x,t) \in \overline{\mathcal{R}_\eta} \times \overline{\mathcal{T}_\nu}$. Then, for such (x,t) ;*

$u(x,t) \simeq \frac{1}{2}(u(x+t,0) + u(x-t,0) + \int_{[x-t,x+t]} u^0_t(w)d\mu_\eta(w)$

In particularly, if F satisfies the standard wave equation on $\overline{\mathcal{R}}^2$, with the property that $F^0, F^0_t \subset C^\infty(\mathcal{R})$, then if u satisfies the nonstandard equation with initial conditions $\{(F^0)_\eta, (F^0_t)_\eta\}$, we have that u is S-continuous for finite $(x,t) \in \overline{\mathcal{R}_\eta} \times \overline{\mathcal{T}_\nu}$, and;

$^{\circ}u(x,t)=F(^{\circ}x,^{\circ}t)$

Proof. By the previous lemma, all finite derivatives of the form u_{x^n} and u_{tx^n} are uniformly bounded. Factoring the equation $u_{tt}-u_{xx}=0$, we obtain that;

$(\frac{\partial}{\partial t}+\frac{\partial}{\partial x})(\frac{\partial}{\partial t}-\frac{\partial}{\partial x})u=0$

for $(x,t)\in\overline{\mathcal{R}_\eta}\times(\overline{\mathcal{T}_\nu}\setminus\frac{\nu^2-1}{\nu},\frac{\nu^2-2}{\nu})$. Setting $v=u_t-u_x$, with $(x,t)\in\overline{\mathcal{R}_\eta}\times(\overline{\mathcal{T}_\nu}\setminus\frac{\nu^2-1}{\nu},\frac{\nu^2-2}{\nu})$, we have that;

$v_t+v_x=0$

$v^0=u_t^0-u_x^0$, $(***)$

the first line of $(***)$ holding when $(x,t)\in\overline{\mathcal{R}_\eta}\times(\overline{\mathcal{T}_\nu}\setminus\frac{\nu^2-1}{\nu},\frac{\nu^2-2}{\nu})$. Given finite (x,t), we define $H_{x,t}:\overline{\mathcal{T}_\xi}\rightarrow{}^*\mathcal{R}$ by ;

$H_{x,t}(s)=v(x-\frac{[s\xi]}{\xi},t-\frac{[s\xi]}{\xi})=v(x-\frac{[s\xi]}{\xi},\frac{[t\nu]}{\nu}-\frac{[s\xi]}{\xi})$

Using $(***)$ and the following Lemmas, 0.7 and 0.8, for finite $\{x_0,t_0,s\}$, with $x(s)=-\frac{[s\xi]}{\xi}$ and $y(s)=-\frac{[s\xi]}{\xi}$, so that $x^D(s)=y^D(s)=-1$, we have that;

$\frac{dH}{dt}{}^s_{x_0,t_0}\simeq(-\frac{\partial v}{\partial x}-\frac{\partial v}{\partial t})|_{x_0-s,t_0-s}\simeq 0$

so that;

$(u_t-u_x)(x_0,t_0)=v(x_0,t_0)=H_{x_0,t_0}(0)\simeq H_{x_0,t_0}(s_0)$

(where s_0 is chosen so that $\frac{[s_0\xi]}{\xi}=\frac{[t_1\nu]}{\nu}\simeq\frac{[t_0\nu]}{\nu}$; writing $\xi=\frac{\nu}{\kappa}$, with κ infinite, we have $\frac{[s_0\xi]}{\xi}=\frac{[t_0\nu]}{\nu}$, iff $\frac{\kappa[\frac{s_0\nu}{\kappa}]}{\nu}=\frac{[t_0\nu]}{\nu}$ iff $[\frac{s_0\nu}{\kappa}]=\frac{[t_0\nu]}{\kappa}$. Replacing $[t_0\nu]$ with $[t_0\nu]+r$, where $0\leq r<\kappa$, and $[t_1\nu]=[t_0\nu]+r$, so that $\kappa|[t_1\nu]$ and, as $|\frac{r}{\nu}|<|\frac{\kappa}{\nu}|=|\frac{1}{\xi}|\simeq 0$, we must have $\frac{[t_0\nu]}{\nu}\simeq\frac{[t_1\nu]}{\nu}$. We can then solve $[\frac{s_0\nu}{\kappa}]=\frac{[t_1\nu]}{\kappa}$, taking $\frac{[t_1\nu]}{\nu}\leq s_0<\frac{[t_1\nu]+\kappa}{\nu}$)

$=v(x_0-\frac{[s_0\xi]}{\xi},\frac{[t_0\nu]}{\nu}-\frac{[s_0\xi]}{\xi})$

$\simeq v(x_0-\frac{[s_0\xi]}{\xi},\frac{[t_1\nu]}{\nu}-\frac{[s_0\xi]}{\xi})$

$= v(x_0 - \frac{[s_0\xi]}{\xi}, 0)$

$= v(x_0 - \frac{[t_1\nu]}{\nu}, 0)$

$\simeq v(x_0 - \frac{[t_0\nu]}{\nu}, 0)$

Here, we have used the facts that v is S-continuous in t, and v^0 is S-continuous in x, consequent on $(u_t - u_x)$ S-continuous in t or u_{tt} and u_{tx} bounded, and $(u_t - u_x)^0$ is S-continuous in x or u^0_{tx} and u^0_{xx} bounded. These results follow as $u_{tt} = u_{xx}$, both $u_{tx} = u_{xt}$ and u_{xx} are bounded by the result of the previous lemma.

$= u_t - u_x(x_0 - \frac{[t_0\nu]}{\nu}, 0)$

$= u^0_t(x_0 - \frac{[t_0\nu]}{\nu}) - u^0_x(x_0 - \frac{[t_0\nu]}{\nu})$, $(* * **)$

Rewriting $(* * **)$ as $(u_t - u_x)(x', t') \simeq h(x', t')$

with $h(x', t') = u^0_t(x' - \frac{[t'\nu]}{\nu}) - u^0_x(x' - \frac{[t'\nu]}{\nu})$

define $\Theta_{x',t'} : \overline{\mathcal{T}_\xi} \to {}^*\mathcal{R}$ by ;

$\Theta_{x',t'}(s) = u(x' + \frac{[s\xi]}{\xi}, t' - \frac{[s\xi]}{\xi})$

Then, using $(* * **)$,the definition of h, and Lemmas 0.7 and 0.8 again, this time with $x(s) = \frac{[s\xi]}{\xi}$, so that $x^D(s) = 1$;

$\frac{d\Theta}{dt}{}^s_{x',t'} \simeq (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t})|_{x' + \frac{[s\xi]}{\xi}, t' - \frac{[s\xi]}{\xi}}$

$\simeq (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t})|_{x' + \frac{[s\nu]}{\nu}, t' - \frac{[s\nu]}{\nu}}$

(S-continuity of u_x in x and u_t in t, consequent on u_{xx} and u_{tt} bounded, follows from the previous lemma.)

$\simeq -h(x' + \frac{[s\nu]}{\nu}, t' - \frac{[s\nu]}{\nu})$

$= -u^0_t(x' + \frac{[s\nu]}{\nu} - \frac{[t'\nu]-[s\nu]}{\nu}) + u^0_x(x' + \frac{[s\nu]}{\nu} - \frac{[t'\nu]-[s\nu]}{\nu})$

$= -u^0_t(x' + \frac{2[s\nu]}{\nu} - \frac{[t'\nu]}{\nu}) + u^0_x(x' + \frac{2[s\nu]}{\nu} - \frac{[t'\nu]}{\nu})$, (†)

We have, as above, that;

$\Theta_{x',t'}(0)=u(x',t')$

$\Theta_{x',t'}(t')\simeq u(x'+\frac{[t'\nu]}{\nu},t'-\frac{[t'\nu]}{\nu})=u(x'+\frac{[t'\nu]}{\nu},0)$, $(*****)$

Using Lemma 0.8, $(*****)$ and (†), we obtain;

$u(x'+\frac{[t'\nu]}{\nu},0)-u(x',t')$

$\simeq\Theta_{x',t'}(t')-\Theta_{x',t'}(0)$

$=\int_{[0,\frac{[t'\nu]}{\nu})}\frac{d\Theta}{dt}^{w}_{x',t'}d\mu_\xi(w)$

$\simeq\int_{[0,\frac{[t'\nu]}{\nu})}(-u^0_t(x'+\frac{2[s\nu]}{\nu}-\frac{[t'\nu]}{\nu})+u^0_x(x'+\frac{2[s\nu]}{\nu}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)$

so that, using Lemma 0.5 and Lemma 0.9;

$u(x',t')$

$\simeq u(x'+\frac{[t'\nu]}{\nu},0)+\int_{[0,\frac{[t'\nu]}{\nu})}(u^0_t(x'+\frac{2[s\nu]}{\nu}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)$

$-\int_{[0,\frac{[t'\nu]}{\nu})}u^0_x(x'+\frac{2[s\nu]}{\nu}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)$

$=u(x'+\frac{[t'\nu]}{\nu},0)+\int_{[0,\frac{[t'\nu]}{\nu})}(u^0_t(x'+\frac{2[s\xi]}{\xi}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)$

$-\int_{[0,\frac{[t'\nu]}{\nu})}u^0_x(x'+\frac{2[s\xi]}{\xi}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)$

$=u(x'+\frac{[t'\nu]}{\nu},0)+\int_{[0,\frac{[t'\nu]}{\nu})}(u^0_t(x'+\frac{2[s\xi]}{\xi}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)$

$-\int_{[0,\frac{[t'\nu]}{\nu})}u^0_x(x'+\frac{2[s\xi]}{\xi}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)$

Now let $s=h_\rho(u)=\frac{\frac{[\rho u]}{\rho}+\frac{[t'\nu]}{\nu}-x'}{2}$, where $\xi=\kappa\rho$ and $\kappa\in{}^*\mathcal{N}$ is infinite, then, letting $w_{1,\xi}(s)=u^0_t(x'+2\frac{[s\xi]}{\xi}-\frac{[t'\nu]}{\nu})$, we have, by refinement, that;

$(w_{1,\xi}\circ h_\rho)(u)=u^0_t(x'+2(\frac{\frac{[\rho u]}{\rho}+\frac{[t'\nu]}{\nu}-x'}{2})-\frac{[t'\nu]}{\nu})=u^0_t(\frac{[\rho u]}{\rho})$

$h^D_\rho(u)=\frac{1}{2}$

and, using Lemma 0.9;

$\int_{[0,\frac{[t'\nu]}{\nu})}(u_t^0(x'+\frac{2[s\xi]}{\xi}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)=\frac{1}{2}\int_{[x'-\frac{[t'\nu]}{\nu},x'+\frac{[t'\nu]}{\nu})}u_t^0(\frac{[\rho u]}{\rho})d\mu_\rho(u)$

$-\int_{[0,\frac{[t'\nu]}{\nu})}u_x^0(x'+\frac{2[s\xi]}{\xi}-\frac{[t'\nu]}{\nu})d\mu_\xi(s)=-\frac{1}{2}\int_{[x'-\frac{[t'\nu]}{\nu},x'+\frac{[t'\nu]}{\nu})}u_x^0(\frac{[\rho u]}{\rho})d\mu_\rho(u)$

so that;

$u(x',t')\simeq u(x'+\frac{[t'\nu]}{\nu},0)+\frac{1}{2}\int_{[x'-\frac{[t'\nu]}{\nu},x'+\frac{[t'\nu]}{\nu})}u_t^0(\frac{[\rho u]}{\rho})d\mu_\rho(u)$

$-\frac{1}{2}\int_{[x'-\frac{[t'\nu]}{\nu},x'+\frac{[t'\nu]}{\nu})}u_x^0(\frac{[\rho u]}{\rho})d\mu_\rho(u)$, $(****)$

Using Lemma 0.10, we have that;

$\frac{1}{2}\int_{[x'-\frac{[t'\nu]}{\nu},x'+\frac{[t'\nu]}{\nu})}u_x^0(\frac{[\rho u]}{\rho})d\mu_\rho(u)$

$\simeq\frac{1}{2}(u^0(x'+\frac{[t'\nu]}{\nu})-u^0(x'-\frac{[t'\nu]}{\nu}))$, $(*****)$

Combining $(****)$ and $(*****)$, we obtain that;

$u(x',t')\simeq u(x'+\frac{[t'\nu]}{\nu},0)-\frac{1}{2}(u^0(x'+\frac{[t'\nu]}{\nu})-u^0(x'-\frac{[t'\nu]}{\nu}))$

$+\frac{1}{2}\int_{[x'-\frac{[t'\nu]}{\nu},x'+\frac{[t'\nu]}{\nu})}u_t^0(\frac{[\rho u]}{\rho})d\mu_\rho(u)$

$=\frac{1}{2}(u^0(x'+\frac{[t'\nu]}{\nu})-u^0(x'-\frac{[t'\nu]}{\nu}))$

$+\frac{1}{2}\int_{[x'-\frac{[t'\nu]}{\nu},x'+\frac{[t'\nu]}{\nu})}u_t^0(\frac{[\rho u]}{\rho})d\mu_\rho(u)$

$\simeq\frac{1}{2}(u^0(x'+t')-u^0(x'-t')$

$+\frac{1}{2}\int_{[x'-t',x'+t'}u_t^0(w)d\mu_\eta(w)$

as required.

□

Remarks 0.4. *For finite values, this result matches d'Alembert's solution from the standard case, so the nonstandard wave equation is a reasonable model for pursuing a diffusion approach. This would help to understand photon paths, for example, in Maxwell's equation for zero charge and density.*

Lemma 0.5. *Let $F:{}^*\mathcal{R}\rightarrow{}^*\mathcal{R}$ be internal, and $\eta=\kappa\xi$ with $\kappa\in{}^*\mathcal{N}$, then;*

$((F)_\eta)_\xi = (F)_\xi$

Proof. Let $x = \frac{i}{\xi}$, with $i \in {}^*\mathcal{Z}$, then;

$((F)_\eta)_\xi(x) = F_\eta(x) = F_\eta(\frac{i}{\xi}) = F_\eta(\frac{i\kappa}{\eta})$

$= F(\frac{i\kappa}{\eta}) = F(\frac{i}{\xi}) = (F)_\xi(\frac{i}{\xi}) = (F)_\xi(x)$

As both $((F)_\eta)_\xi$ and $(F)_\xi$ are μ_ξ measurable, this proves the result. □

Lemma 0.6. *Let $f \in V(\overline{\mathcal{R}_\eta})$, with f^D S-continuous and bounded, then if $\epsilon \simeq 0$, with $\epsilon\sqrt{\eta}$ infinite, we have that;*

$f^{D_\epsilon} \simeq f^D$

Proof. Let $\epsilon = \frac{n}{\sqrt{\eta}} + \delta$, with $0 \leq \delta < \frac{1}{\sqrt{\eta}}$, and $n \in {}^*\mathcal{N}$ infinite, then;

$f^{D_\epsilon}(x) = \frac{f(x+\epsilon)-f(x)}{\epsilon}$

$= \frac{f(x+\frac{n}{\sqrt{\eta}})-f(x)}{\frac{n}{\sqrt{\eta}}+\delta}$, as $f \in V(\overline{\mathcal{R}_\eta})$

Without loss of generality, we have that;

$\frac{n}{n+1}\frac{f(x+\frac{n}{\sqrt{\eta}})-f(x)}{\frac{n}{\sqrt{\eta}}} < \frac{f(x+\frac{n}{\sqrt{\eta}})-f(x)}{\frac{n}{\sqrt{\eta}}+\delta} < \frac{f(x+\frac{n}{\sqrt{\eta}})-f(x)}{\frac{n}{\sqrt{\eta}}}$

and;

$\frac{f(x+\frac{n}{\sqrt{\eta}})-f(x)}{\frac{n}{\sqrt{\eta}}}$

$= \frac{1}{n}{}^*\sum_{k=0}^{n-1}\frac{f(x+\frac{k+1}{\sqrt{\eta}})-f(x+\frac{k}{\sqrt{\eta}})}{\frac{1}{\sqrt{\eta}}}$

$= \frac{1}{n}{}^*\sum_{k=0}^{n-1} f^D(x+\frac{k}{\sqrt{\eta}})$

$\simeq f^D(x)$

as f^D is S-continuous, so that;

$f^D(x) \simeq \frac{n}{n+1}f^D(x) < f^{D_\epsilon}(x) < f^D(x)$

as $\frac{n}{n+1} \simeq 1$ and $f^D(x)$ is finite, giving that $f^D(x) \simeq f^{D_\epsilon}(x)$

□

Lemma 0.7. *Let $f \in V(\overline{\mathcal{R}_\eta} \times \overline{\mathcal{T}_\nu})$, $x : \overline{\mathcal{T}_\xi} \rightarrow \overline{\mathcal{R}_\eta}$, $y : \overline{\mathcal{T}_\xi} \rightarrow \overline{\mathcal{T}_\nu}$, be measurable, with the forward derivatives $\{f_x^D, f_{xy}^D, f_t^D, x^D, y^D\}$ bounded. Then, if $H \in V(\overline{\mathcal{T}_\nu})$, with $H(s) = f(x(s), y(s))$, then;*

$$H^D(s) \simeq f_x^D(x(s))x^D(s) + f_y^D(y(s))y^D(s)$$

when D relative to ξ is taken so that $\frac{\sqrt{\eta}}{\xi}$ and $\frac{\nu}{\xi}$ are infinite.

Proof. We have that;

$$H^D(s) = \nu(H(s + \tfrac{1}{\xi}) - H(s))$$

$$= \nu(f(x(s + \tfrac{1}{\xi}), y(s + \tfrac{1}{\xi})) - f(x(s), y(s)))$$

$$= \nu(f(x(s + \tfrac{1}{\xi}), y(s + \tfrac{1}{\xi})) - f(x(s), y(s + \tfrac{1}{\xi}))$$

$$+\nu(f(x(s), y(s + \tfrac{1}{\xi})) - f(x(s), y(s)))$$

$$= \frac{(f(x(s+\frac{1}{\xi}), y(s+\frac{1}{\xi})) - f(x(s), y(s+\frac{1}{\xi})))}{x(s+\frac{1}{\xi}) - x(s)}\xi(x(s + \tfrac{1}{\xi}) - x(s))$$

$$+\frac{(f(x(s), y(s+\frac{1}{\xi})) - f(x(s), y(s)))}{y(s+\frac{1}{\xi}) - y(s)}\xi(y(s + \tfrac{1}{\xi}) - y(s))$$

$$= \frac{(f(x(s+\frac{1}{\xi}), y(s+\frac{1}{\xi})) - f(x(s), y(s+\frac{1}{\xi})))}{x(s+\frac{1}{\xi}) - x(s)}x^D(s) + \frac{(f(x(s), y(s+\frac{1}{\xi})) - f(x(s), y(s)))}{y(s+\frac{1}{\xi}) - y(s)}y^D(s)$$

$$= \frac{(f(x(s)+\epsilon, y(s+\frac{1}{\xi})) - f(x(s), y(s+\frac{1}{\xi})))}{\epsilon}x^D(s) + \frac{(f(x(s), y(s)+\delta) - f(x(s), y(s)))}{\delta}y^D(s)$$

$$= f_x^{D_\epsilon}(x(s), y(s + \tfrac{1}{\xi}))x^D(s) + f_y^{D_\delta}(x(s), y(s))y^D(s)$$

with $\epsilon = x(s + \frac{1}{\xi}) - x(s) \simeq 0$ and $\delta = y(s + \frac{1}{\xi}) - y(s) \simeq 0$, as x^D and y^D are bounded.

We have $x^D(s) = \xi\epsilon = C$, $y^D(s) = \delta\epsilon = D$, with C, D finite, so that $\epsilon\sqrt{\eta} = \frac{C\sqrt{\eta}}{\xi}$ and $\delta\nu = \frac{D\nu}{\epsilon}$ are infinite, by the hypotheses. Applying the result of Lemma 0.6, we have $f_x^{D_\epsilon} \simeq f_x^D$, and $f_y^{D_\delta} \simeq f_y^D$, so;

$$H^D(s) \simeq f_x^D(x(s), y(s + \tfrac{1}{\xi}))x^D(s) + f_y^D(x(s), y(s))y^D(s)$$

As f_{xy}^D is bounded (finite position), f_x^D is S-continuous in y, so that;

$H^D(s) \simeq f_x^D(x(s), y(s))x^D(s) + f_y^D(x(s), y(s))y^D(s)$

as required.

□

Lemma 0.8. *If* $H \in V(\overline{\mathcal{T}_\xi})$ *has the property that;*

$H^D(s) \simeq 0$

for $s \in \overline{\mathcal{T}_\xi}$ *with* s *finite, then;*

$H(0) \simeq H(s)$

for $s \in \overline{\mathcal{T}_\xi}$ *with* s *finite.*

If $G, R \subset V(\overline{\mathcal{T}_\xi})$ *have the property that;*

$G^D(s) \simeq R(s)$

for $s \in \overline{\mathcal{T}_\xi}$ *with* s *finite, then;*

$G(s) \simeq G(0) + \int_{[0, \frac{[s\xi]}{\xi})} R(w) d\mu_\xi(w)$

for $s \in \overline{\mathcal{T}_\xi}$ *with* s *finite.*

Proof. Using the definition of D, we have that;

$|H(s) - H(0)| = |\frac{1}{\xi} {}^* \sum_{k=0}^{[s\xi]-1} \xi(H(\frac{k+1}{\xi}) - H(\frac{k}{\xi}))|$

$= |\int_{[0, \frac{[s\xi]}{\xi})} H^D(w) d\mu_\xi(w)|$

$\leq \int_{[0, \frac{[s\xi]}{\xi})} |H^D(w)| d\mu_\xi(w)$

$\leq \epsilon \int_{[0, \frac{[s\xi]}{\xi})} d\mu_\xi(w)$

$\leq \epsilon \frac{[s\xi]}{\xi}$

where $\epsilon \in \mathcal{R}_{>0}$ is arbitrary. As $\frac{[s\xi]}{\xi}$ is finite, we conclude that;

$|H(s) - H(0)| \simeq 0$

as required for the first result. For the second result;

$|G(s)-G(0)-\int_{[0,\frac{[s\xi]}{\xi})}R(w)d\mu_\xi(w)|$

$=|\frac{1}{\xi}{}^*\sum_{k=0}^{[s\xi]-1}\xi(H(\frac{k+1}{\xi})-H(\frac{k}{\xi}))-\int_{[0,\frac{[s\xi]}{\xi})}R(w)d\mu_\xi(w)|$

$=|\int_{[0,\frac{[s\xi]}{\xi})}H^D(w)d\mu_\xi(w)-\int_{[0,\frac{[s\xi]}{\xi})}R(w)d\mu_\xi(w)|$

$\leq\int_{[0,\frac{[s\xi]}{\xi})}|(H^D-R)(w)|d\mu_\xi(w)$

$\leq\epsilon\int_{[0,\frac{[s\xi]}{\xi})}d\mu_\xi(w)$

$\leq\epsilon\frac{[s\xi]}{\xi}$

where $\epsilon\in\mathcal{R}_{>0}$ is arbitrary. As $\frac{[s\xi]}{\xi}$ is finite, we conclude that;

$|G(s)-G(0)-\int_{[0,\frac{[s\xi]}{\xi})}R(w)d\mu_\xi(w)|\simeq 0$ as required.

□

Lemma 0.9. *Integration by Substitution Suppose* $\{a,b\}\subset\mathcal{R}$*, and let* $f\in C([a,b])$ *be continuous with corresponding* $f_\eta\in V(\overline{\mathcal{R}_\eta}\cap{}^*[a,b))$*, suppose that* $h:[h^{-1}(a),h^{-1}(b)]\rightarrow[a,b]$ *is continuous and increasing, so invertible, with corresponding* $h_\xi\in V(\overline{\mathcal{R}_\xi}\cap{}^*[h^{-1}(a),h^{-1}(b)))$*, and forward derivative* h_ξ^D*, then;*

$$\int_{\overline{\mathcal{R}_\eta}\cap^*[a,b)}f_\eta(x)d\mu_\eta(x)\simeq\int_{\overline{\mathcal{R}_\xi}\cap^*[h^{-1}(a),h^{-1}(b))}(f_\eta\circ h_\xi)(y)h_\xi^D(y)d\mu_\xi(y)$$

provided $\frac{\eta}{\xi}$ *is infinite.*

Proof. For bounded $f\in V(\overline{\mathcal{R}_\eta}\cap^*[a,b))$, we define the measure $\mu_{\eta,l,f}$ by;

$$\mu_{\eta,l,f}([\tfrac{i}{\xi},\tfrac{i+1}{\xi}))=(f_\eta\circ h_\xi)(\tfrac{i}{\xi})\mu_\eta[\tfrac{[\eta h_\xi(\frac{i}{\xi})]}{\eta},\tfrac{[\eta h_\xi(\frac{i+1}{\xi})]}{\eta})$$

and extend linearly to $*$-finite unions of intervals in the corresponding $*$-algebra $\mathfrak{B}$. We have that $\mu_\eta[\frac{[\eta h_\xi(\frac{i}{\xi})]}{\eta},\frac{[\eta h_\xi(\frac{i+1}{\xi})]}{\eta})$

$=\frac{[\eta h_\xi(\frac{i+1}{\xi})]}{\eta}-\frac{[\eta h_\xi(\frac{i}{\xi})]}{\eta}$

$=(\frac{h_\xi(\frac{i+1}{\xi})-\delta}{\eta})-(\frac{h_\xi(\frac{i}{\xi})-\delta'}{\eta})$

$= (\frac{h_\xi(\frac{i+1}{\xi})-h_\xi(\frac{i}{\xi})-\delta''}{\eta})$

where $0 \leq \delta < 1$, $0 \leq \delta' < 1$ and $0 \leq |\delta''| < 1$.

so that;

$|\mu_\eta[\frac{[\eta h_\xi(\frac{i}{\xi})]}{\eta}, \frac{[\eta h_\xi(\frac{i+1}{\xi})]}{\eta}) - (h_\xi(\frac{i+1}{\xi}) - h_\xi(\frac{i}{\xi}))| \leq |\frac{\delta''}{\eta}| \leq \frac{1}{\eta}$

and;

$\mu_\eta[\frac{[\eta h_\eta(\frac{i}{\eta})]}{\eta}, \frac{[\eta h_\eta(\frac{i+1}{\eta})]}{\eta}) \simeq (h_\eta(\frac{i+1}{\eta}) - h_\eta(\frac{i}{\eta}))$

and, as f_η is bounded, with $|f_\eta| \leq C$;

$|f_\eta(h_\xi(\frac{i}{\xi}))\mu_\eta[\frac{[\eta h_\xi(\frac{i}{\xi})]}{\eta}, \frac{[\eta h_\xi(\frac{i+1}{\xi})]}{\eta}) - f_\eta(h_\xi(\frac{i}{\xi}))(h_\xi(\frac{i+1}{\xi}) - h_\xi(\frac{i}{\xi}))|$

$\leq |\frac{C\delta''}{\eta}|$

$\leq \frac{C}{\eta} \simeq 0$, (†)

In particularly, letting $\kappa = [\xi g^{-1}(b)] - [\xi g^{-1}(a)]$, we have that $\frac{C\kappa}{\eta} \simeq 0$ if $\frac{\eta}{\xi}$ is infinite, (††) ;

Then, using (†);
$f_\eta(h_\xi(\frac{i}{\xi}))\mu_\eta[\frac{[\eta h_\xi(\frac{i}{\xi})]}{\eta}, \frac{[\eta h_\xi(\frac{i+1}{\xi})]}{\eta})$

$\simeq f_\eta(h_\xi(\frac{i}{\xi}))(h_\xi(\frac{i+1}{\xi}) - h_\xi(\frac{i}{\xi}))$

$= \frac{1}{\xi} f_\eta(h_\xi(\frac{i}{\xi}))h^D_{xi}(\frac{i}{\xi})$, (∗)

We claim that for f as in the statement of the lemma;

$\int_{[a,b)} f_\eta d\mu_\eta = \int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,f}$ (†), in which case, we obtain the result, as, using (∗) and (††);

$\int_{[a,b)} f_\eta(x) d\mu_\eta(x)$

$= \int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,f}$

$= {}^*\sum_{[\xi h^{-1}(a)]\leq i\leq[\xi h^{-1}(b)]}\mu_{\eta,l,f}([\frac{i}{\xi},\frac{i+1}{\xi}))$

$\simeq {}^*\sum_{[\xi h^{-1}(a)]\leq i\leq[\xi h^{-1}(b)]}f_\eta(h_{xi}(\frac{i}{\xi}))\frac{1}{\xi}h_\xi^D(\frac{i}{\xi})$

$= \frac{1}{\xi}{}^*\sum_{[\xi h^{-1}(a)]\leq i\leq[\xi h^{-1}(b)]}f_\eta(h_\xi(\frac{i}{\xi}))h_\xi^D(\frac{i}{\xi})$

$= \int_{[h^{-1}(a),h^{-1}(b))} f_\eta(h_\xi(y)h_\xi^D(y)d\mu_\xi(y)$

$= \int_{[h^{-1}(a),h^{-1}(b))}(f_\eta\circ h_\xi)(y)h_\xi^D(y)d\mu_\xi(y)$

In order to show (†), we first consider the case when $g=\chi_{(c,d)}$ with $(c,d)\subset[a,b)$. Then, using the argument above and the fact that $\chi_{(c,d)}$ is bounded;

$\int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,g_\eta}$

$\simeq {}^*\sum_{[\eta h^{-1}(a)]\leq i\leq[\eta h^{-1}(b)]}(f\circ h_\xi)(\frac{i}{\xi})(h_\xi(\frac{i+1}{\xi})-h_\xi(\frac{i}{\xi}))$

$= {}^*\sum_{r_c\leq i\leq r_d}(h_\xi(\frac{i+1}{\xi})-h_\xi(\frac{i}{\xi}))$

$= h_\xi(\frac{r_d+1}{\xi})-h_\xi(\frac{r_c}{\xi})$

$\simeq (d-c)$

$\simeq \int_{[a,b)}\chi_{(c,d)}d\mu_\eta$

$= \int_{[a,b)} f d\mu_\eta.$

as h is continuous, where $r_c=min(\{i:h_\xi(\frac{i}{\xi})\geq c\})$, $r_d=max(\{i: h_\xi(\frac{i}{\xi})\leq d\})$,

By a similar argument, we can show that, if $f_r=\lambda_1\chi_{(c_1,d_1)}+\ldots\lambda_r\chi_{(c_r,d_r)}$ is a finite combination of characteristic functions, then;

$\int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,f_{r,\eta}}\simeq\int_{[a,b)} f_{r,\eta}d\mu_\eta$ (♯)

Now, considering the case when f is continuous on $[a,b]$, we can find a sequence $\{f_r : r\in\mathcal{N}\}$, such that $lim_{r\rightarrow\infty}f_r=f$ pointwise, and each f_r has the property (♯).

We claim first that $lim_{r\rightarrow\infty}{}^\circ(\int_{[a,b)} f_{r,\eta}d\mu_\eta)={}^\circ(\int_{[a,b)} f_\eta d\mu_\eta)$, (♯♯). This follows as;

$lim_{r\rightarrow\infty}{}^{\circ}(\int_{[a,b)} f_{r,\eta}d\mu_{\eta})$

$= lim_{r\rightarrow\infty}\int_{[a,b)} {}^{\circ}f_{r,\eta}dL(\mu_{\eta})$

$= lim_{r\rightarrow\infty}\int_{[a,b)} f_{r}d\mu$

$= \int_{[a,b)} fd\mu$

$= \int_{[a,b)} {}^{\circ}f_{\eta}dL(\mu_{\eta})$

$= {}^{\circ}(\int_{[a,b)} f_{\eta}d\mu_{\eta})$

using the definition of the Riemann integral, together with the facts that f_{η} and $\{f_{r,\eta} : r \in \mathcal{N}\}$, are S-continuous, S-integrable, and piecewise S-continuous, S-integrable respectively.

We claim, secondly, that;

$lim_{r\rightarrow\infty}{}^{\circ}(\int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,f_{r,\eta}}) = {}^{\circ}(\int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,f_{\eta}})$, $(\sharp\sharp\sharp)$

This follows, by observing that;

$|\int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,f_{r,\eta}} - \int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,f_{\eta}}|$

$= |{}^{*}\sum_{[\xi h^{-1}(a)]\leq i\leq[\xi h^{-1}(b)]}(\mu_{\eta,l,f_{r,\eta}}([\frac{i}{\xi},\frac{i+1}{\xi})) - \mu_{\eta,l,f_{\eta}}([\frac{i}{\xi},\frac{i+1}{\xi})))$

$\leq {}^{*}\sum_{[\xi h^{-1}(a)]\leq i\leq[\xi h^{-1}(b)]}|f_{r,\eta}\circ h_{\xi}(\frac{i}{\xi}) - f_{\eta}\circ h_{\xi}(\frac{i}{\xi})||h_{\xi}(\frac{i+1}{\xi}) - h_{\xi}(\frac{i}{\xi})|$

$= \int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,|f_{\eta}-f_{r,\eta}|}$

$\leq C\epsilon_{r}$

when $|f_{\eta} - f_{r,\eta}| \leq \epsilon_{r}$, and $C \simeq b - a$, where $\lim_{r\rightarrow\infty}\epsilon_{r} = 0$ if we assume uniform convergence. Now define $U(s_{\eta}) = \int_{[a,b)} s_{\eta}d\mu_{\eta}$, and $T(s_{\eta}) = \int_{[h^{-1}(a),h^{-1}(b))} d\mu_{\eta,l,s_{\eta}}$. Then, we have, using $(\sharp), (\sharp\sharp), (\sharp\sharp\sharp)$, that;

$|U(f_{\eta}) - T(f_{\eta})|$

$= |U(f_{\eta}) - U(f_{r,\eta}) + U(f_{r,\eta}) - T(f_{r,\eta}) + T(f_{r,\eta}) - T(f_{\eta})|$

$\leq |U(f_{\eta}) - U(f_{r,\eta})| + |U(f_{r,\eta}) - T(f_{r,\eta})| + |T(f_{r,\eta}) - T(f_{\eta})|$

$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$

As ϵ was arbitrary, we obtain the result.

□

Lemma 0.10. *Let* $\{u, u^D, u^{D^2}\} \subset V(\overline{\mathcal{T}_\nu})$ *be bounded, and let* $\nu = \kappa\xi$, *with* $\kappa \in {}^*\mathcal{N}$ *and* κ *infinite. Let* $(u^D)_\xi \in V(\overline{\mathcal{T}_\xi})$ *be the measurable counterpart of* u^D. *Then, if* $\{a, b\} \subset \overline{\mathcal{T}_\xi}$ *are finite ;*

$\int_{(a,b)} (u^D)_\xi(x) d\mu_\xi)(x) \simeq u(b) - u(a)$

Proof. For $0 \leq j \leq \kappa - 1$, let $(u^D)_{\xi,j}(x) = \nu(u(x + \frac{j+1}{\nu}) - u(x + \frac{j}{\nu}))$. We claim that for $0 \leq j \leq \kappa-2$, $(u^D)_{\xi,j} \simeq (u^D)_{\xi,j+1}$, (†). We have that;

$|(u^D)_{\xi,j+1} - (u^D)_{\xi,j}|$

$= |\nu(u(x + \frac{j+2}{\nu}) - u(x + \frac{j+1}{\nu})) - \nu(u(x + \frac{j+1}{\nu}) - u(x + \frac{j}{\nu}))|$

$= |\nu(u(x + \frac{j+2}{\nu}) - 2u(x + \frac{j+1}{\nu}) + u(x + \frac{j}{\nu}))|$

$= \frac{1}{\nu}|\nu^2(u(x + \frac{j+2}{\nu}) - 2u(x + \frac{j+1}{\nu}) + u(x + \frac{j}{\nu}))|$

$= \frac{1}{\nu}|(u^{D^2}(x + \frac{j}{\nu}))|$

$\leq \frac{C}{\nu}$

with $C \in \mathcal{R}$. By the triangle inequality, we have that, for $0 \leq j_1 \leq j_2 \leq \kappa - 1$;

$|(u^D)_{\xi,j_2} - (u^D)_{\xi,j_1}| \leq \frac{C\kappa}{\nu} = \frac{C}{\xi} \simeq 0$

It follow that;

$(u^D)_\xi \simeq \frac{1}{\kappa} {}^*\sum_{j=0}^{\kappa-1} (u^D)_{\xi,j}$

and;

$\int_{(a,b)} (u^D)_\xi(x) d\mu_\xi)(x)$

$\simeq \int_{(a,b)} \frac{1}{\kappa} {}^*\sum_{0 \leq j \leq \kappa-1} (u^D)_{\xi,j}$

$= \frac{1}{\kappa\xi}{}^{*}\sum[a\xi] \leq i \leq [b\xi]^{*}\sum_{0\leq j\leq \kappa-1}\nu(u(\frac{i}{\xi}+\frac{j+1}{\nu})-u(\frac{i}{\xi}+\frac{j}{\nu}))$

$= \frac{\nu}{\kappa\xi}(u(\frac{[b\xi]+1}{\xi})-u((\frac{[a\xi]}{\xi})))$

$= u(\frac{[b\xi]+1}{\xi})-u((\frac{[a\xi]}{\xi}))$

$\simeq u(b)-u(a)$

as u is S-continuous.

$\square$

References

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