A NONSTANDARD POISSON SUMMATION FORMULA

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ABSTRACT. We formulate and prove a nonstandard Poisson Summation formula.

We formulate a nonstandard version of the Poisson Summation formula, which might be useful in further applications;

Definition 0.1. *We let $\eta = \nu^2$, where ν is an odd prime. We let;*

$\overline{\mathcal{R}_\eta} = \{\frac{i}{\sqrt{\eta}} : i \in {}^*\mathcal{Z}, -\frac{(\eta-1)}{2} \leq i \leq \frac{(\eta-1)}{2}\}$

$\overline{\mathcal{T}_\eta} = [0,1)_\eta = \{\frac{i}{\sqrt{\eta}} : i \in {}^*\mathcal{Z}, 0 \leq i \leq \sqrt{\eta} - 1\}$

$\overline{\mathcal{N}_\eta} = \{i \in {}^*\mathcal{Z} : 0 \leq i \leq \sqrt{\eta} - 1\}$

$\overline{\mathcal{Z}_\eta} = \{i \in {}^*\mathcal{Z} : -\sqrt{\eta} \leq i \leq \sqrt{\eta} - 1\}$

We recall the rescaled inversion theorems from [2] *and* [3]*;*

If $f \in V(\overline{\mathcal{R}_\eta})$, then, for $y \in \overline{\mathcal{R}_\eta}$;

$\mathcal{F}_\eta(f)(y) = \int_{\overline{\mathcal{R}_\eta}} f(x) exp_\eta(-2\pi i y x) d\mu_\eta(x)$

$f(x) = \int_{\overline{\mathcal{R}_\eta}} \mathcal{F}_\eta(f)(y) exp_\eta(2\pi i x y) d\mu_\eta(y)$

If $g \in V(\overline{\mathcal{T}_\eta})$, then, for $m \in \overline{\mathcal{N}_\eta}$;

$\mathcal{F}_\eta(g)(m) = \int_{\overline{\mathcal{T}_\eta}} g(x) exp_{\sqrt{\eta}}(-2\pi i m x) d\mu_\eta(x)$

$g(x) = {}^*\sum_{m \in \overline{\mathcal{N}_\eta}} \mathcal{F}_\eta(g)(m) exp_{\sqrt{\eta}}(2\pi i m x)$

Let $f \in S(\mathcal{R})$, with corresponding f_η, then we define $g \in V(\overline{\mathcal{T}_\eta})$ by;

$g(x) = {}^*\sum_{n \in \overline{\mathcal{Z}_t}} f_\eta(x + n)$

where $t=[\frac{\sqrt{\eta}}{2}-1]=\frac{\sqrt{\eta}-3}{2}$. .

Lemma 0.2. *We have that;*

$^{*}\sum_{m\in\overline{\mathcal{Z}_t}}f_\eta(m)\simeq {}^{*}\sum_{m\in\overline{\mathcal{Z}_\eta},|m|\leq[\frac{\sqrt{\eta}}{2}]}\mathcal{F}_\eta(f_\eta)(m)$

In particular;

$\sum_{m\in\mathcal{Z}}\mathcal{F}(f)(m)=\sum_{m\in\mathcal{Z}}f(m)$

where $\mathcal{F}$ *denotes the usual Fourier transform.*

Proof. For $m\in\overline{\mathcal{N}_\eta}$, with $0\leq m\leq[\frac{\sqrt{\eta}}{2}]$, we have;

$\mathcal{F}_\eta(g)(m)=\int_{\mathcal{T}_\eta}({}^{*}\sum_{n\in\overline{\mathcal{Z}_t}}f_\eta(x+n))exp_{\sqrt{\eta}}(-2\pi ixm)d\mu_\eta(x)$

$\int_{\mathcal{T}_\eta}({}^{*}\sum_{n\in\overline{\mathcal{Z}_t}}f_\eta(x+n))exp_{\sqrt{\eta}}(-2\pi i(x+n)m)d\mu_\eta(x)$

$=\int_{\mathcal{R}_\eta}f_\eta(x))exp_{\sqrt{\eta}}(-2\pi ixm)d\mu_\eta(x)$

$-\int_{\frac{\sqrt{\eta}-3}{2}}^{\frac{\eta-1}{2\sqrt{\eta}}}f_\eta(x)exp_{\sqrt{\eta}}(-2\pi ixm)d\mu_\eta(x)$

$-\int_{-\frac{\eta-1}{2\sqrt{\eta}}}^{-\frac{\sqrt{\eta}-3}{2}}f_\eta(x)exp_{\sqrt{\eta}}(-2\pi ixm)d\mu_\eta(x)$

$=\int_{\mathcal{R}_\eta}f_\eta(x))exp_{\sqrt{\eta}}(-2\pi ixm)d\mu_\eta(x)+\epsilon$

$=\int_{\mathcal{R}_\eta}f_\eta(x))exp_{\sqrt{\eta}}(-2\pi ix\frac{m\sqrt{\eta}}{\sqrt{\eta}})d\mu_\eta(x)+\epsilon$

$=\int_{\mathcal{R}_\eta}f_\eta(x))exp_\eta(-2\pi ixm)d\mu_\eta(x)+\epsilon$

$=\mathcal{F}_\eta(f_\eta)(m)+\epsilon$

where $|\epsilon|\leq\int_{\frac{\sqrt{\eta}-3}{2}}^{\frac{\eta-1}{2\sqrt{\eta}}}|f_\eta(x)|d\mu_\eta(x)+\int_{-\frac{\eta-1}{2\sqrt{\eta}}}^{-\frac{\sqrt{\eta}-3}{2}}|f_\eta(x)|d\mu_\eta(x)$

$\leq\int_{\frac{\sqrt{\eta}-3}{2}}^{\frac{\eta-1}{2\sqrt{\eta}}}\frac{C}{|x_\eta|^3}d\mu_\eta(x)+\int_{-\frac{\eta-1}{2\sqrt{\eta}}}^{-\frac{\sqrt{\eta}-3}{2}}\frac{C}{|x_\eta|^3}d\mu_\eta(x)$, ($C\in\mathcal{R}$, as $f\in S(\mathcal{R})$)

$\leq 2[-\frac{C}{2x^2}]_{\frac{\sqrt{\eta}-5}{2}}^{\frac{\eta-1}{2\sqrt{\eta}}}$

$\leq 2\frac{C}{2(\frac{\sqrt{\eta}}{4})^2}=\frac{D}{\eta}$, where $D=16C$

For $m\in\overline{\mathcal{N}_\eta}$, with $[\frac{\sqrt{\eta}}{2}]<m\leq\sqrt{\eta}-1$, writing $m=\sqrt{\eta}-r$, where $1\leq r\leq\sqrt{\eta}-[\frac{\sqrt{\eta}}{2}]$, we have that;

$exp_\eta(-2\pi imx)$

$=exp_\eta(-2\pi i(\sqrt{\eta}-r)x)$

$=exp_\eta(-2\pi i(-r)x)$

$=exp_\eta(-2\pi i(m-\sqrt{\eta})x)$

An identical calculation gives that;

$\mathcal{F}_\eta(g)(m)=\mathcal{F}_\eta(f_\eta)(m-\sqrt{\eta})+\epsilon$

where $|\epsilon|\leq\frac{D}{\eta}$

By the inversion theorem, we have that;

$g(x)={}^*\sum_{m\in\overline{\mathcal{N}_\eta}}\mathcal{F}_\eta(g)(m)exp_{\sqrt{\eta}}(2\pi ixm)$

$={}^*\sum_{m\in\overline{\mathcal{N}_\eta},0\leq m\leq[\frac{\sqrt{\eta}}{2}]}(\mathcal{F}_\eta(f_\eta)(m)+\epsilon)exp_{\sqrt{\eta}}(2\pi ixm)$

$+{}^*\sum_{m\in\overline{\mathcal{N}_\eta},[\frac{\sqrt{\eta}}{2}]<m\leq\sqrt{\eta}-1}(\mathcal{F}_\eta(f_\eta)(m-\sqrt{\eta})+\epsilon)exp_{\sqrt{\eta}}(2\pi ixm)$

$={}^*\sum_{m\in\overline{\mathcal{Z}_\eta},0\leq m\leq[\frac{\sqrt{\eta}}{2}]}(\mathcal{F}_\eta(f_\eta)(m)+\epsilon)exp_{\sqrt{\eta}}(2\pi ixm)$

$+{}^*\sum_{m\in\overline{\mathcal{Z}_\eta},-[\frac{\sqrt{\eta}}{2}]\leq m\leq -1}(\mathcal{F}_\eta(f_\eta)(m)+\epsilon)exp_{\sqrt{\eta}}(2\pi ix(\sqrt{\eta}-m))$

$\simeq{}^*\sum_{m\in\overline{\mathcal{Z}_\eta},0\leq m\leq[\frac{\sqrt{\eta}}{2}]}\mathcal{F}_\eta(f_\eta)(m)exp_{\sqrt{\eta}}(2\pi ixm)$

$+{}^*\sum_{m\in\overline{\mathcal{Z}_\eta},-[\frac{\sqrt{\eta}}{2}]\leq m\leq -1}(\mathcal{F}_\eta f_\eta)(m)exp_{\sqrt{\eta}}(2\pi ix(\sqrt{\eta}-m))$

as $|{}^*\sum_{m\in\overline{\mathcal{Z}_\eta},0\leq m\leq[\frac{\sqrt{\eta}}{2}]}\epsilon exp_{\sqrt{\eta}}(2\pi ixm)$

$+{}^*\sum_{m\in\overline{\mathcal{Z}_\eta},-[\frac{\sqrt{\eta}}{2}]\leq m\leq -1}\epsilon exp_{\sqrt{\eta}}(2\pi ix(\sqrt{\eta}-m))|$

$\leq\epsilon\sqrt{\eta}=\frac{D\sqrt{\eta}}{\eta}=\frac{D}{\sqrt{\eta}}\simeq 0$

In particular;

$g(0) = {}^*\sum_{m\in\overline{\mathcal{Z}_t}} f_\eta(m) \simeq {}^*\sum_{m\in\overline{\mathcal{Z}_\eta}, |m|\leq[\frac{\sqrt{\eta}}{2}]} \mathcal{F}_\eta(f_\eta)(m)$, $(**)$

as required.

For the final part, observe that, as $f \in S(R)$, $|f_\eta(m)| \leq \frac{E}{m^2}$, where $E \in \mathcal{R}$. Moreover, by definition of f_η, and the fact that $m\sqrt{\eta}$ is an integer, we have that $f_\eta(m) = f(m)$. Let $L = {}^*\sum_{\mathcal{Z}} f(m)$, let ϵ be arbitrary, and choose $m_0 \in \mathcal{N}$ with $\frac{E}{m_0-1} < \frac{2\epsilon}{3}$, and $L - {}^*\sum_{|m|\leq m_0} f(m) < \frac{\epsilon}{3}$. Then;

$|L - {}^*\sum_{m\in\overline{\mathcal{Z}_t}} f_\eta(m)|$

$= |L - {}^*\sum_{|m|\leq m_0} f(m) - {}^*\sum_{|m|>m_0, m\in\overline{\mathcal{Z}_t}} f_\eta(m)|$

$\leq |L - {}^*\sum_{|m|\leq m_0} f(m)| + |{}^*\sum_{|m|>m_0, m\in\overline{\mathcal{Z}_t}} f_\eta(m)|$

$< \frac{\epsilon}{3} + {}^*\sum_{|m|>m_0, m\in\overline{\mathcal{Z}_t}} \frac{E}{m^2}|$

$< \frac{\epsilon}{3} + \int_{m_0-1}^{\infty} \frac{E}{m^2}$ (by transfer)

$= \frac{\epsilon}{3} + [\frac{-E}{m}]_{m_0-1}^{\infty}$ (by transfer)

$= \frac{\epsilon}{3} + \frac{E}{m_0-1}$

$= \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$

As ϵ was arbitrary, we have that, ${}^*\sum_{m\in\overline{\mathcal{Z}_t}} f_\eta(m) \simeq {}^*\sum_{m\in\mathcal{Z}} f(m)$, and as ${}^*\sum_{m\in\mathcal{Z}} f(m)$ is standard, ${}^\circ({}^*\sum_{m\in\overline{\mathcal{Z}_t}} f_\eta(m)) = {}^*\sum_{m\in\mathcal{Z}} f(m)$, $(\dagger)$. Observe now that, as f_η is S-continuous and S-integrable, for $m \in \mathcal{Z}$ finite, ${}^\circ\mathcal{F}_\eta(f_\eta)(m) = \mathcal{F}(f)(m)$. Moreover, by Lemma 0.19 of [4], we have that;

$\mathcal{F}_\eta((f_\eta)^{D^2})(m) = \chi_\eta^2(m)\mathcal{F}_\eta(f_\eta)(m)$

where D denotes the forward derivative and;

$\chi_\eta(m) = \sqrt{\eta}(exp_\eta(-\frac{2\pi i m}{\sqrt{\eta}}) - 1)$

By Lemma 0.20 of [4], we have the estimate, $4m^2 \leq \chi_\eta^2(m) \leq 64m^2$, for $m \in \overline{\mathcal{Z}_\eta}$, with $|m| \leq [\frac{\sqrt{\eta}}{2}]$. By Lemma 0.21 of [4];

$|(f_\eta)^{D^2}(x)| \leq \frac{F}{|x_\eta|^2}$, for $|x| \geq 1$, $x \in \overline{\mathcal{R}_\eta}$

$(f_\eta)^{D^2}(x)| \leq G$, for $|x| < 1$, $x \in \overline{\mathcal{R}_\eta}$, (*)

Using $(*)$ and the above Lemmas, it follows that;

$|\mathcal{F}_\eta(f_\eta)(m)| = |\frac{\mathcal{F}_\eta((f_\eta)^{D^2})(m)}{\chi_\eta^2(m)}| \leq |\frac{\mathcal{F}_\eta((f_\eta)^{D^2})(m)}{4m^2}| \leq \frac{H}{m^2}$

where $H \in \mathcal{R}$. Now, proceeding as above, we conclude that;

$^\circ({}^*\sum_{m \in \overline{\mathcal{Z}_\eta}, |m| \leq [\frac{\sqrt{\eta}}{2}]} f_\eta(m)) = {}^*\sum_{m \in \mathcal{Z}} \mathcal{F}(f)(m)$, (††)

Combining (†), (††) and$(**)$, we obtain the summation formula ${}^*\sum_{m \in \mathcal{Z}} f(m) = {}^*\sum_{m \in \mathcal{Z}} \mathcal{F}(f)(m)$ as required.

□

The above result assumes that $f \in S(\mathcal{R})$. We make the following definition;

Definition 0.3. *We say that f is piecewise differentiable and analytic on $[0,1]$ if there exist finitely many points $\{p_1, \ldots, p_r\}$, with $0 < p_1 < \ldots < p_r < 1$, such that f is analytic on the intervals $(0, p_1)$, $(p_r, 1)$ and (p_j, p_{j+1}), for $1 \leq j \leq r-1$, left and right differentiable at the points $\{0, p_1, \ldots, p_r, 1\}$, and continuous on $[0,1]$, with the endpoints identified.*

With this definition, we have the following nonstandard version of Dirichlet's Theorem, the proof of which follows easily from the results in [1].

Theorem 0.4. *If g is piecewise differentiable and analytic on $[0,1]$, with measurable counterpart g_η on $\overline{\mathcal{T}_\eta}$, then, for $x \in {}^*[0,1]$, with;*

$\mathcal{Z}_{\sqrt{\eta}} = \{i \in {}^*\mathcal{Z} : -\frac{\sqrt{\eta}-1}{2} \leq i < \frac{\sqrt{\eta}-1}{2}\}$

$S(k, g_\eta) = {}^*\sum_{|m| \leq k, m \in \mathcal{Z}_{\sqrt{\eta}}} \mathcal{F}_\eta(g_\eta)(m) exp_{\sqrt{\eta}}(2\pi i m x)$

and notation as in Definition 0.1, we have that there exists Q infinite, $Q = \eta^{\frac{1}{4}}$, independent of x, such that $S(k, g_\eta)(x) \simeq g_\eta(x)$, for all $0 < k \leq Q$, k infinite.

Definition 0.5. *We say that $h \in P(\mathcal{R})$ if;*

(i). $\sum_{n \in \mathcal{Z}} h(x+n)$ is piecewise differentiable and analytic on $[0,1]$.

(ii). h is bounded and there exists a constant $C \in \mathcal{R}$ *such that* $|h(x)| \leq \frac{C}{|x|^2}$*, for* $|x| > 1$.

We then have the following nonstandard version of the Poisson summation formula;

Theorem 0.6. *If* $h \in P(\mathcal{R})$*, with measurable counterpart* h_η *on* $\overline{\mathcal{R}_\eta}$*, then;*

there exists Q *infinite,* $Q = \eta^{\frac{1}{4}}$*, such that;*

$^*\sum_{|m| \leq k, m \in \mathcal{Z}_{\sqrt{\eta}}} h_\eta(m) \simeq {}^*\sum_{|m| \leq k, m \in \mathcal{Z}_{\sqrt{\eta}}} \mathcal{F}_\eta(h_\eta)(m)$

for all k *infinite with* $0 < k \leq Q$.

In particular;

$\sum_{m \in \mathcal{Z}} h(m) = \sum_{m \in \mathcal{Z}} \mathcal{F}(h)(m)$

Proof. The proof follows the structure of Lemma 0.2. We let;

$g_\eta(x) = {}^*\sum_{n \in \mathcal{Z}_t} h_\eta(x + n)$

$h_{per}(x) = \sum_{n \in \mathcal{Z}} h(x + n)$

By condition (ii) of 0.5, we have that $g_\eta(x) \simeq h_{per,\eta}$ on $^*[0, 1]$. More specifically, we have that;

$|g_\eta(x) - h_{per,\eta}(x)|$

$= |{}^*\sum_{n \in {}^*\mathcal{Z}} h_\eta(x + n) - {}^*\sum_{n \in \mathcal{Z}_t} h_\eta(x + n)|$

$\leq \frac{E}{t} = \frac{F}{\sqrt{\eta}}$

for appropriate constants $\{E, F\} \subset \mathcal{R}_{>0}$. As in Lemma 0.2, with a slight correction in the error term, we have that;

$|\mathcal{F}_\eta(g_\eta)(m) - \mathcal{F}_\eta(h_\eta)(m)| \leq \frac{C}{\sqrt{\eta}}$

for some $C \in \mathcal{R}_{>0}$ and $m \in \mathcal{Z}_{\sqrt{\eta}}$. We have that;

$|\mathcal{F}_\eta(h_{per,\eta})(m) - \mathcal{F}_\eta(g_\eta)(m)|$

$\leq \int_{\mathcal{T}_\eta} |h_{per,\eta}(x) - g_\eta(x)| d\mu_\eta(x)$

$\leq \frac{F}{\sqrt{\eta}}$

so that;

$|\mathcal{F}_\eta(h_{per,\eta})(m) - \mathcal{F}_\eta(h_\eta)(m)|$

$\leq \frac{F}{\sqrt{\eta}} + \frac{C}{\sqrt{\eta}}$

$= \frac{G}{\sqrt{\eta}}$

where $G = F + C$

Using condition (i) of Definition 0.5 and Theorem 0.4, we can find an infinite $Q = \eta^{\frac{1}{4}}$ with;

$h_{per,\eta}(0) \simeq {}^*\sum_{|m|\leq k, m\in\mathcal{Z}_{\sqrt{\eta}}} \mathcal{F}_\eta(h_{per,\eta})(m)$

for all infinite k with $0 < k \leq Q$. As $\frac{QF}{\sqrt{\eta}} \simeq 0$, we obtain that;

$h_{per,\eta}(0) \simeq {}^*\sum_{|m|\leq k, m\in\mathcal{Z}_{\sqrt{\eta}}} \mathcal{F}_\eta(h_\eta)(m)$

for all infinite k with $0 < k \leq Q$. Finally;

$g(0) \simeq h_{per,\eta}(0) \simeq {}^*\sum_{|m|\leq k, m\in\mathcal{Z}_{\sqrt{\eta}}} h_\eta(m)$

by condition (i) of Definition 0.5 and infinite k with $0 < k \leq Q$. Combining these results, gives that;

${}^*\sum_{|m|\leq k, m\in\mathcal{Z}_{\sqrt{\eta}}} h_\eta(m) \simeq {}^*\sum_{|m|\leq k, m\in\mathcal{Z}_{\sqrt{\eta}}} \mathcal{F}_\eta(h_\eta)(m)$

for all infinite k with $0 < k \leq Q$, as required. For the deduction of the standard Poisson summation formula, first observe that from condition (ii) in Definition 0.5 and the proof of Lemma 0.2, we have that;

${}^\circ({}^*\sum_{|m|\leq k, m\in\mathcal{Z}_{\sqrt{\eta}}} h_\eta(m)) = \sum_{m\in\mathcal{Z}} h(m)$

Now, extend the standard infinite sequence, $(\sum_{|m|\leq n}\mathcal{F}(h)(m))$ to an internal sequence $(s_n)_{0\leq n\leq Q}$. By condition (i), we have that $\mathcal{F}(h)(m)\simeq\mathcal{F}_\eta(h_\eta)(m)$, for $m\in\mathcal{Z}$. In particular, by overflow, there exists Q' infinite, with $s_n\simeq {}^*\sum_{|m|\leq n, m\in\mathcal{Z}_{\sqrt{\eta}}}\mathcal{F}_\eta(h_\eta)(m)$, for all n infinite with $0<n\leq Q'\leq Q$. For such n, using the previous result in the Theorem, we have that;

$$s_n\simeq {}^\circ({}^*\sum_{|m|\leq n, m\in\mathcal{Z}_{\sqrt{\eta}}}\mathcal{F}_\eta(h_\eta)(m))$$

$$= {}^\circ({}^*\sum_{|m|\leq n, m\in\mathcal{Z}_{\sqrt{\eta}}}h_\eta(m))=\sum_{m\in\mathcal{Z}}h(m)$$

Using Robinson's result on the limit criteria, see [5], we obtain that;

$$\sum_{m\in\mathcal{Z}}\mathcal{F}(h)(m)=\sum_{m\in\mathcal{Z}}h(m)$$

as required.

□

References

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