A THEORY OF DIVISORS FOR ALGEBRAIC CURVES

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ABSTRACT. The purpose of this paper is two-fold. We first prove a series of results, concerned with the notion of Zariski multiplicity, mainly for non-singular algebraic curves. These results are required in [6], where, following Severi, we introduced the notion of the "branch" of an algebraic curve. Secondly, we use results from [6], in order to develop a refined theory of g_n^r on an algebraic curve. This refinement depends critically on replacing the notion of a point with that of a "branch". We are then able to construct a theory of divisors, *generalising* the corresponding theory in the special case when the algebraic curve is *non-singular*, which is *birationally invariant*.

1. Introduction

In this paper, we use the same definition of an algebraic curve as in [6]. Namely, an algebraic curve C is a closed, irreducible subvariety of dimension 1 in P^w, for some $w \geq 1$, where P^w denotes projective space of dimension w. We will often abbreviate the terminology of "algebraic curve" to just "curve". The advantage of developing a birationally invariant theory of divisors for such curves depends mainly on the viewpoint of the "Italian School" of algebraic geometry. Namely, that there are a number of benefits in studying the geometry of *plane* algebraic curves, $(*)$, and that any algebraic curve C is birational to a plane algebraic curve C', (see, for example, Theorem 1.33 of [6]). It is not the purpose of this paper to discuss the question raised in $(*)$, leaving this point of view for another occasion. The results of this paper cover *all* characteristics of the underlying algebraically closed field L. However, we will make it clear when a result depends on the assumption that L has non-zero characteristic.

2. Smooth Curves

Before looking at this section, the reader is strongly advised to consult the paper [8] for relevant notation and terminology. In particular, the reader should be acquainted with the statement of Theorem 3.3

Thanks to Francesco Severi and the Lamb.

from [8]. We first recall the following theorem (which is Theorem 6.6 in [8]);

Theorem 2.1. *Let hypotheses be as in Theorem 3.3 of* [8]*, with the additional assumption that $char(L) = 0$ and F, D are* smooth *curves. Then the notions of Zariski multiplicity and algebraic multiplicity coincide.*

As we need to refer to the proof of this result from [8] later in the paper, for the convenience of the reader, we repeat it below. The reader should, however, consult [8] for relevant notation.

Proof. As D has a non-constant meromorphic function, we can write D as a finite cover of $P^1(L)$. As we have checked both algebraic multiplicity and Zariski multiplicity are multiplicative over composition (in [8]), a straightforward calculation shows that we need only check the notions agree for the branched finite cover $\pi : F \to P^1(L)$. (1)

Now consider this cover restricted to A^1, let x be the canonical coordinate with $ord_a(\pi^*(x)) = m$, so we have that $\pi^* x = h^m u$, for u a unit in $\mathcal{O}_a$ and h a uniformiser at a. (2)

As u is a unit and $char(L) = 0$, the equation $z^m = u$ splits in the residue field of $\mathcal{O}_a^\wedge$. By Hensel's Lemma and Theorem 5.5 of [8], it is solvable in $\mathcal{O}_a^\wedge$. By the definition of $\mathcal{O}_a^\wedge$, we can find an etale morphism $\pi : (U, b) \to (F, a)$ containing such a solution in the local ring $\mathcal{O}_b$. We may assume that U is irreducible and moreover, as π is etale, that U is smooth. (3)

Now we can embed U in a projective smooth curve F' and, as F' is smooth, extend the morphism π to a projective morphism from F' to F. (4)

We claim that $(ba) \in graph(\pi) \subset F' \times F$ is unramified in the sense of Zariski structures. For this we need the following fact whose algebraic proof relies on the fact that etale morphisms are flat, see [8];

Fact 2.2. *Any etale morphism can be locally presented in the form*

$$\begin{array}{ccc} V & \xrightarrow{g} & Spec((A[T]/f(T))_d) \\ \downarrow \pi & & \downarrow \pi' \\ U & \xrightarrow{h} & Spec(A) \end{array}$$

where $f(T)$ is a monic polynomial in $A[T]$, $f'(T)$ is invertible in $(A[T]/f(T))_d$ and g, h are isomorphisms. (5)

Using Lemma 4.6 of [8] and the fact that the open set V is smooth, we may safely replace $graph(\pi)$ by $\overline{graph(\pi')} \subset F'' \times F$ where F'' is the projective closure of $Spec((A[T]/f(T))$, F is the projective closure of $Spec(A)$ and $\overline{graph(\pi')}$ is the projective closure of $graph(\pi')$ and show that $(g(b)a)$ is Zariski unramified. Note that over the open subset $U = Spec(A) \subset F$, $\overline{graph(\pi')} = Spec((A[T]/f(T)$ as this is closed in $U \times F''$. For ease of notation, we replace $(g(b)a)$ by (ba). (6)

Suppose that f has degree n. Let $\sigma_1 \ldots \sigma_n$ be the elementary symmetric functions in n variables $T_1, \ldots T_n$. Consider the equations

$\sigma_1(T_1, \ldots, T_n) = a_1$

$\ldots$

$\sigma_n(T_1, \ldots, T_n) = a_n$ (*)

where $a_1, \ldots a_n$ are the coefficients of f with appropriate sign. These cut out a closed subscheme $C \subset Spec(A[T_1 \ldots T_N])$. Suppose $(ba) \in graph(\pi') = Spec(A[T]/f(T))$ is ramified in the sense of Zariski structures, then I can find $(a'b_1b_2) \in \mathcal{V}_{abb}$ with $(a'b_1)$,$(a'b_2) \in Spec(A(T)/f(T))$ and b_1, b_2 distinct. Then complete (b_1b_2) to an n-tuple $(b_1b_2c'_1 \ldots c'_{n-2})$ corresponding to the roots of f over a'. The tuple $(a'b_1b_2c'_1 \ldots c'_{n-2})$ satisfies C, hence so does the specialisation $(abbc_1 \ldots c_{n-2})$. Then the tuple $(bbc_1 \ldots c_{n-2})$ satisfies $(*)$ with the coefficients evaluated at a. However such a solution is unique up to permutation and corresponds to the roots of f over a. This shows that f has a double root at (ab) and therefore $f'(T)|_{ab} = 0$. As (ab) lies inside $Spec(A[T]/f(T))_d$, this contradicts the fact that f' is invertible in $A[T]/f(T))_d$. (7)

In (2) we may therefore assume that $\pi^*x = h^m$ for h a local uniformiser at a. Now we have the sequence of ring inclusions given by

$$L[x] \to L[x,y]/(y^m - x) \to R$$

$$x \mapsto \pi^*x, y \mapsto h$$

where R is the coordinate ring of F in some affine neighborhood of a. It follows that we can factor our original map such that F is etale near a over the projective closure of $y^m - x = 0$. (8)

Again, repeating the argument from (4) to (7), we just need to check that the projective closure of $y^m - x$ has multiplicity m at 0 considered as a cover of $P^1(\bar{k})$. This is trival, let $\epsilon \in \mathcal{V}_0$ be generic over $\mathcal{M}$,then as we are working in characteristic 0 we can find distinct $\epsilon_1, \ldots \epsilon_m$ in $\mathcal{M}_*$ solving $y^m = \epsilon$. By specialisation, each $\epsilon_i \in \mathcal{V}_0$. (9) □

The purpose of this section is essentially to find an analogous result to Theorem 2.1 when $char(L) = p \neq 0$. An analogous result was given in [8], however, the proof was flawed. We correct this difficulty here. We obtained similar results, in [8], under different assumptions, by the straightforward method of counting points in the fibres. In this section, we need to use more sophisticated local methods, which will be explained below. We first make the following remark concerning the Frobenius morphism;

Remarks 2.3. *Frobenius*

Given a smooth curve C, defined over a field of characteristic p, with function field $L(C)$, we let $L(C)^{1/p}$ be the field obtained by extracting p^{th} roots of $L(C)$ in some fixed algebraic closure. We denote by C_p the unique (up to isomorphism) smooth curve, having function field $L(C)^{1/p}$. Corresponding to the inclusion $i : L(C) \to L(C)^{1/p}$, we obtain a morphism $Frob : C_p \to C$, which, by some abuse of the standard terminology, (the standard terminology is L-linear Frobenius), we will refer to as Frobenius. Although $L(C)$ and $L(C)^{1/p}$ are clearly isomorphic as fields, they may not be isomorphic over L. Hence, C and C_p are not necessarily isomorphic curves. The Frobenius morphism may be explicitly realised as follows;

Let C be embedded in P^n, for some n, defined by the homogeneous polynomials $\{f_1, \ldots, f_m\}$. Let C' be the variety defined by $\{\overline{f_1}, \ldots, \overline{f_m}\}$, *where, for* $1 \leq j \leq m$, $\overline{f_j}$ *is the homogeneous polynomial obtained by applying inverse Frobenius to the coefficients. Then, by a straightforward calculation using Jacobians,* C' *defines a smooth curve. The morphism Frobenius;*

$$Fr : P^n \rightarrow P^n$$

$$Fr([X_0 : \ldots : X_n]) = [X_0^p : \ldots : X_n^p]$$

restricts to define a morphism $Fr : C' \rightarrow C$. *Let* Rat_k *denote the rational functions of degree* k *on* P^n. *Then* Fr *induces a map;*

$$Fr^* : Rat_k \rightarrow Rat_{kp}$$

by the formula;

$$(Fr^*F)(X_0, \ldots, X_n) = F(X_0^p, \ldots, X_n^p)$$

For a homogeneous polynomial f_j *defining* C, *we have that;*

$$Fr^*(f_j) = (\overline{f}_j)^p$$

Hence, Fr^* *restricts to define an* L*-linear map;*

$$Fr^* : L(C) \rightarrow L(C')$$

One can also define a map;

$$Fr^{-1*} : L[X_0, \ldots, X_n] \rightarrow L[X_0^{1/p}, \ldots, X_n^{1/p}]$$

by the formula;

$$(Fr^{-1*}F)(X_0, \ldots, X_n) = F(X_0^{1/p}, \ldots, X_n^{1/p})$$

For a homogeneous polynomial $\overline{f_j}$ *defining* C', *we have that;*

$$Fr^{-1*}(\overline{f_j}) = (f_j)^{1/p}$$

Hence, Fr^{-1*} *restricts to define an* L*-linear isomorphism;*

$Fr^{-1*}: L(C') \rightarrow L(C)^{1/p}$ (†)

We have that $Fr^{-1*} \circ Fr^* = Id$*, restricted to* Rat_k*, hence;*

$Fr^{-1*} \circ Fr^* : L(C) \rightarrow L(C') \rightarrow L(C)^{1/p}$ (††)

is the inclusion map. Using the fact that C_p *and* C' *are nonsingular projective curves, by* (†) *we obtain an isomorphism* $\theta : C_p \rightarrow C'$*. By* (††)*, we have that;*

$Fr \circ \theta = Frob : C_p \rightarrow C$

Hence, without loss of generality, we can identify the morphisms Fr *and the more abstractly defined morphism* $Frob$*.*

We now make the following further remark.

Remarks 2.4. *Given the hypotheses of Theorem 2.1, with the modification that* $char(L) = p \neq 0$*, we define a point* $(ab) \in F$ *to be wildly ramified if* $mult^{alg}_{(ab)}(F/D)$ *is divisible by* p*. Theorem 2.1 holds excluding wildly ramified points,* $(*)$*. In order to see this, we first replace the argument* (1)*, by showing that, for any* given *point* $a \in D$*, we can find a finite morphism* f *from* D *to* $P^1(L)$*, such that* f *is etale in an open neighborhood of* a*;*

As a *is a non-singular, we can find a uniformising element* t *in the local ring* $O_{a,D}$ *of* D*. Considering* t *as an element of the function field* $L(D)$*, we obtain an embedding* $L(t) \subset L(D)$*, which, as* D *is non-singular, determines a unique morphism* f *from* D *to* $P^1(L)$*. Restricting the morphism to* $A^1(L)$ *and letting* x *be the canonical coordinate, we have that* $f^*(x) = t$*, hence* $ord_a(f^*(x)) = 1$*. This shows that* f *is etale in an open neighborhood of* a *by Theorem 5.2 and Remarks 5.3 of* [8]*.* (†)

As etale morphisms have multiplicity coprime to p*, it is sufficient to check the result* $(*)$ *for a branched cover* $\pi : F \rightarrow P^1(L)$*. If* $a \in F$ *is not wildly ramified for this cover, then we can follow through arguments* (2) *and* (3) *of Theorem 2.1. The argument from* (4) *to* (8) *is the same and we obtain the result of* (9) *again using the fact that* m *there is coprime to* p*. This proves the result* $(*)$*.*

Theorem 2.1 also holds with the modification that $char(L) = p \neq 0$ *and the cover* $pr : F \rightarrow D$ *is* seperable*. However, the proof requires* *more sophisticated methods, which we consider below. We can, however, handle a special case by an elementary counting argument. First observe that we can replace the argument* (1) *by observing that there exists a seperable morphism f from D to $P^1(L)$. This either follows from the argument* (†) *above or using the classical result that the function field $L(D)$ admits a seperating transcendence basis over L, (see p27 of* [2]*). Hence, it is sufficient to check the result for a finite seperable cover $\pi : F \rightarrow P^1(L)$. By a classical result, (see Proposition 2.2, p300, of* [2]*), there exist finitely many ramification points, in particularly finitely many wild ramification points $\{a_1, \ldots, a_n\}$, for the cover π. By the previous proof, we need only check the result of Theorem 2.1 for these finitely many points.*

Special Case. a is a wild ramification point for the cover with the property that that there exist no other wild ramification points in the fibre $\pi^{-1}(\pi(a))$.

As both F and $P^1(L)$ are non-singular, the finite morphism π is flat, by Lemma 5.11 of [8]*. By a result in* [3]*, (Corollary of Proposition 2, p218), we have that;*

$\sum_{y\in\pi^{-1}(x)} mult_y^{alg}(F/P^1)$ *is independent of $x \in P^1(L)$, and equals the cardinality of a generic fibre.*

By Lemma 4.3 of [8]*, a corresponding result also holds for Zariski multiplicities. Hence, by the result of the previous proof in this remark, the claim follows.*

Unfortunately, one can have;

a is a wild ramification point for the cover with the property that there exist other wild ramification points $\{a_1, \ldots, a_r\}$, distinct from a, in the fibre $\pi^{-1}(\pi(a))$.

It seems difficult to find any way of reducing this scenario to the special case. However, one can still use a local method, which is done in the following Theorem.

Theorem 2.5. *Let hypotheses be as in Theorem 2.1, with the modification that $char(L) = p \neq 0$ and the cover $pr : F \rightarrow D$ is seperable. Then the notions of Zariski multiplicty and algebraic multiplicity coincide.*

Proof. By the previous remark, it is sufficient to consider the case when D is $P^1(L)$. Let $a \in F$, such that, without loss of generality, $pr(a) = 0$ in the restriction of pr to $A^1(L)$. As a is non-singular, we can find polynomials $\{f_1, \ldots, f_{n-1}\}$ in the variables $\{x_1, \ldots, x_n\}$ of an affine co-ordinate system A^n, such that a corresponds to the origin O of this system and F is defined locally by;

$$f_1(x_1, \ldots, x_n) = \ldots = f_{n-1}(x_1, \ldots, x_n) = 0$$

with;

$$Jac(\tfrac{f_1,\ldots,f_{n-1}}{x_2,\ldots,x_n})|_{\bar{0}} \neq 0$$

We may then apply the implicit function theorem, (see for example p179 of [1]), in order to find power series $\{\eta_1, \ldots, \eta_{n-1}\}$, in the variable t, with $\eta_j(t) = 0$, for $1 \leq j \leq n-1$, such that;

$$f_j(t, \eta_1(t), \ldots, \eta_{n-1}(t)) = 0, \text{ for } 1 \leq j \leq n-1. \ (*)$$

By $(*)$, we clearly have that the total transcendence degree of $\{t, \eta_1(t), \ldots, \eta_{n-1}(t)\}$ over L is equal to 1. Hence, we have that $\{\eta_1(t), \ldots, \eta_{n-1}(t)\}$ are algebraic over $L(t)$. This implies, by the remarks at the beginning of Section 3 of [5], that they belong to the Henselisation of $L[t]_0$, hence they define functions on some etale cover $(U, 0_{lift})$ with coordinate ring $L[t]^{ext}$ of $(A^1, 0)$. We have the ring map;

$$\frac{L[x_1,\ldots,x_n]}{<f_1,\ldots,f_{n-1}>} \rightarrow_i R = \frac{L[x_1]^{ext}[x_2,\ldots,x_n]}{<x_2-\eta(x_1),\ldots,x_n-\eta_{n-1}(x_1)>}$$

which corresponds to an etale cover (U', a_{lift}) of (F, a). We also have an isomorphism;

$$R \rightarrow_\gamma L[t]^{ext}; \ x_1 \mapsto t, x_2 \mapsto \eta_1(t), \ldots, x_n \mapsto \eta_{n-1}(t)$$

which corresponds to an isomorphism between $(U, 0_{lift})$ and (U', a_{lift}). Now consider the composition;

$$\theta : (U, 0_{lift}) \rightarrow (U', a_{lift}) \rightarrow (F, a) \rightarrow_{pr} (A^1, 0)$$

By the general method of [8], we can define both the algebraic and Zariski multiplicities of these covers. By Theorem 1.4 and Lemma 2.2 of [5], we have that;

$Mult_{(0,a)}(F/D) = Mult_{(0,0_{lift})}(U/A^1)$

By Theorem 1.8 of [5], we also have that;

$mult^{alg}_{(0,a)}(F/D) = mult^{alg}_{(0,0_{lift})}(U/A^1)$

Hence, the theorem is shown by proving that Zariski multiplicity and algebraic multiplicity coincide at $(0, 0_{lift})$ for the seperable cover θ. Suppose that the algebraic multiplicity is m, then, if t is the canonical coordinate for A^1 at 0, we have that;

$\theta^* t = t^m u(t)$ for a unit $u(t) \in L[[t]] \cap L(t)^{alg}$

By the usual factoring argument, see (8) of Theorem 2.1, it is sufficent to check that the Zariski multiplicity of the seperable cover ϕ determined by;

$L[s] \rightarrow \frac{L[t]^{ext}[s]}{<t^m u(t)-s>}$

is equal to m at $(0, 0_{lift})$ as well. This is done by the general method of Lemmas 4.5 and 4.6 of [5]. We apply Weierstrass preparation to $t^m u(t) - s$, see [1] for the power series version of this result, in order to obtain the factorisation;

$t^m u(t) - s = u(t,s)(t^m + c_1(s)t^{m-1} + \ldots + c_m(s)) = u(t,s)g(t,s)$

where $c_j(s) \in L[[s]] \cap L(s)^{alg}$, $c_j(s) = 0$ for $1 \leq j \leq m$ and $u(t,s) \in L[[s,t]] \cap L(s,t)^{alg}$ is a unit, see Lemma 3.2 of [5]. As is done in Lemma 4.6 of [5], we obtain the etale cover determined by;

$\frac{L[t]^{ext}[s]}{<t^m u(t)-s>} \rightarrow \frac{L[t,s]^{ext}}{<u(t,s)g(t,s)>}$

By the argument there, it is sufficient to determine when the Weierstrass factor $g(t,s)$ determines a generically reduced cover. Using the method of resultants in Lemma 4.5 of [5], this occurs if and only if $\frac{\partial g}{\partial t}$ is not identically zero. If $\frac{\partial g}{\partial t}$ is identically zero, we obtain the factorisation $g(t,s) = h(t^p, s)$. This clearly implies that the original cover ϕ is inseperable, which is a contradiction. The theorem is then proved.

□

We now have;

Theorem 2.6. *Let hypotheses be as in Theorem 2.1, with the modification that* $char(L) = p \neq 0$*. If* e *denotes the Zariski multiplicity and* d *the algebraic multiplicity at* $a \in F$*, then* $d = ep^n$ *and* π *factors as* $F \rightarrow_h F' \rightarrow_g D$ *with* $h = Frob^n$ *and* g *having algebraic multiplicity* e *at* $h(a)$*.*

Proof. As in Theorem 6.3 of [8], we can factor π into a purely inseperable morphism $h : F \rightarrow F'$ and a seperable morphism $g : F' \rightarrow D$ with F' a smooth projective curve. We then have a corresponding sequence of field extensions $L(D) \subset L(F') \subset L(F)$, with $L(F)$ a purely inseperable extension of $L(F')$. As $L(F)$ is a purely inseperable field extension of $L(F')$, it has degree p^n for some $n \geq 1$. Hence, $L(F) = L(F')^{1/p^n}$ and we may, without loss of generality, assume that $h = Frob^n$, see also Proposition 2.5 (p302) of [2]. By the previous theorem, the notions of Zariski multiplicity and algebraic multiplicity coincide for the morphism g. By Remarks 2.3, the Frobenius morphism $Frob$ may be identified with Fr, without effecting Zariski or algebraic multiplicities. Clearly, Fr is a bijection on points, hence it is Zariski unramified. Fr has algebraic multiplicity p everywhere, as, for any point $x \in F'$, we can choose a local uniformiser t at x such that $Fr^*(t) = t^p$. It follows that h has algebraic multiplicity p^n everywhere and is Zariski unramified. The result now follows immediately from Lemma 4.5 and Remarks 5.7 of [8].

□

We now give a local version of Theorem 2.1 in the general case of algebraic curves over a field L with $char(L) = 0$ and find an analogous version of Theorem 2.5, in the case when $char(L) = p \neq 0$.

Theorem 2.7. *Let hypotheses be as in Theorem 3.3 of* [8]*, with the additional assumption that* $char(L) = 0$ *and* D *is a* smooth *curve. Let* pr *be the projection map of* F *onto* D*. Then, if* $(ab) \in F$ *is non-singular;*

$$Mult_{ab}(F/D) = mult_{ab}^{alg}(F/D)$$

that is Zariski multiplicity and algebraic multiplicity coincide. In particular, the cover (F/D) *is Zariski unramified at* (ab) *iff there exists an open* $U \subset F$*, containing* (ab)*, such that* $pr : U \rightarrow D$ *is etale.*

Proof. For the first part of the theorem, we follow the proof of Theorem 2.1, the difference between the hypotheses there is that we do *not* assume that F is smooth. Using the fact that D is smooth and the result of Theorem 2.1, we may, without loss of generality, assume that

$D = P^1(L)$. Now, one can follow through the proof of Theorem 2.1, using the fact that (ab) is non-singular, in order to obtain the result. One should make the modification that Zariski multiplicity is well defined for any finite cover $F' \to F$ at (abc) lying over (ab). This follows from an easy extension of Theorem 3.3 (in [8]), to show that a nonsingular open subvariety of an irreducible projective variety of dimension 1 is presmooth (see [4]). For the second part of the theorem, suppose that there exists an open $U \subset F$, containing (ab), such that $pr : U \to D$ is etale. As (ab) is non-singular, we may assume that U defines a non-singular open subvariety of F. Following the argument of Theorem 2.1, from the end of (4) to the end of (7), we obtain that the cover (F/D) is Zariski unramified at (ab). For the converse, assume that the cover is Zariski unramified at (ab). By Theorem 5.2, Remarks 5.3 of [8] and the fact that (ab) is non-singular, it is sufficient to prove that $d(pr) : (m_{(ab)}/m^2_{(ab)})^* \to (m_a/m^2_a)^*$ is an isomorphism. Equivalently, we need to show that the algebraic multiplicity $mult^{alg}_{(ab)}(F/D)$ of pr at $(ab) \in F$ equals 1. This follows from the first part of the theorem.

□

Theorem 2.8. *Let hypotheses be as in Theorem 3.3 of* [8]*, with the additional assumption that* $char(L) = p \neq 0$*,* D *is a* smooth *curve and the projection map* pr *of* F *onto* D *is seperable. Then, if* $(ab) \in F$ *is non-singular;*

$$Mult_{ab}(F/D) = mult^{alg}_{ab}(F/D)$$

that is Zariski multiplicity and algebraic multiplicity coincide. In particular, the cover (F/D) *is Zariski unramified at* (ab) *iff there exists an open* $U \subset F$*, containing* (ab)*, such that* $pr : U \to D$ *is etale.*

Proof. Here, the hypotheses are the same as Theorem 2.5, with the modification that we do *not* assume F is smooth. The proof is similar to the previous theorem. By Remarks 2.4, we can assume that $D = P^1(L)$. Using the fact that (ab) is non-singular, one can either follow through the proof of Theorem 2.1, if (ab) is not wildly ramified for the cover, or one can use the method in Theorem 2.5, if (ab) is wildly ramified for the cover. For the second part, one can use the same reasoning as in the previous theorem.

□

Remarks 2.9. *This last result is required for the proof of Lemma 2.10 from* [6] *under suitable assumptions, when* $char(L) = p \neq 0$*. The reader should consult the final section on Frobenius from the paper* [6]*.*

We finish this section with the following result;

Theorem 2.10. *Let $G(X,Y)=0$ define an irreducible plane algebraic curve C, with a non-singular point at $(0,0)$. Let $(T,\eta(T))$ be a power series representation of this point. Then, for any plane, possibly reduced, algebraic curve $F(X,Y)=0$ passing through $(0,0)$;*

$F(T,\eta(T))\equiv 0$ iff F contains C as a component.

Otherwise, $I(G,F,(0,0))=ord_T F(T,\eta(T))$.

Proof. The proof partly uses the methods of [5]. For the first part, note that if F contains C as a component, then by the Nullstellensatz, there exists $H(X,Y)$ such that $F(X,Y)=H(X,Y)G(X,Y)$. It then follows trivially that $F(T,\eta(T))\equiv 0$. For the converse direction, suppose that $F(T,\eta(T))\equiv 0$. As in Lemma 4.17 of [5], we may interpret the equation $Y-\eta(X)$ as defining a curve C_1 on some etale extension $i:(A^2_{et},(00)^{lift})\rightarrow(A^2,(00))$ such that $i(C_1)\subset C$. The vanishing of $F(X,Y)$ on C_1 then implies that F intersects C in an open dense subset. Therefore, as both F and C define Zariski closed sets, F must contain C as a component. For the second part of the theorem, we may therefore assume that F has finite intersection with C and $ord_T F(T,\eta(T))$ is defined. Suppose that $F(X,Y)$ has degree d and consider F as part of the family of degree d curves Q_d. Without loss of generality, we may suppose that $F(X,Y)=H(X,Y,\bar{v}^0)$ where, for $\bar{v}\in Par_{Q_d}$, $H(X,Y,\bar{v})$ defines an algebraic curve of degree d. Similarily, we can write $G(X,Y)$ in the form $G(X,Y,\bar{u}^0)$ for some non-varying constant $\bar{u}^0$. As in Lemma 4.17 of [5], we have the sequence of maps;

$$L[\bar{v}]\rightarrow\frac{L[X,Y][\bar{v}]}{<G(X,Y,\bar{u}_0),H(X,Y,\bar{v})>}\rightarrow\frac{L[X]^{ext}[Y][\bar{v}]}{<Y-\eta(X),H(X,Y,\bar{v})>}$$

which corresponds to a sequence of finite covers;

$$F_1\rightarrow F'(\bar{u}^0,V)\rightarrow Spec(L[\bar{v}])$$

One checks that the left hand morphism is etale at $(\bar{v}^0,(00)^{lift})$, by direct calculation. We use the fact that F is non-singular at (00), therefore the completion of the local rings $\frac{L[X,Y]}{<G(X,Y,\bar{u}_0)>}_{(00)}$ and $\frac{L[X]^{ext}[Y]}{<Y-\eta(X)>}_{(00)}$ are in both cases equal to the formal power series ring $L[[X]]$.

We now compute the Zariski multiplicity of the cover $F_1 \rightarrow Spec(L[\bar{v}])$ at $(\bar{v}^0, (00)^{lift})$ $(*)$. We are given the formal power series $H(X, \eta(X), \bar{v}) \in L[[X, \bar{v}]]$. Let $d = ord_X H(X, \eta(X), \bar{v}_0)$. Then, by Weierstrass preparation in several variables, see [1], we can find $H_1(X, \bar{v})$ and $U(X, \bar{v})$ in $L[[X, \bar{v}]]$ such that;

$$H(X, \eta(X), \bar{v}) = H_1(X, \bar{v})U(X, \bar{v})$$

and $U(0, \bar{v}_0) \neq 0$ and

$$H_1(X, \bar{v}) = X^d + c_1(\bar{v})X^{d-1} + \ldots c_d(\bar{v})$$

with $c_j(\bar{v}_0) = 0$ for $1 \leq j \leq d$. Now use the proofs of Lemma 4.5 and 4.6 from [5] and the fact that the cover;

$$Spec(H_1(X, \bar{v})) \rightarrow Spec(L[\bar{v}])$$

is generically reduced to show the Zariski multiplicity of the cover $(*)$ is exactly d. This proves that the Zariski multiplicity of the cover;

$$F'(\bar{u}^0, V) \rightarrow Spec(L[\bar{v}])$$

at $((0, 0), \bar{v}^0)$ is exactly d as well. By the general result of the paper [5], that;

$$I(C_{\bar{u}^0}, C_{\bar{v}^0}, (00)) = RightMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0})$$

when $C_{\bar{u}^0}$ defines a reduced algebraic curve, the result of the theorem follows.

□

Remarks 2.11. *This last Theorem was required in the proof of Theorem 6.1 of* [6]*. It is also required in the proof of Remarks 4.8 below.*

3. A refined theory of g_n^r

The purpose of this section is to refine the general theory of g_n^r, given in [6], in order to take into account the notion of a branch for a projective algebraic curve. We will rely heavily on results proved in [6]. We also refer the reader there for the relevant notation. We will make *no* assumptions on the characteristic of the base field L. As usual, by an algebraic curve, we always mean a projective irreducible variety of dimension 1.

Definition 3.1. *Let $C \subset P^w$ be a projective algebraic curve of degree d and let Σ be a linear system of dimension R, contained in the space of algebraic forms of degree e on P^w. Let ϕ_λ belong to Σ, having finite intersection with C. Then, if $p \in C \cap \phi_\lambda$ and γ_p is a branch centred at p, we define;*

$$I_p(C, \phi_\lambda) = I_{italian}(p, C, \phi_\lambda)$$

$$I_p^\Sigma(C, \phi_\lambda) = I_{italian}^\Sigma(p, C, \phi_\lambda)$$

$$I_p^{\Sigma,mobile}(C, \phi_\lambda) = I_{italian}^{\Sigma,mobile}(p, C, \phi_\lambda)$$

$$I_{\gamma_p}(C, \phi_\lambda) = I_{italian}(p, \gamma_p, C, \phi_\lambda)$$

$$I_{\gamma_p}^\Sigma(C, \phi_\lambda) = I_{italian}^\Sigma(p, \gamma_p, C, \phi_\lambda)$$

$$I_{\gamma_p}^{\Sigma,mobile}(C, \phi_\lambda) = I_{italian}^{\Sigma,mobile}(p, \gamma_p, C, \phi_\lambda)$$

where $I_{italian}$ was defined in [6].

It follows that, as λ varies in Par_Σ, we obtain a series of weighted sets;

$$W_\lambda = \{n_{\gamma_{p_1}^1}, \ldots, n_{\gamma_{p_1}^{n_1}}, \ldots, n_{\gamma_{p_m}^1}, \ldots, n_{\gamma_{p_m}^{n_m}}\}$$

where;

$\{p_1, \ldots, p_i, \ldots, p_m\} = C \cap \phi_\lambda$, for $1 \leq i \leq m$,

$\{\gamma_{p_i}^1, \ldots, \gamma_{p_i}^{j(i)}, \ldots, \gamma_{p_i}^{n_i}\}$, for $1 \leq j(i) \leq n_i$, consists of the branches of C centred at p_i

and

$$I_{\gamma_{p_i}^{j(i)}}(C, \phi_\lambda) = n_{\gamma_{p_i}^{j(i)}}$$

By the branched version of the Hyperspatial Bezout Theorem, see [6], the total weight of any of these sets, which we will occasionally abbreviate by $C \sqcap \phi_\lambda$, is always equal to de. Let r be the least integer such that every weighted set W_λ is defined by a linear subsystem

$\Sigma' \subset \Sigma$ of dimension r.

Definition 3.2. *We define;*

$Series(\Sigma) = \{W_\lambda : \lambda \in Par_\Sigma\}$

$dimension(Series(\Sigma)) = r$

$order(Series(\Sigma)) = de$

We then claim the following;

Theorem 3.3. .

(i). $r \leq R$, with equality iff every weighted set W_λ of the series is cut out by a single *form of Σ.*

(ii). $r \lneqq R$ iff there exists a form ϕ_λ in Σ, containing all of C.

Proof. We first show the equivalence of (i) and (ii). Suppose that (i) holds and $r \lneqq R$. Then, we can find a weighted set W and distinct elements $\{\lambda_1, \lambda_2\}$ of Par_Σ such that $W = W_{\lambda_1} = W_{\lambda_2}$. Let $\{\phi_{\lambda_1}, \phi_{\lambda_2}\}$ be the corresponding algebraic forms of Σ and consider the pencil $\Sigma_1 \subset \Sigma$ defined by these forms. We claim that;

$W = C \sqcap (\mu_1\phi_{\lambda_1} + \mu_2\phi_{\lambda_2})$, for $[\mu_1 : \mu_2] \in P^1$ $(*)$

This follows immediately from the results in [6] that the condition of multiplicity at a branch is *linear* and the branched version of the Hyperspatial Bezout Theorem. Now choose a point $p \in C$, which is not a base point for any of the branches in W. Then, the condition that an algebraic form ϕ_λ passes through p defines a hyperplane condition on Par_e, hence, intersects Par_{Σ_1} in a point. Let ϕ_{λ_0} be the algebraic form in Σ_1 defined by this parameter. Then, by $(*)$, we have that;

$W \cup \{p\} \subseteq C \sqcap \phi_{\lambda_0}$

Hence, the total multiplicity of intersection of ϕ_{λ_0} with C is at least equal to $de + 1$. By the branched version of the Hyperspatial Bezout Theorem, C must be contained in ϕ_{λ_0}. Conversely, suppose that (i) holds and there exists a form ϕ_{λ_0} in Σ containing all of C. Let W be cut out by ϕ_{λ_1} and consider the pencil $\Sigma_1 \subset \Sigma$ generated by $\{\phi_{\lambda_0}, \phi_{\lambda_1}\}$. By the same argument as above, we can find ϕ_{λ_2} in Σ_1, distinct from ϕ_{λ_1}, which also cuts out W. Hence, by (i), we must have that $r \lneq R$. Therefore, (ii) holds.

The argument that (ii) implies (i) is similar.

We now prove that (ii) holds. Using the Hyperspatial Bezout Theorem, the condition on Par_Σ that a form ϕ_λ contains C is linear. Let H be the linear subsystem of Σ, consisting of forms containing C and let $h = dim(H)$. Let $K \subset \Sigma$ be a maximal linear subsystem, having finite intersection with C. Then K has no form in common with H and $dim(K) = R-h-1$. We claim that every weighted set in $Series(\Sigma)$ is cut out by a unique form from K. For suppose that $W = C \sqcap \phi_\lambda$ is such a weighted set and consider the linear system defined by $< H, \phi_\lambda >$. If ϕ_μ belongs to this system and has finite intersection with C, then clearly $(C \cap \phi_\lambda) = (C \cap \phi_\mu)$. Using linearity of multiplicity at a branch and the Hyperspatial Bezout Theorem again (by convention, a form containing C has infinite multiplicity at a branch), we must have that $(C \sqcap \phi_\lambda) = (C \sqcap \phi_\mu)$. Now consider $K \cap < H, \phi_\lambda >$. We have that;

$$\begin{aligned} codim(K\cap < H, \phi_\lambda >) &\leq codim(K) + codim(< H, \phi_\lambda >) \\ &= (h+1) + (R-(h+1)) = R. \end{aligned}$$

Hence, $dim(K\cap < H, \phi_\lambda >) \geq 0$. We can, therefore, find a form ϕ_μ belonging to K such that $W = (C \sqcap \phi_\mu)$. We need to show that ϕ_μ is the unique form in K defining W. This follows by the argument given above. It follows immediately that $r = dim(K) = R-h-1$. Hence, $r \lneq R$ iff $h \geq 0$. Therefore, (ii) is shown.

□

Using this theorem, we give a more refined definition of a g_n^r.

Definition 3.4. *Let $C \subset P^w$ be a projective algebraic curve. By a g_n^r on C, we mean the collection of weighted sets, without repetitions, defined by $Series(\Sigma)$ for* some *linear system Σ, such that $r = dimension(Series(\Sigma))$ and $n = order(Series(\Sigma))$. If a branch γ_p^j appears with multiplicity at least s in every weighted set of a g_n^r, as just defined, then we allow the possibility of removing some multiplicity contribution $s' \leq s$ from each weighted set and adjusting n to $n' = n - s'$.*

Remarks 3.5. *The reader should observe carefully that a g_n^r is defined independently of a particular linear system. However, by the previous theorem, for any g_n^r, there exists a $g_{n'}^r$ with $n \leq n'$ such that the following property holds. The $g_{n'}^r$ is defined by a linear system of dimension r, having finite intersection with C, such that each there is a bijection between the weighted sets W in the $g_{n'}^r$ and the W_λ in $Series(\Sigma)$. The original g_n^r is obtained from the $g_{n'}^r$ by removing some fixed branch contribution.*

We now reformulate the results of Section 2 and Section 5 in [6] for this new definition of a g_n^r. In order to do this, we require the following definition;

Definition 3.6. *Suppose that $C \subset P^w(L)$ is a projective algebraic curve and $C^{ext} \subset P^w(K)$ is its non-standard model. Let a g_n^r be given on C, defined by a linear system Σ after removing some fixed branch contribution. We define the extension $g_n^{r,ext}$ of the g_n^r to the nonstandard model C^{ext} to be the collection of weighted sets, without repetitions, defined by $Series(\Sigma)$ on C^{ext}, after removing the same fixed point contribution. Note that, by definability of multiplicity at a branch, see Theorem 6.5 of* [6], *if γ_p^j is a branch of C and;*

$I_{italian}(p, \gamma_p^j, C, \phi_\lambda) \geq k,\ (\lambda \in Par_{\Sigma(L)})$

then;

$I_{italian}(p, \gamma_p^j, C, \phi_\lambda) \geq k,\ (\lambda \in Par_{\Sigma(K)})$

Hence, it is *possible to remove the same fixed point contribution of $Series(\Sigma)$ on C^{ext}. See also the proof of Lemma 3.7.*

It is a remarkable fact that, after introducing the notion of a branch, the definition is independent of the particular linear system Σ. This is the content of the following lemma;

Lemma 3.7. *The previous definition is independent of the particular choice of linear system Σ defining the g_n^r.*

Proof. We divide the proof into the following cases;

Case 1. $\Sigma \subset \Sigma'$;

By the proof of Theorem 3.3, we can find a linear system $\Sigma_0 \subset \Sigma \subset \Sigma'$ of dimension r, having finite intersection with C, such that the g_n^r is defined by removing some fixed contribution from Σ_0. Here, we have also used the fact that the base point contributions (at a branch) of $\{\Sigma_0, \Sigma, \Sigma'\}$ are the same. Again, by Theorem 3.3, if $W_{\lambda'}$ is a weighted set defined by Σ' on C^{ext}, then it appears as a weighted set $V_{\lambda''}$ defined by Σ_0 on C^{ext}. Hence, it appears as a weighted set $V_{\lambda''}$ defined by Σ on C^{ext}. By the converse argument and Remarks 3.5 on base branch contributions, the proof is shown.

Case 2. Σ are Σ' are both linear systems of dimension r, having finite intersection with C, such that $degree(\Sigma) = degree(\Sigma') = n$;

By Theorem 3.3, every weighted set W in the g_n^r is defined uniquely by weighted sets W_{λ_1} and V_{λ_2} in $Series(\Sigma_1)$ and $Series(\Sigma_2)$ respectively. Let (C^{ns}, Φ^{ns}) be a non-singular model of C. Using the method of Section 5 in [6] to avoid the technical problem of presentations of Φ^{ns} and base point contributions, we may, without loss of generality, assume that there exist finite covers $W_1 \subset Par_\Sigma \times C^{ns}$ and $W_2 \subset Par_{\Sigma'} \times C^{ns}$ such that;

$$j_{k,\Sigma}(\lambda, p_j) \equiv Mult_{(W_1/Par_\Sigma)}(\lambda, p_j) \geq k \text{ iff } I_{italian}(p, \gamma_p^j, C, \phi_\lambda) \geq k$$

$$j_{k,\Sigma'}(\lambda', p_j) \equiv Mult_{(W_2/Par_{\Sigma'})}(\lambda', p_j) \geq k \text{ iff } I_{italian}(p, \gamma_p^j, C, \psi_{\lambda'}) \geq k$$

Then consider the sentences;

$$(\forall \lambda \in Par_\Sigma)(\exists! \lambda' \in Par_{\Sigma'})\forall x \in C^{ns}[\textstyle\bigwedge_{k=1}^n (j_k(\lambda, x) \leftrightarrow j_k(\lambda', x))]$$

$$(\forall \lambda' \in Par_\Sigma)(\exists! \lambda \in Par_\Sigma)\forall x \in C^{ns}[\textstyle\bigwedge_{k=1}^n (j_k(\lambda', x) \leftrightarrow j_k(\lambda, x))] \text{ (*)}$$

in the language of $< P^1(L), C_i >$, considered as a Zariski structure with predicates $\{C_i\}$ for Zariski closed subsets defined over L, (see [4]). We have, again by results of [4] or [7], that $< P^1(L), C_i > \prec < P^1(K), C_i >$, for the nonstandard model $P(K)$ of $P(L)$. It follows immediately from the algebraic definition of j_k in [4], that, for any weighted set $W_{\lambda'_1}$ defined by $Series(\Sigma)$ on C^{ext}, there exists a unique weighted set $V_{\lambda'_2}$ defined by $Series(\Sigma')$ on C^{ext} such that $W_{\lambda'_1} = V_{\lambda'_2}$, and conversely. Hence, the proof is shown.

Case 3. Σ are Σ' are both linear systems of dimension r, having finite intersection with C;

Let $n_1 = degree(\Sigma)$ and $n_2 = degree(\Sigma')$. Then the original g_n^r is obtained from $Series(\Sigma)$, by removing a fixed point contribution of multiplicity $n_1 - n$, and, is obtained from $Series(\Sigma')$, by removing a fixed point contribution of multiplicity $n_2 - n$. We now imitate the proof of Case 2, with the slight modification that, in the construction of the sentences given by $(*)$, we make an adjustment of the multiplicity statement at the finite number of branches where a fixed point contribution has been removed. The details are left to the reader. □

Now, using Definition 3.6, we construct a specialisation operator $sp : g_n^{r,ext} \rightarrow g_n^r$. We first require the following simple lemma;

Lemma 3.8. *Let $C \subset P^w(L)$ be a projective algebraic curve and let $C^{ext} \subset P^w(K)$ be its nonstandard model. Let $p' \in C^{ext}$ be a non-singular point, with specialisation $p \in C$. Then there exists a unique branch γ_p^j such that $p' \in \gamma_p^j$.*

Proof. We may assume that $p' \neq p$, otherwise p would be non-singular and, by Lemma 5.4 of [6], would be the origin of a single branch γ_p. Let (C^{ns}, Φ) be a non-singular model of C, then p' must belong to the canonical set $V_{[\Phi]}$, hence there exists a unique $p'' \in C^{ns}$ such that $\Phi(p'') = p'$. By properties of specialisations, $p'' \in C^{ns} \cap \mathcal{V}_{p_j}$ for some $p_j \in \Gamma_{[\Phi]}(x,p)$. Hence, by definition of a branch given in Definition 5.15 of [6], we must have that $p' \in \gamma_p^j$. The uniqueness statement follows as well. □

We now make the following definition;

Definition 3.9. *Let $C \subset P^w(L)$ be a projective algebraic curve and let $C^{ext} \subset P^w(K)$ be its non-standard model. Given a g_n^r on C with extension $g_n^{r,ext}$ on C^{ext}, we define the specialisation operator;*

$sp : g_n^{r,ext} \rightarrow g_n^r$

by;

$sp(\gamma_{p'}) = \gamma_p^j$, *for* $p' \in NonSing(C^{ext})$ *and* γ_p^j *as in Lemma 3.8.*

$sp(\gamma_p^j)=\gamma_p^j$, *for* $p\in Sing(C^{ext})=Sing(C)$ *and* $\{\gamma_p^1,\ldots,\gamma_p^j,\ldots,\gamma_p^s\}$ *enumerating the branches at* p.

$sp(n_1\gamma_{p_1}^{j_1}+\ldots+n_r\gamma_{p_r}^{j_r})=n_1 sp(\gamma_{p_1}^{j_1})+\ldots+n_r sp(\gamma_{p_r}^{j_r})$,

for a linear combination of branches with $n_1+\ldots+n_r=n$

It is also a remarkable fact that, after introducing the notion of a branch, the specialisation operator sp is well defined. This is the content of the following lemma;

Lemma 3.10. *Let hypotheses be as in the previous definition, then, if* W *is a weighted set belonging to* $g_n^{r,ext}$, *its specialisation* $sp(W)$ *belongs to* g_n^r.

Proof. We may assume that there exists a linear system Σ, having finite intersection with C, such that $dimension(\Sigma)=r$ and $degree(\Sigma)=n_1$, with the g_n^r and $g_n^{r,ext}$ both defined by $Series(\Sigma)$, after removing some fixed branch contribution W_0 of multiplicity n_1-n. Let W be a weighted set of the $g_n^{r,ext}$, then $W\cup W_0=(C\sqcap\phi_{\lambda'})$, for some unique $\lambda'\in Par_\Sigma$. We claim that $sp(W\cup W_0)=C\sqcap\phi_\lambda$, for the specialisation $\lambda\in Par_\Sigma$ of λ' $(*)$. As $sp(W_0)=W_0$, it then follows immediately from linearity of sp, that $sp(W)$ belongs to the g_n^r as required. We now show $(*)$. Let $p\in C$ and let γ_p be a branch centred at p. By γ_p^{ext}, we mean the branch at p, where p is considered as an element of C^{ext}. We now claim that;

$$I_{\gamma_p}(C,\phi_\lambda)=I_{\gamma_p^{ext}}(C,\phi_{\lambda'})+\textstyle\sum_{p'\in(\gamma_p\setminus p)}I_{\gamma_{p'}^{ext}}(C,\phi_{\lambda'})\ (**)$$

Let $(C^{ns},\Phi)\subset P^{w'}(L)$ be a non-singular model of C, such that γ_p corresponds to $C^{ns,ext}\cap\mathcal{V}_q$, where $q\in\Gamma_{[\Phi]}(x,p)$ and $\mathcal{V}_q$ is defined relative to the specialisation from $P(K)$ to $P(L)$. Let $C^{ns,ext,ext}\subset P^{w'}(K')$ be a non-standard model of $C^{ns,ext}$, such that γ_q^{ext} corresponds to $C^{ns,ext,ext}\cap\mathcal{V}_q$, where $\mathcal{V}_q$ is defined relative to the specialisation from $P(K')$ to $P(K)$. Then, for $p'\in(\gamma_p\setminus p)$, we can find $q'\in\mathcal{V}_q\cap C^{ns,ext}$ such that $\gamma_{p'}$ corresponds to $\mathcal{V}_{q'}\cap C^{ns,ext,ext}$. We may choose a suitable presentation Φ_{Σ_1} of Φ, such that $Base(\Sigma_1)$ is disjoint from $\Gamma_{[\Phi]}(x,p)$, and, therefore, disjoint from $\Gamma_{[\Phi]}(x,p')$, for $p'\in(\gamma_p\setminus p)$. Let $\{\overline{\phi_\lambda}\}$ denote the lifted family of on C^{ns} from the presentation $\Phi_{\Sigma'}$. In this case, we have, by results of [6], that;

$I_{\gamma_p}(C,\phi_\lambda)=I_q(C^{ns},\overline{\phi_\lambda})$

$I_{\gamma_p^{ext}}(C,\phi_{\lambda'})=I_q(C^{ns},\overline{\phi_{\lambda'}})$

$I_{\gamma_{p'}^{ext}}(C,\phi_{\lambda'})=I_{q'}(C^{ns},\overline{\phi_{\lambda'}})$ (1)

By summability of specialisation, see [6] and [5];

$I_q(C^{ns},\overline{\phi_\lambda})=I_q(C^{ns},\overline{\phi_{\lambda'}})+\sum_{q'\in C^{ns}\cap(\mathcal{V}_q\setminus q)}I_{q'}(C^{ns},\overline{\phi_{\lambda'}})$ (2)

Combining (1) and (2), the result $(**)$ follows, as required. Now, suppose that a branch γ_p occurs with non-trivial multiplicity in $sp(C\sqcap\phi_{\lambda'})$. By Definition 3.9, the contribution must come from either $I_{\gamma_p^{ext}}(C,\phi_{\lambda'})$ or $I_{\gamma_{p'}^{ext}}(C,\phi_{\lambda'})$, for some $p'\in(\gamma_p\setminus p)$. Applying sp to $(**)$, one sees that the branch γ_p occurs with multiplicity $I_{\gamma_p}(C,\phi_\lambda)$. It follows that $sp(C\sqcap\phi_{\lambda'})=C\sqcap\phi_\lambda$, hence $(*)$ is shown. The lemma then follows.

□

We can now reformulate the results of Section 2 and Section 5 of [6] in the language of this refined theory of g_n^r. We first make the following definition;

Definition 3.11. *Let $C\subset P^w$ be a projective algebraic curve and let a g_n^r be given on C. Let W be a weighted set in this g_n^r or its extension $g_n^{r,ext}$ and let γ_p be a branch centred at p. Then we say that;*

γ_p is s-fold (s-plo) for W if it appears with multiplicity at least s.

γ_p is multiple for W if it appears with multiplicity at least 2.

γ_p is simple for W if it is not multiple.

γ_p is counted (contato) s-times in W if it appears with multiplicity exactly s.

γ_p is a base branch of the g_n^r if it appears in every *weighted set.*

γ_p is s-fold for the g_n^r if it is s-fold in W for every *weighted set W of the g_n^r.*

γ_p is counted s-times for the g_n^r if it is s-fold for the g_n^r and is counted s-times in some weighted set W of the g_n^r.

We then have the following;

Theorem 3.12. *Local Behaviour of a g_n^r*

Let C be a projective algebraic curve and let a g_n^r be given on C. Let γ_p be a branch centred at p, such that γ_p is counted s-times for the g_n^r. If γ_p is counted t times in a given weighted set W, then there exists a weighted set W' in $g_n^{r,ext}$ such that $sp(W') = W$ and $sp^{-1}(t\gamma_p)$ consists of the branch γ_p counted s-times and $t-s$ other distinct branches $\{\gamma_{p_1}, \ldots, \gamma_{p_{t-s}}\}$, each counted once in W'.

Proof. Without loss of generality, we may assume that the g_n^r is defined by a linear system Σ of dimension r, having finite intersection with C. Let W be the weighted set defined by ϕ_λ in Σ. Suppose that $s = 0$, then γ_p is not a base branch for Σ. Hence, by Lemma 5.25 of [6], we can find $\lambda' \in \mathcal{V}_\lambda$, generic in Par_Σ, and distinct $\{p_1, \ldots, p_t\} = C^{ext} \cap \phi_{\lambda'} \cap (\gamma_p \setminus p)$ such that the intersections at these points are transverse. Let W' be the weighted set defined by $\phi_{\lambda'}$ in $g_n^{r,ext}$. By the proof of $(*)$ in Lemma 3.10, we have that $sp(W') = W$. By the construction of sp in Definition 3.9, we have that $sp^{-1}(t\gamma_p)$ consists of the distinct branches $\{\gamma_{p_1}, \ldots, \gamma_{p_t}\}$, each counted once in W'. If $s \geq 1$, then γ_p is a base branch for Σ. By Lemma 5.27 of [6], we have that $I_{italian}^{\Sigma,mobile}(p, \gamma_p, C, \phi_\lambda) = t - s$. The result then follows by application of Lemma 5.28 in [6] and the argument given above. ☐

We now note the following;

Lemma 3.13. *Let a g_n^r be given on a projective algebraic curve C. Let W_0 be* any *weighted set on C with total multiplicity n'. Then the collection of weighted sets given by $\{W \cup W_0\}$ for the weighted sets W in the g_n^r defines a $g_{n+n'}^r$.*

Proof. Let the original g_n^r be obtained from a linear system Σ of dimension r and degree n'', having finite intersection with C, after removing some fixed branch contribution J of total multiplicity $n'' - n$. Let $\{\phi_0, \ldots, \phi_r\}$ be a basis for Σ and let $\{n_1\gamma_{p_1}^{j_1}, \ldots, n_m\gamma_{p_m}^{j_m}\}$ be the branches appearing in W_0 with total multiplicity $n_1 + \ldots + n_m = n'$ (†). Let $\{H_1, \ldots, H_m\}$ be hyperplanes passing through the points $\{p_1, \ldots, p_m\}$ and let G be the algebraic form of degree n' defined by $H_1^{n_1} \cdot \ldots \cdot H_m^{n_m}$. Let Σ' be the linear system of dimension r defined by the basis

$\{G \centerdot \phi_0, \ldots, G \centerdot \phi_r\}$. As we may assume that C is not contained in any hyperplane section, Σ' has finite intersection with C. We claim that $g^r_{n''}(\Sigma) \subset g^r_{n''+n'deg(C)}(\Sigma')$, in the sense that every weighted set W_λ defined by $g^r_{n''+n'deg(C)}(\Sigma')$ is obtained from the corresponding V_λ in $g^r_{n''}(\Sigma)$ by adding a *fixed* weighted set $W_1 \supset W_0$ of total multiplicity $n'deg(C)$ $(*)$. The proof then follows as we can recover the original g^r_n by removing the fixed branch contribution $J \cup (W_1 \setminus W_0)$ from $g^r_{n''+n'deg(C)}(\Sigma')$. In order to show $(*)$, let W_1 be the weighted set defined by $C \sqcap G$. By the branched version of the Hyperspatial Bezout Theorem, see Theorem 5.13 of [6], this has total multiplicity $n'deg(C)$. We claim that $W_0 \subset W_1$ $(**)$. Let γ^j_p be a branch appearing in $(\dagger)$ with multiplicity s. By construction, we can factor G as $H^s \centerdot R$, where H is a hyperplane passing through s. We need to show that;

$$I_{\gamma^j_p}(C, H^s \centerdot R) \geq s$$

or equivalently,

$$I_{p_j}(C^{ns}, \overline{H^s \centerdot R}) = I_{p_j}(C^{ns}, \overline{H}^s \centerdot \overline{R}) \geq s$$

for a suitable presentation C^{ns} of a non-singular model of C, see Lemma 5.12 of [6], where we have used the "lifted" form notation there. Using the method of conic projections, see section 4 of [6], we can find a plane projective curve C' birational to C^{ns}, such that the point p_j corresponds to a non-singular point q of C' and;

$$I_{p_j}(C^{ns}, \overline{H}^s \centerdot \overline{R}) = I_q(C', \overline{\overline{H}^s \centerdot \overline{R}}) = I_q(C', \overline{\overline{H}}^s \centerdot \overline{\overline{R}})$$

The result then follows by results of the paper [5] for the intersections of plane projective curves. This shows $(**)$. We now need to prove that, for an algebraic form ϕ_λ in Σ and a branch γ^j_p of C;

$$I_{\gamma^j_p}(C, \phi_\lambda \centerdot G) = I_{\gamma^j_p}(C, \phi_\lambda) + I_{\gamma^j_p}(C, G)$$

This follows by exactly the same argument, reducing to the case of intersections between plane projective curves and using the results of [5]. The result is then shown.

□

Theorem 3.14. *Birational Invariance of a* g^r_n

Let $\Phi : C_1 \rightsquigarrow C_2$ be a birational map between projective algebraic curves. Then, given a g_n^r on C_2, there exists a canonically defined g_n^r on C_1, depending only on the class $[\Phi]$ of the birational map. Conversely, given a g_n^r on C_1, there exists a canonically defined g_n^r on C_2, depending only on the class $[\Phi^{-1}]$ of the birational map. Moreover, these correspondences are inverse.

Proof. By Lemma 5.7 of [6], $[\Phi]$ induces a bijection;

$$[\Phi]^* : \bigcup_{O\in C_2} \gamma_O \rightarrow \bigcup_{O\in C_1} \gamma_O$$

of branches, with inverse given by $[\Phi^{-1}]^*$.

Then $[\Phi]^*$ extends naturally to a map on weighted sets of degree n by the formula;

$$[\Phi]^*(n_1\gamma_{p_1}^{j_1} + \ldots + n_r\gamma_{p_r}^{j_r}) = n_1[\Phi]^*(\gamma_{p_1}^{j_1}) + \ldots + n_r[\Phi]^*(\gamma_{p_r}^{j_r})$$

for a linear combination of branches $\{\gamma_{p_1}^{j_1}, \ldots, \gamma_{p_r}^{j_r}\}$ with $n = n_1 + \ldots + n_r$. Therefore, given a g_n^r on C_2, we obtain a canonically defined collection $[\Phi]^*(g_n^r)$ of weighted sets on C_1 of degree n $(*)$. It is trivial to see that $[\Phi^{-1}]^* \circ [\Phi]^*(g_n^r)$ recovers the original g_n^r on C_2, by the fact the map $[\Phi]^*$ on branches is invertible, with inverse given by $[\Phi^{-1}]^*$. Let C^{ns} be a non-singular model of C_1 and C_2 with morphisms $\Phi_1 : C^{ns} \rightarrow C_1$ and $\Phi_2 : C^{ns} \rightarrow C_2$ such that $\Phi \circ \Phi_1 = \Phi_2$ and $\Phi^{-1} \circ \Phi_2 = \Phi_1$ as birational maps (see the proof of Lemma 5.7 in [6]). We then have that $[\Phi]^*(g_n^r) = [\Phi_1^{-1}]^* \circ [\Phi_2]^*(g_n^r)$. It remains to prove that this collection given by $(*)$ defines a g_n^r on C_1. We will prove first that $[\Phi_2]^*(g_n^r)$ defines a g_n^r on C^{ns} $(\dagger)$. Let the original g_n^r on C_2 be defined by a linear system Σ, having finite intersection with C_2, such that $dimension(\Sigma) = r$ and $degree(\Sigma) = n'$, after removing some fixed branch contribution of multiplicity $n' - n$. We may assume that $n' = n$, as if the fixed branch contribution in question is given by W_0 and $g_n^r \cup W_0 = g_{n'}^r$, then $[\Phi_2]^*(g_n^r) \cup [\Phi_2]^*(W_0) = [\Phi_2]^*(g_{n'}^r)$, hence it is sufficient to prove that $[\Phi_2]^*(g_{n'}^r)$ defines a $g_{n'}^r$. Let W_1 be the fixed branch contribution of the g_n^r on C_2 and let $g_{n''}^r \subset g_n^r$ be obtained by removing this fixed branch contribution. It will be sufficient to prove that $[\Phi_2]^*(g_{n''}^r)$ defines a $g_{n''}^r$ on C^{ns} as $[\Phi_2]^*(g_n^r) = [\Phi_2]^*(g_{n''}^r) \cup [\Phi_2]^*(W_1)$ and we may then use Lemma 3.13. Let Φ_{Σ_1} and Φ_{Σ_2} be presentations of the morphisms Φ_1 and Φ_2. We may assume that $Base(\Sigma_1)$ and $Base(\Sigma_2)$ are disjoint. Let $\{\overline{\phi_\lambda}\}$ denote the lifted family of forms on C^{ns}, defined by the linear system Σ and the presentation Φ_{Σ_2}. We claim that

$[\Phi_2]^*(g^r_{n''})$ is defined by this system after removing its fixed branch contribution. In order to see this, we first show that for any branch γ_p^j of C;

$$I^{\Sigma,mobile}_{\gamma_p^j}(C,\phi_\lambda) = I^{\Sigma,mobile}_{p_j}(C^{ns},\overline{\phi_\lambda}) \ (*) \ (1)$$

where p_j corresponds to γ_p^j in the fibre $\Gamma_{[\Phi_2]}(x,p)$, see Section 5 of [6]. By Definition 2.20 and Lemma 5.23 of [6], we have that;

$$I^{\Sigma,mobile}_{p_j}(C^{ns},\overline{\phi_\lambda}) = Card(C^{ns} \cap (\mathcal{V}_{p_j} \setminus p_j) \cap \overline{\phi_{\lambda'}}) \text{ for } \lambda' \in \mathcal{V}_\lambda, \text{ generic in } Par_\Sigma$$

$$I^{\Sigma,mobile}_{\gamma_p^j}(C,\phi_\lambda) = Card(C \cap (\gamma_p^j \setminus p) \cap \phi_{\lambda'}) \text{ for } \lambda' \in \mathcal{V}_\lambda, \text{ generic in } Par_\Sigma$$

As $(\gamma_p^j \setminus p)$ is in biunivocal correspondence with $(\mathcal{V}_{p_j} \setminus p_j)$ under the morphism Φ_2, we obtain immediately the result $(*)$. Now, using Lemma 5.27 of [6], we have that, if γ_p^j appears in a weighted set W_λ of the $g^r_{n''}$ with multiplicity s, then the corresponding branch γ_{p_j} appears in the weighted set $[\Phi_2]^*(W_\lambda)$ with multiplicity equal to $s = I^{mobile}_{p_j}(C^{ns},\overline{\phi_\lambda})$. Again, using Lemma 5.27 of [6], we obtain that $[\Phi_2]^*(W_\lambda)$ is given by $C^{ns} \sqcap \overline{\phi_\lambda}$, after removing all fixed point contributions of the linear system Σ. We, therefore, obtain that $[\Phi_2]^*(g^r_{n''})$ is defined by Σ, after removing all fixed branch contributions, as required. This proves (†). We now claim that, for the given g^r_n on C^{ns}, $[\Phi_1^{-1}]^*(g^r_n)$ defines a g^r_n on C_1, (††). Let Φ_{Σ_3} be a presentation of the morphism Φ_1^{-1}. If ϕ_λ is a form belonging to the linear system Σ defined on C^{ns}, using the presentations Φ_{Σ_1} and Φ_{Σ_3} of Φ_1 and Φ_1^{-1}, we obtain a lifted form $\overline{\phi_\lambda}$ on C_1 and a lifted form $\overline{\overline{\phi_\lambda}}$ on C^{ns} again. We now claim that, for $p \in C^{ns}$;

$$I^{\Sigma,mobile}_p(C^{ns},\phi_\lambda) = I^{\Sigma,mobile}_p(C^{ns},\overline{\overline{\phi_\lambda}}) \ (2)$$

In order to see this, first observe that we can obtain the lifted system of forms $\{\overline{\overline{\phi_\lambda}}\}$ directly from the linear system Σ_4, obtained by composing bases of the linear systems Σ_1 and Σ_3. The corresponding morphism Φ_{Σ_4} defines a birational map of C^{ns} to itself, which is equivalent to the identity map Id. Now the result follows immediately from Definition 2.20 and Lemma 2.16 of [6], both multiplicities are witnessed inside the canonical set W of Φ_{Σ_4}, which, in this case, is just the domain of definition of Φ_{Σ_4} on C^{ns}, see Definition 1.30 of [6]. Now, returning to the proof of (††), we may suppose that the given g^r_n on C^{ns} is defined by the linear system Σ, after removing all fixed branch contributions.

Combining (1) and (2), we have that, for a branch γ_p^j of C_1;

$$I_{\gamma_p^j}^{\Sigma,mobile}(C_1,\overline{\phi_\lambda})=I_{p_j}^{\Sigma,mobile}(C^{ns},\overline{\overline{\phi_\lambda}})=I_{p_j}^{\Sigma,mobile}(C^{ns},\phi_\lambda)$$

The result now follows from the same argument as above, using Lemma 5.27 of [6]. This completes the theorem.

Remarks 3.15. *Using the quoted Theorem 1.33 of* [6]*, one can use the Theorem to reduce calculations involving g_n^r on projective algebraic curves to calculations on plane projective curves. This idea is central to the philosophy of the "Italian School" of algebraic geometry.*

□

We finally note the following;

Lemma 3.16. *For a given g_n^r, we always have that $r\leq n$.*

Proof. The proof is almost identical to Lemma 2.24 of [6]. We leave the details to the reader.

□

4. A Theory of Complete Linear Series on an Algebraic Curve

We now develop further the theory of g_n^r on an algebraic curve C, analogously to classical results for divisors on non-singular algebraic curves. We will first assume that C is a plane projective algebraic curves. We will first assume that C is a plane projective algebraic curve, defined by some homogeneous polynomial $F(X,Y,Z)$. Without loss of generality, we will use the coordinates $x=X/Z$ and $y=Y/Z$ for local calculations on the curve C, defined in this system by $f(x,y)=0$. Using Theorem 3.14, we will later derive general results for g_n^r on an algebraic curve from the corresponding calculations for the plane case.

We consider first the case when $r=1$. By results of the previous section, a g_n^1 is defined by a pencil Σ of algebraic curves $\{\phi(x,y)+\lambda\phi'(x,y)=0\}_{\lambda\in P^1}$ (in affine coordinates), after removing some fixed branch contribution, where, by convention, we interpret the algebraic curve $\phi(x,y)+\infty\phi'(x,y)=0$ to be $\phi'(x,y)=0$. We assume that the g_n^1 is, in fact, cut out by this pencil. Now suppose that γ_p is a branch of C. We may assume that p corresponds to the origin O of the affine coordinate system (x,y), (use a linear transformation and the result of Lemma 4.1) By Theorem 6.1 of [6], we can find algebraic power series

$\{x(t), y(t)\}$, with $x(t) = y(t) = 0$, parametrising γ_p. We can now substitute the power series in order to obtain a formal expression of the form;

$$\frac{\phi(x(t),y(t))}{\phi'(x(t),y(t))} = \frac{t^i u(t)}{t^j v(t)} = t^{i-j}u(t)v(t)^{-1}, \text{ where } \{u(t), v(t), u(t)v(t)^{-1}\} \text{ are units in } L[[t]].$$

We then define;

(i). $ord_{\gamma_p}(\frac{\phi}{\phi'}) = i - j$,

$val_{\gamma_p}(\frac{\phi}{\phi'}) = 0$, if $i > j$, ($\frac{\phi}{\phi'}$ has a zero of order $i - j$)

(ii). $ord_{\gamma_p}(\frac{\phi}{\phi'}) = j - i$,

$val_{\gamma_p}(\frac{\phi}{\phi'}) = \infty$, if $i < j$, ($\frac{\phi}{\phi'}$ has a pole of order $j - i$)

(iii). $ord_{\gamma_p}(\frac{\phi}{\phi'}) = ord_t(h(t) - h(0))$,

$val_{\gamma_p}(\frac{\phi}{\phi'}) = h(0)$, if $i = j$ and $h(t) = u(t)v(t)^{-1}$

Observe that in all cases, ord_{γ_p} gives a *positive* integer, while val_{γ_p} determines an element of P^1. In order to see that this construction does not depend on the particular power series representation of the branch, we require the following lemma;

Lemma 4.1. *Let $\{C, \gamma_p, \phi, \phi', g_n^1, \Sigma\}$ be as defined above, then;*

$ord_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}(C, \phi - \lambda\phi')$, *if γ_p is not a base branch for the g_n^1 and $\frac{\phi}{\phi'}(p) = val_{\gamma_p}(\frac{\phi}{\phi'}) = \lambda$.*

$ord_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}^{\Sigma,mobile}(C, \phi - \lambda\phi')$, *if γ_p is a base branch for the g_n^1 and $\lambda = val_{\gamma_p}(\frac{\phi}{\phi'})$ is unique such that, for $\mu \neq \lambda$;*
$I_{\gamma_p}(C, \phi - \lambda\phi') > I_{\gamma_p}(C, \phi - \mu\phi')$.

Proof. Suppose that γ_p is not a base branch for the g_n^1, then $\frac{\phi}{\phi'}(p) = \lambda$ is well defined, if we interpret $(c/0) = \infty$ for $c \neq 0$, and $\phi - \lambda\phi'$ is the unique curve in the pencil passing through p. It is trivial to check, using the facts that $\phi(p) = \phi(x(0), y(0))$ and $\phi'(p) = \phi'(x(0), y(0))$, that, in all cases, $val_{\gamma_p}(\frac{\phi}{\phi'}) = \lambda$ as well. By Theorem 6.1 of [6], we have that;

$I_{\gamma_p}(C,\phi-\lambda\phi')=ord_t[(\phi-\lambda\phi')(x(t),y(t))]$

If $\lambda=0$, then $\phi(p)=0$ and $\phi'(p)\neq 0$, hence, by a straightforward algebraic calculation, $\phi(x(t),y(t))=t^iu(t)$, for some $i\geq 1$, and $\phi'(x(t),y(t))=v(t)$ for $\{u(t),v(t)\}$ units in $L[[t]]$. Therefore, $ord_{\gamma_p}(\frac{\phi}{\phi'})=ord_t\phi(x(t),y(t))$ and the result follows.

If $\lambda=\infty$, then $\phi(p)\neq 0$ and $\phi(p)=0$, hence, $\phi(x(t),y(t))=u(t)$ and $\phi'(x(t),y(t))=t^jv(t)$, for some $j\geq 1$, and $\{u(t),v(t)\}$ units in $L[[t]]$. Therefore, $ord_{\gamma_p}(\frac{\phi}{\phi'})=ord_t\phi'(x(t),y(t))$ and the result follows.

If $\lambda\neq\{0,\infty\}$, then $\phi(x(t),y(t))=u(t)$ and $\phi'(x(t),y(t))=v(t)$ with $\{u(t),v(t)\}$ units in $L[[t]]$. As $v(t)$ is a unit in $L[[t]]$, we have that;

$ord_t(\frac{u(t)}{v(t)}-\frac{u(0)}{v(0)})=ord_t(v(t)(\frac{u(t)}{v(t)}-\frac{u(0)}{v(0)}))=ord_t(u(t)-\frac{u(0)}{v(0)}v(t))$

Hence, by definition of ord_{γ_p};

$ord_{\gamma_p}(\frac{\phi}{\phi'})=ord_t[(\phi-\lambda\phi')(x(t),y(t))]$

and the result follows.

Now suppose that γ_p is a base branch for the g_n^1, then $\phi(p)=\phi'(p)=0$ and we have that $\phi(x(t),y(t))=t^iu(t)$ and $\phi'(x(t),y(t))=t^jv(t)$, for some $i,j\geq 1$ and $\{u(t),v(t)\}$ units in $L[[t]]$. Again, we divide the proof into the following cases;

$i>j$. In this case, by definition, $val_{\gamma_p}(\frac{\phi}{\phi'})=0$. We compute;

$ord_t(\phi(x(t),y(t))-\lambda\phi'(x(t),y(t)))=ord_t(t^iu(t)-\lambda t^jv(t))$

When $\lambda=0$, we obtain, by Theorem 6.1 of [6], that $I_{\gamma_p}(C,\phi)=i$ and, for $\lambda\neq 0$, that $I_{\gamma_p}(C,\phi-\lambda\phi')=j$. Using Lemma 5.27 of [6], we obtain that $I_{\gamma_p}^{\Sigma,mobile}(C,\phi)=i-j=ord_{\gamma_p}(\frac{\phi}{\phi'})$, as required.

$i<j$. In this case, by definition, $val_{\gamma_p}(\frac{\phi}{\phi'})=\infty$. The computation for ord_{γ_p} is similar, with the critical value being $\lambda=\infty$.

$i = j$. We compute;

$$ord_t(\phi(x(t), y(t)) - \lambda\phi'(x(t), y(t))) = ord_t[t^i(u(t) - \lambda v(t))]$$

Again, there exists a unique value of $\lambda = \frac{u(0)}{v(0)} = val_{\gamma_p}(\frac{\phi}{\phi'}) \neq \{0, \infty\}$ such that $ord_t(u(t) - \lambda v(t)) = k \geq 1$. By the same calculation as above, we have that $I_{\gamma_p}^{\Sigma,mobile}(C, \phi - \lambda\phi') = k$, for this critical value of λ. By a similar algebraic calculation to the above, using the fact that $v(t)$ is a unit, we also compute $ord_{\gamma_p}(\frac{\phi}{\phi'}) = k$, hence the result follows.

□

We now show the following;

Lemma 4.2. *Given any algebraic curve $C \subset P^w$, with function field $L(C)$, for a non-constant rational function $f \in L(C)$ and a branch γ_p, we can unambiguously define $ord_{\gamma_p}(f)$ and $val_{\gamma_p}(f)$.*

Proof. The proof is similar to the above. We may, without loss of generality, assume that p corresponds to the origin of a coordinate system $(x_1, \ldots, x_w)$. Using Theorem 6.1 of [6], we can find algebraic power series $(x_1(t), \ldots, x_w(t))$ parametrising the branch γ_p. By the assumption that f is non-constant, we can find a representation of f as a rational function $\frac{\phi(x_1,\ldots,x_w)}{\phi'(x_1,\ldots,x_w)}$ in this coordinate system, such that the pencil Σ defined by $\{\phi, \phi'\}$ has finite intersection with C, hence defines a g_n^1. Using the method above, we can define $ord_{\gamma_p}(\frac{\phi}{\phi'})$ and $val_{\gamma_p}(\frac{\phi}{\phi'})$ for this representation. The proof of Lemma 4.1 shows that these are defined independently of the particular power series parametrising the branch. We need to check that they are also defined independently of the particular representation of f. Suppose that $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ are algebraic forms with the property that $\frac{\phi_1}{\phi_2} = \frac{\phi_3}{\phi_4}$ as rational functions on C. We claim that, for any branch γ_p of C, $ord_{\gamma_p}(\frac{\phi_1}{\phi_2}) = ord_{\gamma_p}(\frac{\phi_3}{\phi_4})$ and $val_{\gamma_p}(\frac{\phi_1}{\phi_2}) = val_{\gamma_p}(\frac{\phi_3}{\phi_4})$, $(*)$. In order to see this, let $U \subset NonSing(C)$ be an open subset of C, on which $\frac{\phi_1}{\phi_2}$ and $\frac{\phi_3}{\phi_4}$ are defined and equal. Let g_n^1 and g_m^1 on C be defined by the pencils $\Sigma_1 = \{\phi_1 - \lambda\phi_2\}_{\lambda\in P^1}$ and $\Sigma_2 = \{\phi_3 - \lambda\phi_4\}_{\lambda\in P^1}$. Let $V = U \setminus Base(\Sigma_1) \cup Base(\Sigma_2)$. Then $V \subset U$ is also an open subset of C, which we will refer to as the canonical set. Now, suppose that $\gamma_p \subset V$. We will prove $(*)$ for this branch. As both $\frac{\phi_1}{\phi_2}$ and $\frac{\phi_3}{\phi_4}$ are defined and equal at p, using the argument in Lemma 4.1, we have that $val_{\gamma_p}(\frac{\phi_1}{\phi_2}) = val_{\gamma_p}(\frac{\phi_3}{\phi_4})$. It is therefore sufficient, again by Lemma 4.1, to show that;

$$I_{\gamma_p}(C,\phi_1-\lambda\phi_2)=I_{\gamma_p}(C,\phi_3-\lambda\phi_4),\text{ for }\frac{\phi_1}{\phi_2}(p)=\frac{\phi_3}{\phi_4}(p)=\lambda\ (\dagger)$$

Suppose that $I_{\gamma_p}(\phi_1-\lambda\phi_2)=m$, then, by Lemma 5.25 of [6], we can find $\lambda'\in\mathcal{V}_\lambda\cap P^1$ and $\{p_1,\ldots,p_m\}=V\cap\mathcal{V}_p\cap(\phi_1-\lambda'\phi_2)=0$ witnessing this multiplicity. As $\{p,p_1,\ldots,p_m\}$ lie inside V, we also have that $\{p_1,\ldots,p_m\}\subset V\cap\mathcal{V}_p\cap(\phi_3-\lambda'\phi_4)=0$, hence $I_{\gamma_p}(C,\phi_3-\lambda\phi_4)\geq m$. The result ($\dagger$) then follows from the converse argument.

Now, suppose that γ_p is one of the finitely many branches of C, not lying inside V. We will just consider the case when γ_p is a base branch for both the g_n^1 and the g_m^1 defined above, the other cases being similar. In order to prove ($*$) for this branch, it is sufficient, by Lemma 4.1, to show that;

$$I_{\gamma_p}^{\Sigma_1,mobile}(C,\phi_1-\lambda\phi_2)=I_{\gamma_p}^{\Sigma_2,mobile}(C,\phi_3-\mu\phi_4),\text{ for the critical values }\{\lambda,\mu\}$$

and that the critical values $\{\lambda,\mu\}$ coincide, ($\dagger\dagger$).

Using the argument to prove ($\dagger$), witnessing the corresponding multiplicities in the canonical set V, it follows that for *any* $\nu\in P^1$;

$$I_{\gamma_p}^{\Sigma_1,mobile}(C,\phi_1-\nu\phi_2)=I_{\gamma_p}^{\Sigma_2,mobile}(C,\phi_3-\nu\phi_4),\ (\dagger\dagger\dagger)$$

If the critical values $\{\lambda,\mu\}$ were distinct, we would have that;

$$\begin{array}{ccc}I_{\gamma_p}^{\Sigma_1,mobile}(C,\phi_1-\lambda\phi_2) & > & I_{\gamma_p}^{\Sigma_1,mobile}(C,\phi_1-\mu\phi_2)\\ || & & ||\\ I_{\gamma_p}^{\Sigma_2,mobile}(C,\phi_3-\lambda\phi_4) & < & I_{\gamma_p}^{\Sigma_2,mobile}(C,\phi_3-\mu\phi_4)\end{array}$$

which is clearly a contradiction. Hence, $\lambda=\mu$ and the result ($\dagger\dagger$) follows from ($\dagger\dagger\dagger$). The lemma is shown. □

Lemma 4.3. *Birational Invariance of ord_{γ_p} and val_{γ_p}*

Let $\Phi:C_1\leftrightsquigarrow C_2$ be a birational map between projective algebraic curves with corresponding isomorphisms $\Phi^*:L(C_2)\rightarrow L(C_1)$ and $[\Phi]^*:\bigcup_{p\in C_2}\gamma_p\rightarrow\bigcup_{q\in C_1}\gamma_q$. Then, for non-constant $f\in L(C_2)$ and γ_p a branch of C_2, $ord_{\gamma_p}(f)=ord_{[\Phi]^*\gamma_p}(\Phi^*f)$ and $val_{\gamma_p}(f)=val_{[\Phi]^*\gamma_p}(\Phi^*f)$.

Proof. Let f be represented as a rational function by $\frac{\phi_1}{\phi_2}$, as in Lemma 4.2, and consider the g_n^1 on C_2, defined by the linear system $\Sigma = \{\phi_1 - \lambda\phi_2\}_{\lambda\in P^1}$. Let Φ_{Σ_1} be a presentation of the birational map Φ. Using this presentation, we may lift the system Σ to a corresponding linear system $\{\overline{\phi_1} - \lambda\overline{\phi_2}\}_{\lambda\in P^1}$. It is trivial to check that $\Phi^* f$ is represented by the rational function $\frac{\overline{\phi_1}}{\overline{\phi_2}}$. The proof of Theorem 3.14 shows that, for a branch γ_p of C_2;

$$I_{\gamma_p}^{\Sigma,mobile}(C_2, \phi_1 - \lambda\phi_2) = I_{[\Phi]^*\gamma_p}^{\Sigma,mobile}(C_1, \overline{\phi_1} - \lambda\overline{\phi_2}),\ (*)$$

We now need to consider the following cases;

Case 1. γ_p and $[\Phi]^*\gamma_p$ are not base branches for Σ on C_2 and C_1.

Case 2. γ_p is not a base branch, but $[\Phi]^*\gamma_p$ is a base branch for Σ on C_2 and C_1.

Case 3. γ_p is a base branch and $[\Phi]^*\gamma_p$ is a base branch for Σ on C_2 and C_1.

For Case 1, we have, by Lemma 4.1 and $(*)$;

$$ord_{\gamma_p}(\tfrac{\phi_1}{\phi_2}) = I_{\gamma_p}(C_2, \phi_1 - \lambda\phi_2) = I_{[\Phi]^*\gamma_p}(C_1, \overline{\phi_1} - \lambda\overline{\phi_2}) = ord_{[\Phi]^*\gamma_p}(\tfrac{\overline{\phi_1}}{\overline{\phi_2}})$$

where $\frac{\phi_1}{\phi_2}(p) = \frac{\overline{\phi_1}}{\overline{\phi_2}}(q) = val_{\gamma_p}(\frac{\phi_1}{\phi_2}) = val_{\gamma_q}(\frac{\overline{\phi_1}}{\overline{\phi_2}}) = \lambda$ and $[\Phi]^*\gamma_p = \gamma_q$.

For Case 3, we have, by Lemma 4.1, $(*)$ and a similar argument to the previous lemma, in order to show the critical value $\lambda = val_{\gamma_p}(\frac{\phi_1}{\phi_2})$ is also the critical value $val_{\gamma_q}(\frac{\overline{\phi_1}}{\overline{\phi_2}})$ for the lifted system at the corresponding branch $[\Phi]^*\gamma_p$, that;

$$ord_{\gamma_p}(\tfrac{\phi_1}{\phi_2}) = I_{\gamma_p}^{\Sigma,mobile}(C_2, \phi_1 - \lambda\phi_2) = I_{[\Phi]^*\gamma_p}^{\Sigma,mobile}(C_1, \overline{\phi_1} - \lambda\overline{\phi_2}) = ord_{[\Phi]^*\gamma_p}(\tfrac{\overline{\phi_1}}{\overline{\phi_2}})$$

Case 2 is similar, we leave the details to the reader.

The lemma now follows from the previous lemma, that the definitions of $ord_{\gamma_p}(f)$, $ord_{[\Phi]^*\gamma_p}(\Phi^* f)$,$val_{\gamma_p}(f)$ and $val_{[\Phi]^*\gamma_p}(\Phi^* f)$ are independent of their particular representations.

□

We now show;

Lemma 4.4. *flatness*

Let C be a projective algebraic curve, then, to any non-constant rational function f on C, we can associate a g_n^1 on C, which we will denote by (f), where $n = deg(f)$.

Proof. We define the weighted set $(f = \lambda)$ as follows;

$(f = \lambda) := \{n_{\gamma_1}, \ldots, n_{\gamma_r}\}$

where $\{\gamma_1, \ldots, \gamma_r\} = \{\gamma : val_\gamma(f) = \lambda\}$ and $n_\gamma = ord_\gamma(f)$.

As λ varies over P^1, we obtain a series of weighted sets W_λ on C. We claim that this series does in fact define a g_n^1. In order to see this, let f be represented as a rational function by $\frac{\phi}{\phi'}$. As before, we consider the pencil Σ of forms defined by $(\phi - \lambda\phi')_{\lambda\in P^1}$. We claim that the series is defined by this system Σ, after removing its fixed branch contribution, $(*)$. In order to see this, we compare the weighted sets $(f = \lambda)$ and $C \sqcap (\phi - \lambda\phi')$. For a branch γ_p which is not a fixed branch of the system Σ, we have, using Lemmas 4.1 and 4.2, that;

$\gamma_p \in (f = \lambda)$ iff $val_{\gamma_p}(f) = \lambda$ iff $\frac{\phi}{\phi'}(p) = \lambda$ iff $p \in C \cap (\phi - \lambda\phi')$

In this case, by Lemmas 4.1 and 4.2, we have that;

$n_{\gamma_p} = ord_{\gamma_p}(f) = ord_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}(C, \phi - \lambda\phi')$

For a branch γ_p which is a fixed branch of the system Σ, we have, by Lemmas 4.1 and 4.2, that;

$\gamma_p \in (f = \lambda)$ iff $val_{\gamma_p}(\frac{\phi}{\phi'}) = \lambda$ iff $p \in C \cap (\phi - \lambda\phi')$ and λ is a critical value for the system Σ at γ_p.

In this case, by Lemmas 4.1 and 4.2, we have that;

$n_{\gamma_p} = ord_{\gamma_p}(f) = ord_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}^{\Sigma,mobile}(C, \phi - \lambda\phi')$ (1)

Let $I_{\gamma_p} = min_{\mu\in P^1} I_{\gamma_p}(C, \phi - \mu\phi')$ be the fixed branch contribution of Σ at γ_p. Then, at the critical value λ for the system Σ;

$I_{\gamma_p}^{\Sigma,mobile}(\phi-\lambda\phi') = I_{\gamma_p}(C,\phi-\lambda\phi') - I_{\gamma_p}$ (2)

Hence, the result $(*)$ follows from $(1),(2)$ and the definition of $C\sqcap(\phi-\lambda\phi')$.

Finally, we show that $n = deg(f)$. Let Γ_f be the correspondence determined by the rational map $f : C \rightsquigarrow P^1$. By classical arguments, $deg(f)$ is equal to the cardinality of the generic fibre $\Gamma_f(\lambda)$, for $\lambda \in P^1$. Fixing a presentation $\frac{\phi}{\phi'}$ for f, if $U \subset NonSing(C)$ is the canonical set for this presentation, one may assume that the generic fibre $\Gamma_f(\lambda)$ lies inside U. By Lemma 2.17 of [6], one may also assume that the corresponding weighted set of the g_n^1 defined by $(f = \lambda)$ consists of n distinct branches, centred at the points of the generic fibre $\Gamma_f(\lambda)$. Therefore, the result follows.

□

Remarks 4.5. *By convention, for a non-zero rational function* $c \in L \setminus \{0\}$*, we define* $(c = 0)$ *and* $(c = \infty)$ *to be the empty weighted sets. The notion of a weighted set in a* g_n^1*, generalises the classical notion of the divisor on a non-singular curve. Using the above theorem, we can make sense of the notion of linear equivalence of weighted sets.*

We make the following definition;

Definition 4.6. *Linear equivalence of weighted sets*

Let C *be an algebraic curve and let* A *and* B *be weighted sets on* C *of the same total multiplicity. We define* $A \equiv B$ *if there exists a* g_n^r *on* C *such that* A *and* B *belong to this* g_n^r *as weighted sets.*

Theorem 4.7. *Let hypotheses be as in the previous definition. If* $A \equiv B$*, then there exists a rational function* g *on* C*, such that* A *is defined by* $(g = 0)$ *and* B *is defined by* $(g = \infty)$*, possibly after adding some fixed branch contribution.*

Proof. If $r = 0$ in the definition, then we must have that $A = B$. Hence, we obtain the statement of the theorem by adding the fixed branch contribution A to the empty g_0^0, defined by $(c = 0) = (c = \infty)$, for a non-constant $c \in L^*$. Otherwise, by the definition of a g_n^r, we may, without loss of generality, find a pencil Σ of algebraic forms, $\{\phi - \lambda\phi'\}_{\lambda \in P^1}$, having finite intersection with C, such that;

$A = C \sqcap (\phi - \lambda_1\phi)$,

$B = C \sqcap (\phi - \lambda_2\phi') \qquad (\lambda_1 \neq \lambda_2)$

Let f be the rational function on C defined by $\frac{\phi}{\phi'}$. If A and B have no branches in common (with multiplicity), (†), then the pencil Σ can have no fixed branches and, by Lemma 4.4, we have that;

$A = (f = \lambda_1)$

$B = (f = \lambda_2) \qquad (\lambda_1 \neq \lambda_2)$

Now we can find an algebraic automorphism α of P^1, taking λ_1 to 0 and λ_2 to ∞. We will assume that $\{\lambda_1, \lambda_2\} \neq \infty$, in which case α can be given, for a coordinate z on P^1, by the Mobius transformation $\frac{z-\lambda_1}{z-\lambda_2}$. The other cases are left to the reader. Let g be the rational function on C defined by $\alpha \circ f$. Now, suppose that γ is a branch of C, with $val_\gamma(f) = \lambda$ and $ord_\gamma(f) = m$. Then, we claim that $val_\gamma(g) = \alpha(\lambda)$ and $ord_\gamma(g) = m$, $(*)$. If $\lambda \neq \{\lambda_2, \infty\}$, using the method before Lemma 4.1, we obtain the following power series representation of g at γ;

$$\begin{aligned}\frac{(\lambda+\mu t^m+o(t^m))-\lambda_1}{(\lambda+\mu t^m+o(t^m))-\lambda_2} &= [(\lambda-\lambda_1)+\mu t^m+o(t^m)] \cdot \frac{1}{(\lambda-\lambda_2)}[1-\frac{\mu}{(\lambda-\lambda_2)}t^m+o(t^m)] \\ &= \frac{\lambda-\lambda_1}{\lambda-\lambda_2} + t^m[\frac{\mu(\lambda-\lambda_2)-\mu(\lambda-\lambda_1)}{(\lambda-\lambda_2)^2}] + o(t^m) \\ &= \frac{\lambda-\lambda_1}{\lambda-\lambda_2} + t^m[\frac{\mu(\lambda_1-\lambda_2)}{(\lambda-\lambda_2)^2}] + o(t^m)\end{aligned}$$

and the claim $(*)$ follows from the assumption that $\lambda_1 \neq \lambda_2$. If $\lambda = \lambda_2$, we obtain the following power series representation of g at γ;

$$\frac{(\lambda+\mu t^m+o(t^m))-\lambda_1}{(\mu t^m+o(t^m))} = \frac{1}{t^m} \cdot [(\lambda-\lambda_1)+\mu t^m+o(t^m)] \cdot [\mu+o(1)]^{-1}$$

which gives that $val_\gamma(g) = \infty = \alpha(\lambda_2)$ and $ord_\gamma(g) = m$, using the fact that $\lambda \neq \lambda_1$. Finally, if $\lambda = \infty$, the Mobius transformation at ∞ is given by $\frac{\frac{1}{z}-\lambda_1}{\frac{1}{z}-\lambda_2} = \frac{1-\lambda_1 z}{1-\lambda_2 z}$ and g may be represented at γ by $\frac{\phi-\lambda_1\phi'}{\phi-\lambda_2\phi'}$. We then obtain the power series representation of g at γ;

$$\begin{aligned}\frac{(t^i u(t)-\lambda_1 t^{i+m}v(t))}{(t^i u(t)-\lambda_2 t^{i+m}v(t))} &= \frac{(u(t)-\lambda_1 t^m v(t))}{(u(t)-\lambda_2 t^m v(t))} = \frac{[1-\lambda_1 t^m \frac{v(t)}{u(t)}]}{[1-\lambda_2 t^m \frac{v(t)}{u(t)}]} \\ &= 1+(\lambda_2-\lambda_1)t^m w(t)+o(t^m), \text{ for } \{u(t), v(t), w(t)\} \\ &\qquad \text{units in } L[[t]]\end{aligned}$$

which gives that $val_\gamma(g) = 1 = \alpha(\infty)$ and $ord_\gamma(g) = m$, using the fact that $\lambda_1 \neq \lambda_2$ again. This gives the claim $(*)$. It follows that the weighted sets $(f = \lambda)$ correspond exactly to the weighted sets $(g = \alpha(\lambda))$, in particularly the g_n^1 defined by (f) and (g), as in Lemma 4.4, is the same. With this new parametrisation of the g_n^1, we then have that;

$$A = (g = 0)$$

$$B = (g = \infty)$$

Hence, the result follows, with the assumption (†). If A and B have branches in common, with multiplicity, we let $A \cap B$ denote the weighted set consisting of these common branches (with multiplicity). Then, the same argument holds, replacing A by $A \setminus B = A - (A \cap B)$ and B by $B \setminus A = B - (A \cap B)$. After adding the fixed branch contribution $(A \cap B)$ to the g_n^1 defined by (g), we then obtain the result. Note that, by Lemma 3.13, this addition defines a $g_{n+n'}^1$, where n' is the total multiplicity of $(A \cap B)$.

□

Remarks 4.8. *The definition we have given of linear equivalence of weighted sets on a projective algebraic curve C generalises the modern definition of linear equivalence for effective divisors on a smooth projective algebraic curve. More precisely we have;*

Modern Definition; Let A and B be effective divisors on a smooth projective algebraic curve C, then $A \equiv B$ iff $A - B = div(g)$, for some $g \in L(C)^*$.

See, for example, p161 of [12] *for relevant definitions and notation. We now show that our definition is the same in this case. First, observe that there exists a natural bijection between the set of effective divisors on C, in the sense of* [12]*, and the collection of weighted sets on C, $(*)$. This follows immediately from the fact, given in Lemma 5.29 of* [6]*, that, for each point $p \in C$, there exists a unique branch γ_p, centred at p. Secondly, observe that the notion of $div(g)$, for $g \in L(C)$, as given in* [12]*, is the same as the notion of $div(g)$ which we give in Definition 4.9 below, (taking into account the identification $(*)$),* (†)*. This amounts to checking that, for a point $p \in C$, with corresponding branch γ_p;*

$v_p(g) = ord_{\gamma_p}(g)$ (††)

where $v_p(g)$ is defined in p152 of [12]*. First, one can use the fact, given in Lemma 4.9 of* [6]*, together with remarks from the final section of this paper, that there exists a birational map $\phi : C \leftrightsquigarrow C'$, such that C' is a plane projective algebraic curve, and p corresponds to a non-singular point $p' \in C'$ with $\{p, p'\}$ lying inside the canonical sets associated to ϕ. Using the calculation given below, in Lemma 4.10, for ord_{γ_p}, and the definition of v_p, one can assume that $v_p(g) \geq 0$ and $g \in O_{p,C}$. Let $g' \in L(C')$ denote the corresponding rational function to g on $L(C)$. It is then a trivial algebraic calculation, using the fact that the local rings $O_{p,C}$ and $O_{p',C'}$ are isomorphic, to show that $v_p(g) = v_{p'}(g')$. It also follows from Lemma 4.3 that $ord_{\gamma_p}(g) = ord_{\gamma_{p'}}(g')$. Hence, it is sufficient to check* (††) *for the plane projective curve C'. We may, without loss of generality, assume that $v_{p'}(g') \geq 1$ and that g' is represented in some choice of affine coordinates $\{x, y\}$ by the polynomial $q(x, y)$. If $Q(X, Y, Z)$ denotes the projective equation of this polynomial and p' corresponds to the origin of this coordinate system, then;*

$v_{p'}(g') = I_{p'}(C, Q) = length(\frac{L[x,y]}{<h,q>})$

where h is a defining equation for C' in the coordinate system $\{x, y\}$ and $I_{p'}$ is the algebraic intersection multiplicity. It also follows from Lemma 4.1, that;

$ord_{\gamma_{p'}}(g') = I_{\gamma_{p'}}(C, Q)$

Hence, it is sufficient to check that;

$I_{p'}(C, Q) = I_{\gamma_{p'}}(C, Q)$

This calculation was done in Theorem 2.10, hence (††) *and therefore* (†) *is shown. Thirdly, it remains to check that the definitions of linear equivalence are the same. In order to see this, observe that we can write (for effective divisors or weighted sets A and B);*

$A-B = (A\setminus B)+(A\cap B)]-[(B\setminus A)+(A\cap B)] = (A\setminus B)-(B\setminus A)$, (†††)

If $A \equiv B$ in the sense of weighted sets (Definition 4.6), then the calculation (†††) *(which removes the fixed branch contribution) and*

Theorem 4.7 shows that $A - B = div(g)$, for some rational function $g \in L(C)$, where, here, $div(g)$ is as defined in Definition 4.9. By (†), it then follows that $A \equiv B$ as effective divisors. Conversely, if $A \equiv B$ as effective divisors, then there exists a rational function $g \in L(C)$ such that $A - B = div(g)$, in the sense of the modern definition given above. The above calculations (†††) and (†) then show that $div(g) = (A \setminus B) - (B \setminus A)$, in the sense of Definition 4.9 below. It follows, by Lemma 4.4, that there exists a g_n^1 to which $(A \setminus B)$ and $(B \setminus A)$ belong as weighted sets. Adding the fixed branch contribution $(A \cap B)$ to this g_n^1, we then obtain that $A \equiv B$ in the sense of Definition 4.6, as required.

Definition 4.9. *Let C be a projective algebraic curve and let f be a non-zero rational function on C. Then we define $div(f)$ to be the weighted set $A - B$ where;*

$$A = (f = 0), \quad B = (f = \infty)$$

We now require the following lemma;

Lemma 4.10. *Let C be a projective algebraic curve, and let f and g be non-zero rational functions on C. Then;*

$$div(\tfrac{1}{f}) = -div(f)$$

$$div(fg) = div(f) + div(g)$$

$$div(\tfrac{f}{g}) = div(f) - div(g)$$

Proof. In order to prove the first claim, it is sufficient to show that, for a branch γ of C;

$$val_\gamma(f) = 0 \text{ iff } val_\gamma(\tfrac{1}{f}) = \infty$$

$$val_\gamma(f) = \infty \text{ iff } val_\gamma(\tfrac{1}{f}) = 0$$

and ord_γ is preserved in both cases. This follows trivially from the relevant power series calculation at a branch. Namely, we can represent f by $\frac{\phi}{\phi'}$ and $\frac{1}{f}$ by $\frac{\phi'}{\phi}$. Substituting the branch parametrisation, we obtain that;

$val_\gamma(f) = 0, ord_\gamma(f) = m$ iff $f \sim t^m u(t),\ \ m \geq 1, u(t) \in L[[t]]$ a unit.
iff $\frac{1}{f} \sim t^{-m}u(t)^{-1}$
iff $val_\gamma(f) = \infty, ord_\gamma(f) = m$

and the calculation for $val_\gamma(f) = \infty, ord_\gamma(f) = m$ is similar.

In order to prove the second claim, we need to verify the following cases at a branch γ of C;

Case 1. If $val_\gamma(f) = val_\gamma(g) \in \{0, \infty\}$, $ord_\gamma(f) = m$ and $ord_\gamma(g) = n$

then $val_\gamma(fg) \in \{0, \infty\}$ and $ord_\gamma(fg) = m + n$

Case 2. If $val_\gamma(f) \neq val_\gamma(g) \in \{0, \infty\}$, $ord_\gamma(f) = m$ and $ord_\gamma(g) = n$

then $val_\gamma(fg) \in \{0, \infty\}$ and $ord_\gamma(fg) = |m - n|$

Case 3. If exactly one of $val_\gamma(f)$ and $val_\gamma(g)$ is in $\{0, \infty\}$, with $ord_\gamma(f)$ or $ord_\gamma(g) = m$

then $val_\gamma(fg) \in \{0, \infty\}$, with $ord_\gamma(fg) = m$.

Case 4. If neither of $val_\gamma(f)$ and $val_\gamma(g)$ are in $\{0, \infty\}$

then $val_\gamma(fg)$ is not in $\{0, \infty\}$

If f is represented by $\frac{\phi}{\phi'}$ and g is represented by $\frac{\psi}{\psi'}$, then we can represent fg by $\frac{\phi\psi}{\phi'\psi'}$. The proof of these cases then follow by elementary power series calculations at the branch γ. For example, for Case 2, if $val_\gamma(f) = 0$ and $ord_\gamma(f) = m$, $val_\gamma(g) = \infty$ and $ord_\gamma(g) = n$, then we have;

$$f \sim t^n u(t),\ g \sim t^{-m}v(t),\ fg \sim t^n t^{-m} u(t)v(t) = t^{n-m}w(t),$$
$$\text{for } \{u(t), v(t), w(t)\} \text{ units in } L[[t]].$$

The third claim follows from the first two claims.

□

We now claim the following;

Theorem 4.11. *Transitivity of Linear Equivalence*

Let C' be an algebraic curve. If A, B, C are weighted sets on C' of the same total multiplicity, then, if $A \equiv B$ and $B \equiv C$, we must have that $A \equiv C$.

Proof. By Theorem 4.7, we can find rational functions f and g on C', such that;

$(A \setminus B) - (B \setminus A) = div(f)$

$(B \setminus C) - (C \setminus B) = div(g)$

By Lemma 4.10, we have that;

$div(fg) = (A \setminus B) - (B \setminus A) + (B \setminus C) - (C \setminus B)$

By drawing a Venn diagram, one easily checks that;

$$(A \setminus B) - (B \setminus A) = (A \cap B^c \cap C^c) + (A \cap B^c \cap C) - (A^c \cap B \cap C^c) - (A^c \cap B \cap C)$$

$$+$$

$$(B \setminus C) - (C \setminus B) = (A \cap B \cap C^c) + (A^c \cap B \cap C^c) - (A^c \cap B^c \cap C) - (A \cap B^c \cap C)$$

$$||$$

$$(A \setminus C) - (C \setminus A) = (A \cap B^c \cap C^c) + (A \cap B \cap C^c) - (A^c \cap B^c \cap C) - (A^c \cap B \cap C)$$

Hence, $div(fg) = (A \setminus C) - (C \setminus A)$. Now, given the g_n^1 defined by the rational function fg, as in Lemma 4.4, it follows that $(A \setminus C)$ and $(C \setminus A)$ belong to this g_n^1 as weighted sets. We can now add the fixed branch contribution $A \cap C$ to this g_n^1, giving a $g_{n+n'}^1$, to which A and C belong as weighted sets. Therefore, the result follows.

□

As an immediate corollary, we have;

Theorem 4.12. *Let C be a projective algebraic curve, then $\equiv$ is an equivalence relation on weighted sets for C of a given multiplicity.*

We also have;

Theorem 4.13. *Linear Equivalence preserved by Addition*

Let C' be a projective algebraic curve and suppose that $\{A, B, C, D\}$ are weighted sets on C' with;

$A \equiv B$ *and* $C \equiv D$

then;

$A + C \equiv B + D$

Proof. By Definition 4.6, we can find a g_n^r containing C and D as weighted sets. If s is the total multiplicity of A, then, by Lemma 3.13, we can add the weighted set A as a fixed branch contribution to this g_n^r and obtain a g_{n+s}^r, containing $A + C$ and $A + D$ as weighted sets. Hence, by Definition 4.6 again, we have that;

$A + C \equiv A + D$ (1)

Similarily, one shows, by adding D as a fixed branch contribution to the $g_{n'}^{r'}$ containing A and B as weighted sets, that;

$A + D \equiv B + D$ (2)

The result then follows immediately by combining (1), (2) and using Theorem 4.11.

□

We now develop further the theory of g_n^r on a projective algebraic curve C. We begin with the following definition;

Definition 4.14. *Subordinate g_n^r*

Let $\{g_n^r, g_n^t\}$ be given on C with the same *order n. Then we say that;*

$g_n^r \subseteq g_n^t$

if every *weighted set in g_n^r is included in the weighted sets of the g_n^t.*

We now claim the following;

Theorem 4.15. *Amalgamation of g_n^r*

Let $\{g_n^r, g_n^s\}$ be given on C, having a common weighted set G, then there exists t with $r \leq t, s \leq t$ and a g_n^t such that $g_n^r \subseteq g_n^t$ and $g_n^s \subseteq g_n^t$.

Proof. Assume first that $\{g_n^r, g_n^s\}$ have no fixed branch contribution and are defined exactly by linear systems. Then we can find algebraic forms $\{\phi_0, \psi_0\}$ such that;

$G = (C \sqcap \phi_0 = 0) = (C \sqcap \psi_0 = 0)$

and;

g_n^r is defined by $C \sqcap (\epsilon_0\phi_0 + \epsilon_1\phi_1 + \ldots + \epsilon_r\phi_r = 0)$

g_n^s is defined by $C \sqcap (\eta_0\psi_0 + \eta_1\psi_1 + \ldots + \eta_s\psi_s = 0)$

Now consider the linear system Σ defined by;

$\epsilon\phi_0\psi_0 + \psi_0(\epsilon_1\phi_1 + \ldots + \epsilon_r\phi_r) + \phi_0(\eta_1\psi_1 + \ldots + \eta_s\psi_s) = 0$

and let g_m^t be defined by Σ. As $deg(\psi_0\phi_0) = deg(\psi_0) + deg(\phi_0)$, we have that $m = 2n$. We claim that the fixed branch contribution of g_{2n}^t is exactly G, $(*)$. In order to see this, observe that we can write an algebraic form in Σ as;

$\psi_0\phi_{\bar{\epsilon}} + \phi_0\psi_{\bar{\eta}}$

If γ is a branch counted w-times in G, then, using the proof at the end of Lemma 3.13 and linearity of multiplicity at a branch, see [6];

$I_\gamma(C, \psi_0\phi_{\bar{\epsilon}}) = I_\gamma(C, \psi_0) + I_\gamma(C, \phi_{\bar{\epsilon}}) \geq w$

$I_\gamma(C, \phi_0\psi_{\bar{\eta}}) = I_\gamma(C, \phi_0) + I_\gamma(C, \psi_{\bar{\eta}}) \geq w$

$I_\gamma(C, \psi_0\phi_{\bar{\epsilon}} + \phi_0\psi_{\bar{\eta}}) = min\{I_\gamma(C, \psi_0\phi_{\bar{\epsilon}}), I_\gamma(C, \phi_0\psi_{\bar{\eta}})\} \geq w$ (†)

Hence, γ is w-fold for the g_{2n}^t and G is contained in the fixed branch contribution of the g_{2n}^t. In order to obtain the exactness statement, $(*)$, first observe that, if γ is a fixed branch of the g_{2n}^t, then, in particular, it belongs to $(C \sqcap \phi_0\psi_0 = 0)$. Hence, it belongs either to $(C \sqcap \phi_0 = 0)$ or $(C \sqcap \psi_0 = 0)$. Hence, it belongs to G. Now, using the fact that the original $\{g_n^r, g_n^s\}$ had no fixed branch contribution, we can easily find $\phi_{\bar{\epsilon}_0}$ and $\psi_{\bar{\eta}_0}$ with G disjoint from both $(C \sqcap \phi_{\bar{\epsilon}_0} = 0)$ and $(C \sqcap \psi_{\bar{\eta}_0} = 0)$. Then, by the same argument (†), we obtain, for a branch γ of G;

$$I_\gamma(C, \psi_0\phi_{\bar{\epsilon}_0} + \phi_0\psi_{\bar{\eta}_0}) = w$$

hence, γ is counted w-times in $C \sqcap (\psi_0\phi_{\bar{\epsilon}_0} + \phi_0\psi_{\bar{\eta}_0} = 0)$ and, therefore, $(*)$ holds, as required. Now, as G had total multiplicity n, removing this fixed branch contribution from the g_{2n}^t, we obtain a g_n^t. We then claim that $g_n^r \subseteq g_n^t$ and $g_n^s \subseteq g_n^t$, $(**)$. By Definition 4.14, it is sufficient to check that, if $\{W_1, W_2\}$ are weighted sets appearing in $\{g_n^r, g_n^s\}$, defined by $(C \sqcap \phi_{\bar{\epsilon}} = 0)$ and $(C \sqcap \psi_{\bar{\eta}} = 0)$, then they appear in the g_n^t. We clearly have that both $\psi_0\phi_{\bar{\epsilon}}$ and $\phi_0\psi_{\bar{\eta}}$ belong to Σ and the calculation (†) shows that;

$$C \sqcap (\psi_0\phi_{\bar{\epsilon}} = 0) = W_1 + G$$

$$C \sqcap (\phi_0\psi_{\bar{\eta}} = 0) = W_2 + G$$

Hence, the result $(**)$ follows after removing the fixing branch contribution G. The fact that $r \leq t$ and $s \leq t$ then follows easily from the definition of the dimension of a g_n^r and Theorem 3.3.

Now consider the case when the $\{g_n^r, g_n^s\}$ are defined exactly by linear systems and *have* a fixed branch contribution. Let $G_1 \subseteq G$ and $G_2 \subseteq G$ be these fixed branch contributions and let $G_3 = G_1 \cap G_2$. We claim that the fixed branch contribution of the g_{2n}^t defined by Σ, as given above, in this case is exactly $G_3 + G$. The proof is similar to the above and left to the reader. Now, removing the fixed branch contribution G, we obtain a series g_n^t with fixed branch contribution G_3. A similar proof to the above, left to the reader, shows that this g_n^t contains the original series $\{g_n^r, g_n^s\}$. Finally, we need to consider the case when the $\{g_n^r, g_n^s\}$ are defined, after removing some fixed branch contribution from linear series. Let G_1 and G_2, with total multiplicity r_1 and r_2, be these fixed branch contributions and let $\{g_{n+r_1}^r, g_{n+r_2}^s\}$ be the series obtained from adding these fixed branch contributions to $\{g_n^r, g_n^s\}$. In this case, the linear system Σ, as given above, defines a $g_{2n+r_1+r_2}^t$. We claim that the weighted set $G \cup G_1 \cup G_2$, of total multiplicity $(n+r_1+r_2)$, is contained in the fixed branch contribution of this series. This follows from a similar calculation, using the method above, the details are left to the reader. Removing this weighted set from the $g_{2n+r_1+r_2}^t$, we obtain a g_n^t and a similar calculation shows that this contains the original $\{g_n^r, g_n^s\}$, again the details are left to the reader. $\square$

As a corollary, we have;

Theorem 4.16. *Let a g_n^r be given on C, then there exists a* unique g_n^t *on C, with $r \leq t \leq n$, such that;*

$g_n^r \subseteq g_n^t$

and, for any g_n^s *such that $g_n^r \subseteq g_n^s$, we have that;*

$g_n^s \subseteq g_n^t$

Proof. By Lemma 3.16, we can find $r \leq t \leq n$ and a g_n^t on C, with $g_n^r \subseteq g_n^t$ and t maximal with this property. If $g_n^r \subseteq g_n^s$, then $\{g_n^s, g_n^t\}$ would contain a common weighted set. By Theorem 4.15, we could then find $t' \leq n$ such that $s \leq t'$, $t \leq t'$ and $g_n^s \subseteq g_n^{t'}$, $g_n^t \subseteq g_n^{t'}$. If $g_n^s \subsetneq g_n^t$, then, by elementary dimension considerations, we would have that $t < t' \leq n$ and $g_n^r \subset g_n^{t'}$, contradicting maximality of t. Hence, $g_n^s \subseteq g_n^t$. The uniqueness statement also follows from a similar amalgamation argument, using Theorem 4.15.

$\square$

We can then make the following definition;

Definition 4.17. *We call a g_n^r on C complete if it cannot be strictly contained in a g_n^t of greater dimension. If G is any weighted set on C of total multiplicity n, then we define $|G|$ to be the unique complete g_n^t to which G belongs.*

We then have that;

Theorem 4.18. *Let G be a weighted set on C, then, $G \equiv G'$ if and only if G' belongs to $|G|$. In particular, $G \equiv G'$ if and only if $|G| = |G'|$.*

Proof. The proof of the first part of the theorem is quite straightforward. By definition, if G' belongs to $|G|$, then $G \equiv G'$. Conversely, if $G' \equiv G$, then, by Definition 4.6, we can find a g_n^1, containing the given weighted sets G and G'. By Theorem 4.16, we can find a unique complete g_n^t on C, with $1 \leq t \leq n$, such that $g_n^1 \subseteq g_n^t$. As G belongs to this g_n^t as a weighted set, it follows by Definition 4.17 that $|G| = g_n^t$. Hence,

G' belongs to $|G|$ as required. For the second part, if $G \equiv G'$, then, by the first part, G' belongs to $|G|$. It follows immediately from Definition 4.17 and Theorem 4.16, that $|G| \subseteq |G'|$. Reversing this argument, we have that $|G'| \subseteq |G|$, hence $|G| = |G'|$ as required. Conversely, if $|G| = |G'|$, then clearly $G \equiv G'$ by Definition 4.6. $\square$

We now make the following definition;

Definition 4.19. *Linear System of a Weighted Set*

Let G be a weighted set on a projective algebraic curve C, then we define the Riemann-Roch space $\mathcal{L}(C,G)$ or $\mathcal{L}(G)$ to be the vector space defined as;

$\{g \in L(C)^* : div(g) + G \geq 0\} \cup \{0\}$

where $div(g)$ was defined in Definition 4.9.

Remarks 4.20. *That $\mathcal{L}(G)$ defines a vector space follows easily from Lemma 4.10, the fact that, for non-constant rational functions $\{f, g, f+g\} \subset L(C)$ and a branch γ of C, we have that;*

$ord_\gamma(f+g) \geq min\{ord_\gamma(f), ord_\gamma(g)\}$, $(*)$

where, for this remark only, ord_γ is counted negatively *if val_γ is infinite, and an argument on constants, $(**)$. We now give a brief proof of $(*)$;*

We just consider the following 2 cases;

Case 1. $val_\gamma(f) < \infty$ and $val_\gamma(g) < \infty$

We then have, substituting the relative parametrisations, that;

$f \sim c + c_1t^m + \ldots$ and $g \sim d + d_1t^n + \ldots$, where $ord_\gamma(f) = m \geq 1$, $ord_\gamma(g) = n \geq 1$ and $\{c_1, d_1\} \subset L$ are non-zero. Then;

$f + g \sim (c+d) + c_1t^m + d_1t^n + \ldots$

If $(f+g)-(c+d) \equiv 0$, as an algebraic power series in $L[[t]]$, then $(f+g) = (c+d)$ as a rational function on C, contradicting the assumption. Hence, we obtain that $ord_\gamma(f+g) = min\{ord_\gamma(f), ord_\gamma(g)\}$, if $m \neq n$

or $m = n$ *and* $c_1 + d_1 \neq 0$*, and* $ord_\gamma(f + g) > min\{ord_\gamma(f), ord_\gamma(g)\}$ *otherwise. Hence,* $(*)$ *is shown in this case.*

Case 2. $val_\gamma(f) = val_\gamma(g) = \infty$

We then have that;

$f \sim c_1 t^{-m} + \ldots$ *and* $g \sim d_1 t^{-n} + \ldots$*, where* $ord_\gamma(f) = -m \leq -1$, $ord_\gamma(g) = -n \leq -1$ *and* $\{c_1, d_1\} \subset L$ *are non-zero. Then;*

$f + g \sim c_1 t^{-m} + d_1 t^{-n} + \ldots$

By the assumption that $f + g$ *is not a constant, if* $m = n$ *and* $c_1 + d_1 = 0$*, we must have higher order terms in* t *in the Cauchy series for* $(f + g)$*, hence* $ord_\gamma(f + g) > min\{ord_\gamma(f), ord_\gamma(g)\}$*. Otherwise, we have that* $ord_\gamma(f + g) = min\{ord_\gamma(f), ord_\gamma(g)\}$*, hence* $(*)$ *is shown in this case as well.*

The remaining cases are left to the reader. One should also consider the case of constants, $(**)$*. Technically, one cannot define* ord_γ *for a constant in* L*. However, we did, by convention, define* $div(c) = 0$*, for* $c \in L^*$*, in Remarks 4.5.*

We now show the following;

Lemma 4.21. *For a weighted set* G*,* $dim(\mathcal{L}(G)) = t + 1$*, where* t *is given in Definition 4.17. In particular,* $\mathcal{L}(G)$ *is finite dimensional.*

Proof. Let t be given by Definition 4.17. If $t = 0$, then $G = (0)$ and $\mathcal{L}(G) = L$. This follows easily from the well known fact that the only regular functions on a projective algebraic curve are the constants (see, for example, [12], p59). In this case, we then have that $dim(\mathcal{L}(G)) = 1$, as required. Otherwise, let $t \geq 1$ be given as in Definition 4.17, with the unique complete g_n^t containing G. After adding some fixed branch contribution W, we can find a linear system Σ, having finite intersection with C, with basis $\{\phi_0, \ldots, \phi_j, \ldots, \phi_t\}$ defining this g_n^t. Moreover, we may assume that $C \sqcap \phi_0 = G \cup W$, $(*)$. Let $\{f_1, \ldots, f_j, \ldots, f_t\}$ be the sequence of rational functions on C defined by $f_j = \frac{\phi_j}{\phi_0}$. We claim that;

$div(f_j)+G\geq 0$, for $1\leq j\leq t$ $(**)$

In order to show $(**)$, it is sufficient to prove that, for a branch γ with $val_\gamma(f_j)=\infty$, we have that γ belong to G and, moreover, that γ is counted at least $ord_\gamma(f_j)$ times in G. Let Σ_j be the pencil of forms defined by $(\phi_j-\lambda\phi_0)_{\lambda\in P^1}$. By the proof of Lemma 4.4, we have that $(f_j=\infty)$ is defined by $(C\sqcap\phi_0)$, after removing the fixed branch contribution of this pencil. By $(*)$ and the fact that the fixed branch contribution of Σ_j includes W, we have that $(f_j=\infty)\subseteq G$. Hence, $(**)$ is shown as required. By Definition 4.19, we then have that f_j belongs to $\mathcal{L}(G)$. We now claim that there do *not* exist constants $\{c_0,\ldots,c_j,\ldots,c_t\}\subset L$ such that;

$c_0+c_1f_1+\ldots+c_jf_j+\ldots+c_tf_t=0$ $(***)$

as rational functions on C. If so, we would have that;

$c_0\phi_0+c_1\phi_1+\ldots+c_j\phi_j+\ldots+c_t\phi_t$

vanished identically on C, contradicting the fact that Σ has finite intersection with C. Hence, by $(***)$, $\{1,f_1,\ldots,f_t\}\subset\mathcal{L}(G)$ are linearly independent and $dim(\mathcal{L}(G))\geq t+1$. Conversely, suppose that $dim(\mathcal{L}(G))\geq k+1$, then we can find $\{1,f_1,\ldots,f_j,\ldots,f_k\}\subset\mathcal{L}(G)$ which are linearly independent, $(\dagger)$. By the usual method of equating denominators, we can find algebraic forms $\{\phi_0,\ldots,\phi_k\}$ of the same degree, such that f_j is represented by $\frac{\phi_j}{\phi_0}$, for $1\leq j\leq k$. Let Σ be the linear system defined by this sequence of forms. By $(\dagger)$, Σ has finite intersection with C. Let W, having total multiplicity n', be the fixed branch contribution of this system and let $(C\sqcap\phi_0)=G_0\cup W$. We claim that $G_0\subseteq G$, $(\dagger\dagger)$. Suppose not, then there exists a branch γ with $I_\gamma^{\Sigma,mobile}(C,\phi_0)=s$, where γ is counted strictly less than s-times in G. By the definition of $I_\gamma^{\Sigma,mobile}$, we can find a form ϕ_λ belonging to Σ, distinct from ϕ_0, witnessing this multiplicity. Consider the pencil Σ_λ defined by $(\phi_\lambda-\mu\phi_0)_{\mu\in P^1}$. We then clearly have that $I_\gamma^{\Sigma_\lambda,mobile}(C,\phi_0)=s$ as well, $(\dagger\dagger\dagger)$. Let $f_\lambda=\frac{\phi_\lambda}{\phi_0}$. By the proof of Lemma 4.4, we have that $(f_\lambda=\infty)$ is defined by $(C\sqcap\phi_0)$, after removing the fixed branch contribution of Σ_λ. By $(\dagger\dagger\dagger)$, it follows that the branch γ is counted s-times in $(f_\lambda=\infty)$ and therefore $div(f_\lambda)+G\ngeq 0$. However, f_λ is a linear combination of $\{1,\ldots,f_k\}$, hence $f_\lambda\in\mathcal{L}(G)$, which is a contradiction. Hence, $(\dagger\dagger)$ is shown. Now, consider the g_n^k defined by Σ. Let W' be the weighted set $G\setminus G_0$ of total multiplicity n''. By Lemma 3.13, we can add the weighted set W' to the g_n^k and obtain a $g_{n+n''}^k$ with fixed branch contribution $W' \cup W$. Now, removing the fixed branch contribution W from this $g_{n+n'}^k$, we obtain a $g_{n+n''-n'}^k$ containing G exactly as a weighted set. It follows, from Definition 4.17, that $k \leq t$. Hence, in particular, $dim(\mathcal{L}(G))$ is finite and $dim(\mathcal{L}(G) \leq t+1$. Therefore, the lemma is proved.

□

We now extend the notion of linear equivalence to include virtual, or non-effecive, weighted sets.

Definition 4.22. *We define a generalised weighted set G on C to be a linear combination of branches;*

$$n_1\gamma_{p_1}^{j_1} + \ldots + n_r\gamma_{p_r}^{j_r}$$

where $\{n_1, \ldots, n_r\}$ belong to $\mathcal{Z}$. If $\{n_1, \ldots, n_r\}$ belong to $\mathcal{Z}_{\geq 0}$, we call the weighted set effective. Otherwise, we call the weighted set virtual. We define $n = n_1 + \ldots + n_r$ to be the total multiplicity or degree of G.

Remarks 4.23. *It is an easy exercise to see that there exist well defined operations of addition and subtraction on generalised weighted sets. It is also easy to check that any generalised weighted set G may be written uniquely as $G_1 - G_2$, where $\{G_1, G_2\}$ are* disjoint effective *weighted sets.*

Definition 4.24. *Let A and B be generalised weighted sets on C of the same total multiplicity. Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be the unique effective weighted sets, as given by the previous remark. Then we define;*

$$(A_1 - A_2) \equiv (B_1 - B_2) \text{ iff } (A_1 + B_2) \equiv (B_1 + A_2)$$

and;

$$A \equiv B \text{ iff } (A_1 - A_2) \equiv (B_1 - B_2)$$

Remarks 4.25. *Note that if $\{A'_1, A'_2\}$ and $\{B'_1, B'_2\}$ are* any *effective weighted sets such that;*

$$A = A'_1 - A'_2 \text{ and } B = B'_1 - B'_2$$

$$\text{then } A \equiv B \text{ iff } A'_1 + B'_2 \equiv B'_1 + A'_2$$

The proof is just manipulation of effective weighted sets. We clearly have that;

$A_1 + A'_2 = A'_1 + A_2$ *and* $B_1 + B'_2 = B'_1 + B_2$ $(*)$

We then have;

$$\begin{aligned} A \equiv B \quad & \textit{iff} \quad A_1 + B_2 \equiv B_1 + A_2 \textit{(Definition 4.24)} \\ & \textit{iff } A_1 + A'_2 + B_2 \equiv B_1 + A_2 + A'_2 \textit{ (Theorem 4.13)} \\ & \textit{iff } A'_1 + A_2 + B_2 \equiv B_1 + A_2 + A'_2 \textit{ (by (*))} \\ & \textit{iff} \quad A'_1 + B_2 \equiv B_1 + A'_2 \textit{ (Theorem 4.13)} \\ & \textit{iff } A'_1 + B_2 + B'_1 \equiv B_1 + B'_1 + A'_2 \textit{ (Theorem 4.13)} \\ & \textit{iff } A'_1 + B_1 + B'_2 \equiv B_1 + B'_1 + A'_2 \textit{ (by (*))} \\ & \textit{iff} \quad A'_1 + B'_2 \equiv B'_1 + A'_2 \textit{ (Theorem 4.13)} \end{aligned}$$

We then have;

Theorem 4.26. *Transitivity of Linear Equivalence*

Let C' be an algebraic curve. If A, B, C are generalised weighted sets on C' of the same total multiplicity, then, if $A \equiv B$ and $B \equiv C$, we must have that $A \equiv C$.

Proof. Let $\{A_1, A_2\}$, $\{B_1, B_2\}$ and $\{C_1, C_2\}$ be the effective weighted sets as given by Remarks 4.23. Then, by Definition 4.24, we have that;

$(A_1 + B_2) \equiv (B_1 + A_2)$ and $(B_1 + C_2) \equiv (C_1 + B_2)$

By Theorem 4.13, we have that;

$(A_1 + B_1 + B_2 + C_2) \equiv (C_1 + B_1 + B_2 + A_2)$

It then follows, by Definition 4.6, that there exists a g^1_n, containing $(A_1 + B_1 + B_2 + C_2)$ and $(C_1 + B_1 + B_2 + A_2)$ as weighted sets. Clearly $(B_1 + B_2)$ is contained in the fixed branch contribution of this g^1_n. Removing this fixed branch contribution, we obtain;

$A_1 + C_2 \equiv C_1 + A_2$

By Definition 4.24, we then have that $A \equiv C$ as required.

□

It follows immediately from Theorem 4.12 and Theorem 4.26 that;

Theorem 4.27. *Let C be a projective algebraic curve, then $\equiv$ is an equivalence relation on generalised weighted sets for C of a given total multiplicity.*

Remarks 4.28. *Again, the definition of linear equivalence that we have given for generalised weighted sets on a smooth projective algebraic curve C is equivalent to the modern definition for divisors. More precisely, we have;*

Modern Definition; Let A and B be divisors on a smooth projective algebraic curve C, then $A \equiv B$ iff $A - B = div(g)$, for some $g \in L(C)^*$.

See, for example, p161 of [12] *for relevant definitions and notation. In order to show that our definition is the same, use Remarks 4.8 and the following simple argument;*

$A \equiv B$ as generalised weighted sets iff $A_1 + B_2 \equiv B_1 + A_2$

where $\{A_1, A_2, B_1, B_2\}$ are the effective weighted sets given by Definition 4.24. Then;

$A_1 + B_2 \equiv B_1 + A_2$ iff $(A_1 + B_2) - (B_1 + A_2) = div(g)$ $(g \in L(C)^*)$

by Remarks 4.8, where $div(g)$ is the modern definition. By a straightforward calculation, we have that;

$(A_1 + B_2) - (B_1 + A_2) = A - B$ as divisors or generalised weighted sets.
Hence, the notions of equivalence coincide.

We also have;

Theorem 4.29. *Linear Equivalence Preserved by Addition*

Let C' be a projective algebraic curve and suppose that $\{A, B, C, D\}$ are generalised weighted sets on C' with;

$A \equiv B$ and $C \equiv D$

then;

$A + C \equiv B + D$

Proof. Let $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{C_1, C_2\}$ and $\{D_1, D_2\}$ be effective weighted sets as given by Remarks 4.23 Then, by Definition 4.24, we have that;

$A_1 + B_2 \equiv B_1 + A_2$ and $C_1 + D_2 \equiv D_1 + C_2$

Hence, by Theorem 4.13;

$A_1 + B_2 + C_1 + D_2 \equiv B_1 + A_2 + D_1 + C_2$ $(*)$

We clearly have that;

$A + C = (A_1 + C_1) - (A_2 + C_2)$ and $B + D = (B_1 + D_1) - (B_2 + D_2)$

as an identity of generalised weighted sets. Moreover, as $(A_1 + C_1), (A_2 + C_2), (B_1 + D_1)$ and $(B_2 + D_2)$ are all effective, we can apply Remarks 4.25 and $(*)$ to obtain the result. □

We now make the following definition;

Definition 4.30. *Let G be a generalised weighted set on a projective algebraic curve C, then we define $|G|$ to be the collection of generalised weighted sets G' with $G' \equiv G$. We define order$(|G|)$ to be the total multiplicity (possibly negative) of any generalised weighted set in $|G|$.*

Remarks 4.31. *If G is an* effective *weighted set, the collection defined by Definition 4.30 is* not *the same as the collection given by Definition 4.17, as it includes virtual weighted sets. Unless otherwise stated, we will use Definition 4.17 for* effective *weighted sets. This convention is in accordance with the Italian terminology.*

We now show that the notions of linear equivalence introduced in this section are birationally invariant;

Theorem 4.32. *Let $\Phi : C_1 \leftrightsquigarrow C_2$ be a birational map. Let A and B be generalised weighted sets on C_2, with corresponding generalised weighted sets $[\Phi]^*A$ and $[\Phi]^*B$ on C_1. Then $A \equiv B$, in the sense of either Definition 4.6 or 4.24, iff $[\Phi]^*A \equiv [\Phi]^*B$.*

Proof. Suppose that $A \equiv B$ in the sense of Definition 4.6. Then, there exists a g_n^r on C_2 containing A and B as weighted sets. By Theorem 3.14, there exists a corresponding g_n^r on C_1, containing $[\Phi]^*A$ and $[\Phi]^*B$ as weighted sets. Hence, again by Definition 4.6, $[\Phi]^*A \equiv [\Phi]^*B$. The converse is similar, using $[\Phi^{-1}]^*$. If $A \equiv B$ in the sense of Definition 4.24, then the same argument works.

□

As a result of this theorem, we introduce the following definition;

Definition 4.33. *Let $\Phi : C_1 \leftrightsquigarrow C_2$ be a birational map. Then, given a generalised weighted set A on C_2, we define;*

$$[\Phi]^*|A| = |[\Phi]^*A|$$

where, in the case that A is effective, $|A|$ can be taken either in the sense of Definition 4.17 or Definition 4.30.

Remarks 4.34. *The definition depends only on the complete series $|A|$, rather than its particular representative A. This follows immediately from Definition 4.17, Definition 4.30 and Theorem 4.32.*

We finally introduce the following definition;

Definition 4.35. *Summation of Complete Series*

Let A and B be generalised weighted sets, defining complete series $|A|$ and $|B|$, in the sense of Definition 4.30. Then, we define the sum;

$$|A| + |B|$$

to be the complete series, in the sense of Definition 4.30, containing all generalised weighted sets of the form $A' + B'$ with $A' \in |A|$ and $B' \in |B|$. If A and B are effective *weighted sets with $|A|$, $|B|$ taken in the sense of Definition 4.17, then we make the same definition for the sum in the sense of Definition 4.17.*

Remarks 4.36. *This is a good definition by Theorem 4.13 and Theorem 4.29.*

Definition 4.37. *Difference of Complete Series*

Let A and B be generalised weighted sets, defining complete series $|A|$ and $|B|$, in the sense of Definition 4.30. Then, we define the difference;

$|A| - |B|$

to be the complete series, in the sense of Definition 4.30, containing all generalised weighted sets of the form $A' - B'$ with $A' \in |A|$ and $B' \in |B|$. If A and B are effective *weighted sets with $|A|$, $|B|$ taken in the sense of Definition 4.17, then we can in certain cases define a difference in the sense of Definition 4.17. (This is called the residual series, the reader can look at* [11] *for more details)*

Remarks 4.38. *This is again a good definition, for generalised weighted sets $\{A, B\}$, it follows trivially from the previous definition and the fact that $\{A, -B\}$ are also generalised weighted sets.*

5. A Geometric Formulation of Flatness

Lemma 5.1. *Let $f : C_1 \rightarrow C_2$ be a finite morphism of irreducible projective algebraic curves. Let $O \in C_2$, with $f^{-1}(O) = \{Q_1, \ldots, Q_s\}$. Then f induces a map;*

$[f]^* : \gamma_O \rightarrow P(\bigcup_{1 \leq k \leq s} \gamma_{Q_k})$

Proof. Let $\phi_1 : C_1^{ns} \rightarrow C_1$ and $\phi_2 : C_2^{ns} \rightarrow C_2$ be nonsingular models. Suppose that γ_O^j belongs to γ_O, corresponding to $[\mathcal{V}_{O_j}]$, where $O_j \in (\phi_2)^{-1}(O)$. As ϕ_2 is birational, we obtain a morphism $(\phi_2^{-1} \circ f \circ \phi_1) : (\phi_2^{-1} \circ f \circ \phi_1)^{-1}(V_{[\phi_2]}) \rightarrow C_2^{ns}$ which extends uniquely to a morphism, $(\phi_2^{-1} \circ f \circ \phi_1) : C_1^{ns} \rightarrow C_2^{ns}$, see notation in [6]. We have that $(\phi_2^{-1} \circ f \circ \phi_1)^{-1}(\mathcal{V}_{O_j}) = \bigcup_{Q \in (\phi_2^{-1} \circ f \circ \phi_1)^{-1}(O_j)} \mathcal{V}_Q$. Observing that $\phi_1(Q) \in \{Q_1, \ldots, Q_s\}$, for $Q \in (\phi_2^{-1} \circ f \circ \phi_1)^{-1}(O_j)$, we can then set $[f]^*(\gamma_j^O) = \{[\mathcal{V}_Q] : Q \in (\phi_2^{-1} \circ f \circ \phi_1)^{-1}(O_j)\} \subset P(\bigcup_{1 \leq k \leq s} \gamma_{Q_k})$, see Definition 5.2 of [6]. It is straightforward to show that this definition does not depend on the choice of $\{C_1^{ns}, C_2^{ns}\}$, see Lemma 5.7 of [6]. $\square$

Definition 5.2. *Let notation be as in Lemma 5.1, $\gamma_2 \in \gamma_O$, with $O \in C_2$, and $\gamma_1 \in [f]^*(\gamma_2)$, centred at $P \in C_1$. Let $pr_2 : graph(\phi_2^{-1} \circ f \circ \phi_1) \rightarrow C_2^{ns}$. Then, we define, $Mult_{(\gamma_1/\gamma_2)}(C_1/C_2)$ and $Mult_{(O_1/O_2)}(C_1^{ns}/C_2^{ns})$ to be $Mult_{(O_1,O_2)}(graph(\phi_2^{-1} \circ f \circ \phi_1)/C_2^{ns})$*

where $[\mathcal{V}_{O_1}] = \gamma_1$, $[\mathcal{V}_{O_2}] = \gamma_2$, $\phi_1(O_1) = P$, $\phi_2(O_2) = O$, *and* $Mult_{(O_1,O_2)}$ *is Definition 4.1 in* [8], ([1]). *If* $f(P) = O$, *we define the local geometric multiplicity of* f *at* P *over* γ_2 *to be;*

$$Mult_{(P/\gamma_2)}(C_1/C_2) = \sum_{\gamma_1 \in [f]^*(\gamma_2) \cap \gamma_P} Mult_{(\gamma_1/\gamma_2)}(C_1/C_2)$$

If O *is nonsingular, we refer to this as the local geometric multiplicity of* f *at* P *over* O.

Definition 5.3. *Let notation be as in Lemma 5.1, then, if* $\gamma_2 \in \bigcup_{O \in C_2} \gamma_O$, *we define the total geometric multiplicity of* f *over* γ_2 *to be;*

$$\sum_{\gamma_1 \in [f]^*(\gamma_2)} Mult_{(\gamma_1/\gamma_2)}(C_1/C_2)$$

Lemma 5.4. *Let notation be as in Lemma 5.1, then, the total geometric multiplicity of* f *is independent of the choice of branch* $\gamma_2 \in \bigcup_{O \in C_2} \gamma_O$.

Proof. Using Lemma 4.3 of [8], we have that;

$$\sum_{Q \in (\phi_2^{-1} \circ f \circ \phi_1)^{-1}(O_2)} Mult_{(O_1,O_2)}(C_1^{ns}, C_2^{ns})$$

is independent of the choice of $O_2 \in C_2^{ns}$. It follows, using Definition 5.2 and Lemma 5.1, that;

$$\sum_{\gamma_1 \in [f]^*(\gamma_2)} Mult_{(\gamma_1/\gamma_2)}(C_1/C_2)$$

$$= \sum_{\gamma_1 \in [f]^*(\gamma_2)} Mult_{(Q,O_2)}(C_1^{ns}, C_2^{ns}),\ Q \in C_1^{ns},\ [\mathcal{V}_Q] = \gamma_1,\ [\mathcal{V}_{O_2}] = \gamma_2.$$

$$= \sum_{Q \in (\phi_2^{-1} \circ f \circ \phi_1)^{-1}(O_2)} Mult_{(Q,O_2)}(C_1^{ns}, C_2^{ns})$$

is independent of the choice of $\gamma_2 \in C_2$. □

Definition 5.5. *Let notation be as in Lemma 5.1. Given* $O \in C_2$, *choose affine open sets* $V \subset C_2$ *and* $f^{-1}(O) \subset U \subset C_2$, *with* $f(U) \subset V$. *Let* $m_O \subset k[V]$, *be the corresponding maximal ideal and corresponding ideal* $m'_O \subset k[U]$, *generated by* $f^*(m_O)$, *we define the total algebraic multiplicity of* f *over* O *to be;*

$$length(k[U]_{m'_O}/m'_O)$$

[1]The reader should be aware that the order of the tuple is reversed, due to the usual convention on the definition of the graph.

and, if $f(P)=O$, the local algebraic multiplicity of f at P over O to be;

$length(O_P/m''_O)$

where $m''_O\subset O_P$, is generated by $f^*(m_O)$.

Lemma 5.6. *Let $g_2:Y\rightarrow Z$ be a morphism, with Z nonsingular, $S\in Y$, with $g(S)=O$, then, the local algebraic multiplicity of f at S over O and the local geometric multiplicity of f at S over O coincide.*

Proof. Without loss of generality, assume that, if w is minimal with $Z\subset P^w$, then $w\geq 3$. Fixing a hyperplane W of dimension $w-1$, we have, by Theorem 6.4, that, there exists an open $U\subset P^w$, such that $pr_{P,W}(O)\in pr_{P,W}(Z)$ is nonsingular. Fix a hyperplane H' through O such that $H'\cap U$ is open in H', with $(H'\cap Z)=\{O,P_2,\ldots P_r\}$. Then, clearly, we can choose $P\in(H'\cap U)$ such that $l_{PO}\cap\{P_2,\ldots P_r\}=\emptyset$, so l_{PO} does not otherwise intersect the curve Z. It follows that $pr_{P,W}$ is birational, hence etale, at O, ([2]), with $pr_{P,W}(O)$ nonsingular. Repeating this argument, we can find $Z'\subset P_2$, and a morphism $f_1:Z\rightarrow Z'$, with $f_1(O)$ nonsingular, and f_1 etale at O. For a suitable choice of coordinates (x,y), the projection pr_x extends to a morphism $pr_x:Z'\rightarrow P^1$, which is etale at $f_1(O)$. Letting $f_2=(pr_x\circ f_1)$, we obtain a morphism $f_2:Z\rightarrow P^1$, with f_2 etale at 0. As both Z and P^1 are nonsingular, we have that the local algebraic multiplicity is multiplicative over the composition $(f_2\circ g_2)$, see [8], Theorem 5.10, ([3]). As the morphism f_2 is etale at O, we have that $mult^{alg}_{O,f_2(O)}(Z'/P^1)=1$, and;

$$mult^{alg}_{S,(f_2\circ g_2)(S)}(Y/P^1)$$

$$=mult^{alg}_{S,g_2(S)}(Y/Z)mult^{alg}_{g_2(S),(f_2\circ g_2)(S)}(Z/P^1)$$

$$=mult^{alg}_{S,O}(Y/Z)$$

Hence, we can assume that $Z=P^1$. Moreover, replacing Y by the fiber product $Y\times_Z Z$, we can assume the morphism g_2 is the projection $pr_{n+1,1}$.

Case 1, All the branches of Y are nonsingular. Again, without loss of generality, assume that $Y\subset P^3$. By Theorem 6.2 of [6], we

[2]In the sense that there exists an open subset $O\in V\subset Z$ such that $pr_{P,W}(V)\subset dom(pr^{-1}_{P,W})$

[3]It is only required, in the statement there, that X_1 and X_2 are nonsingular.

have that, for a generic choice of hyperplane H_λ, passing through S, $I_{italian}(S,\gamma_O^j,C,H_\lambda)=1$, for the branches γ_S^j, $1\leq j\leq t$, passing through S. By a simple adaptation of Theorem 4.16 in [6], using Definition 6.3 of the tangent line to a branch in [6], to show that the variety the variety $Tang(Y)=\bigcup_{a\in Y,\gamma\in\gamma_a} l_{a,\gamma}$ is definable and has dimension 2, ([4]) it follows that, for a generic choice of hyperplane, H_λ, passing through S;

a. $I_{italian}(S,\gamma_S^j,C,H_\lambda)=1$, for the branches γ_S^j, $1\leq j\leq t$, passing through S.

b. The remaining intersections $\{p_i : 1\leq i\leq s\}=H_\lambda\cap Y$ are nonsingular, and $I_{italian}(p_i,C,H_\lambda)=1$, $1\leq i\leq s$, $(***)$

We require the following lemma;

Lemma 5.7. *Let $Y\subset P^n$, be an irreducible curve, $n\geq 3$, $x\in Y$, then, for a generic $Q\in P^3$, and a generic hyperplane $H\subset P^{n-1}$, the projection $pr_{Q,H}$ is etale at x, and there exists an open neighborhood U_x, containing x, such that $pr_{Q,H}|_{U_x}$ is birational.*

Proof. Without loss of generality, suppose that $Y\subset P^3$, defined by $f(x,y,z),g(x,y,z)$, and $x=(0,0,0)$. Let $P=[0:0:1:0]$.

a. Fix P moving H; Choose a transformation taking H to $Z=0$, fixing P. Let H be defined by $aX+bY+cZ+dW=0$, $ax+by+cz+d$, in coordinates $\{x=\frac{X}{W},y=\frac{Y}{W},z=\frac{Z}{W}\}$. We have $a(x-x_0)+by+cz=0$ iff $ax+by+cz-ax_0=0$, hence, taking $-ax_0=d$, $x_0=\frac{-d}{a}$, the affine transformation $T_{\frac{d}{a}}$ of L^3, defined by $T_{\frac{d}{a}}(x,y,z)=(x+\frac{d}{a},y,z)$ maps H to the plane $H_{a,b,c}$ defined by $ax+by+cz=0$. We have that $\{(1,0,\frac{-a}{c}),(0,1,\frac{-b}{c}),(0,0,1)\}$ forms a basis for L^3, with $\{(1,0,\frac{-a}{c}),(0,1,\frac{-b}{c})\}\subset H_{a,b,c}$, hence the transformation $T_{H_{a,b,c};z=0}$, defined by $M^{-1}_{H_{a,b,c};z=0}$, where;

$$M_{H_{a,b,c};z=0}=\begin{pmatrix}1&0&0\\0&1&0\\\frac{-a}{c}&\frac{-b}{c}&1\end{pmatrix}$$

[4]Use the fact that we can take a nonsingular model of Y, $Y''\subset P^w$ with a projection $pr_{w,3}:Y''\leftrightsquigarrow Y$, using fibre products, the result of Theorem 4.16 for nonsingular curves, and the fact that $pr(Tang(Y''))=Tang(Y)$

maps the plane $H_{a,b,c}$ to $Z=0$. It follows that $T_{\frac{d}{a}}^{-1}\circ T_{H_{a,b,c};z=0}^{-1}$, maps the plane $z=0$ to H. The curve $Y\subset P^3$ is mapped to the curve Y' defined by;

$$f(T_{\frac{d}{a}}^{-1}\circ T_{H_{a,b,c};z=0}^{-1}|_{(x,y,z)})=g(T_{\frac{d}{a}}^{-1}\circ T_{H_{a,b,c};z=0}^{-1}|_{(x,y,z)})=0$$

using the transformation $T_{H_{a,b,c};z=0}\circ T_{\frac{d}{a}}$. We have that;

$$(T_{\frac{d}{a}}^{-1}\circ T_{H_{a,b,c};z=0}^{-1})|_{(x,y,z)}=(x-\tfrac{d}{a},y,-\tfrac{a}{c}x-\tfrac{b}{c}y+z)$$

so that Y' is defined by;

$$f_{a,b,c,d}=f(x-\tfrac{d}{a},y,-\tfrac{a}{c}x-\tfrac{b}{c}y+z)$$

$$=g_{a,b,c,d}=g(x-\tfrac{d}{a},y,-\tfrac{a}{c}x-\tfrac{b}{c}y+z)=0,\ (*)$$

.

We have that $(T_{H_{a,b,c};z=0}\circ T_{\frac{d}{a}})\circ pr_{P,H}(Y)=pr_{P,Z=0}\circ(T_{H_{a,b,c};z=0}\circ T_{\frac{d}{a}})(Y)$, hence $pr_{P,H}(Y)\cong pr_{P,Z=0}(Y')$. It is, therefore, sufficient, that for a generic choice of (a,b,c,d) in $(*)$, the cover, defined by $(*)$;

$$Sp(\tfrac{L[x,y,z]}{<f_{a,b,c,d},g_{a,b,c,d}>})\rightarrow Sp(\tfrac{L[x,y]}{<res_{x,y}(f_{a,b,c,d},g_{a,b,c,d})}>)$$

is etale at $((T_{H_{a,b,c};z=0}\circ T_{\frac{d}{a}})(0,0,0),pr_{x,y}((T_{H_{a,b,c};z=0}\circ T_{\frac{d}{a}})(0,0,0)))$. We have that;

$$M_{H_{a,b,c};z=0}^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\\frac{a}{c}&\frac{b}{c}&1\end{pmatrix}$$

hence $(T_{H_{a,b,c};z=0}\circ T_{\frac{d}{a}})(0,0,0)=T_{H_{a,b,c};z=0}(\frac{d}{a},0,0)=M_{H_{a,b,c};z=0}^{-1}(\frac{d}{a},0,0)=(\frac{d}{a},0,\frac{d}{c})$. Let Y'' be defined by $Y'((x+\frac{d}{a},y,z+\frac{d}{c}))$, that is;

$$F_{a,b,c,d}=f(x,y,-\tfrac{a}{c}x-\tfrac{b}{c}y+z+\tfrac{d}{c})$$

$$=G_{a,b,c,d}=g(x,y,-\tfrac{a}{c}x-\tfrac{b}{c}y+z+\tfrac{d}{c})=0$$

then, sufficient to prove that;

$$(Sp(\tfrac{L[x,y,z]}{<F_{a,b,c,d},G_{a,b,c,d}>}),(0,0,0))\rightarrow(Sp(\tfrac{L[x,y]}{<res_{x,y}(F_{a,b,c,d},G_{a,b,c,d})}>),(0,0))$$

is etale, for a generic choice of (a,b,c,d).

b. Fix H, moving Q. Again, let $P=[0:0:1:0]$, in coordinates (X,Y,Z,W). Let H_1 be defined by $z=1$, in the coordinate system $W\neq 0$, $\{x=\frac{X}{W},y=\frac{Y}{W},z=\frac{Z}{W}\}$, so H_1 is given by $Z=W$. Without loss of generality, $x=(0,0,1)$ in the coordinate system $W\neq 0$, $x=[0:0:1:1]$. We're given pr_{Q,H_1}. In the coordinate system, $Z\neq 0$, $\{x=\frac{X}{Z},y=\frac{Y}{Z},w=\frac{W}{Z}\}$, H_1 is defined by $w=1$, and P corresponds to the point $(0,0,0)$. Let $Q=[q_1:q_2:q_3:q_4]$, $q_3\neq 0$, corresponding to $(\frac{q_1}{q_3},\frac{q_2}{q_3},\frac{q_4}{q_3})$ in the coordinate system, $Z\neq 0$. The condition that $Q\notin H_1$ is given by $q_4\neq q_3$. The map $T_1\circ R\circ T_{-1}$, where $T_c(x,y,w)=(x+c,y,w)$ and R is defined by M^{-1}, where;

$$M=\begin{pmatrix}1&0&\frac{-q_1}{q_3}\\0&1&\frac{-q_2}{q_3}\\0&0&-(\frac{q_4}{q_3}-1)\end{pmatrix}$$

maps Q to P and fixes H_1. Hence, if Y' is defined by;

$$f((T_1\circ R\circ T_{-1})^{-1}(x,y,z))=g((T_1\circ R\circ T_{-1})^{-1}(x,y,z))=0$$

or;

$$f(\tfrac{x-\frac{q_1}{q_3}}{(1-\frac{q_4}{q_3})},\tfrac{y-\frac{q_2}{q_3}z}{(1-\frac{q_4}{q_3})},\tfrac{z}{(1-\frac{q_4}{q_3})})=g(\tfrac{x-\frac{q_1}{q_3}}{(1-\frac{q_4}{q_3})},\tfrac{y-\frac{q_2}{q_3}z}{(1-\frac{q_4}{q_3})},\tfrac{z}{(1-\frac{q_4}{q_3})})=0$$

using the fact that;

$$((T_1\circ R\circ T_{-1})^{-1})(x,y,w)$$

$$=(T_{-1}\circ M\circ T_1)(x,y,w)=(x-\tfrac{q_1w}{q_3},y-\tfrac{q_2w}{q_3},w-\tfrac{q_4w}{q_3})$$

$$(T_{-1}\circ M\circ T_1)([X:Y:Z:W])$$

$$=[X-\tfrac{q_1}{q_3}W:Y-\tfrac{q_2}{q_3}Z:Z:W(1-\tfrac{q_4}{q_3})]$$

$$=[\tfrac{X-\frac{q_1}{q_3}W}{W(1-\frac{q_4}{q_3})}:\tfrac{Y-\frac{q_2}{q_3}Z}{W(1-\frac{q_4}{q_3})}:\tfrac{Z}{W(1-\frac{q_4}{q_3})}:1]$$

$$(T_{-1}\circ M\circ T_1)(x,y,z)=(\tfrac{x-\frac{q_1}{q_3}}{(1-\frac{q_4}{q_3})},\tfrac{y-\frac{q_2}{q_3}z}{(1-\frac{q_4}{q_3})},\tfrac{z}{(1-\frac{q_4}{q_3})})$$

We have that $pr_{P,H_0}(Y')=pr_{P,H_0}\circ(T_1\circ R\circ T_{-1})(Y)=pr_{Q,H_0}(Y)$, hence sufficient to prove that;

$Sp(\frac{L[x,y,z]}{<f_{q_1,q_2,q_3,q_4},g_{q_1,q_2,q_3,q_4}>}) \rightarrow Sp(\frac{L[x,y]}{<res_{x,y}(f_{q_1,q_2,q_3,q_4},g_{q_1,q_2,q_3,q_4})>})$ (†)

is etale at $(0,0,0),(0,0)$, where;

$f_{q_1,q_2,q_3,q_4}(x,y,z) = f(\frac{x-\frac{q_1}{q_3}}{(1-\frac{q_4}{q_3})}, \frac{y-\frac{q_2}{q_3}(z+1)}{(1-\frac{q_4}{q_3})}, \frac{(z+1)}{(1-\frac{q_4}{q_3})})$

$g_{q_1,q_2,q_3,q_4}(x,y,z) = g(\frac{x-\frac{q_1}{q_3}}{(1-\frac{q_4}{q_3})}, \frac{y-\frac{q_2}{q_3}(z+1)}{(1-\frac{q_4}{q_3})}, \frac{(z+1)}{(1-\frac{q_4}{q_3})})$

Suppose that $deg_z(f_{q_1,q_2,q_3,q_4}(x,y,z)) = s$ and $deg(g_{q_1,q_2,q_3,q_4}(x,y,z)) = r$, with $s \leq r$. Using Theorem 6.4 of [6], for a generic choice of P, the projection pr_{P,H_0} is generally biunivocal. It follows, for such points, that the loci defined by $det(\overline{M}_{2,q_1,q_2,q_3,q_4}) = 0$ and $det(\overline{M}_{1,q_1,q_2,q_3,q_4}) = 0$, intersect in finitely many points. As (q_1,q_2,q_3,q_4) is generic, these points do not include $(0,0,0)$, ([5]). In particular, for such P, the projection is biunivocal and there exists an open neighborhood $pr_z(Y') \supset U_{\overline{q}} \supset (0,0,0)$, for which $det(\overline{M}_{2,q_1,q_2,q_3,q_4})|_{U_{\overline{q}}} \neq 0$. By footnote 6, we have that there exists polynomials $\{v_{q_1,q_2,q_3,q_4}, \lambda_{i,2,q_1,q_2,q_3,q_4} : 0 \leq i \leq s-2\} \subset L[x,y]$, such that, if;

$(\{v_{q_1,q_2,q_3,q_4}|_{(0,0)}, v_{q_1,q_2,q_3,q_4}|_{(0,0)}\}$

$\cup\{\lambda_{i,2,q_1,q_2,q_3,q_4}|_{(0,0)} : 0 \leq i \leq s-2\}) \cap \{0\} = \emptyset$ (††)

then the projection defined by (†) is birational in a neighborhood $V_{\overline{q}}$ of $(0,0,0)$. The condition (††) is guaranteed by choosing (q_1,q_2,q_3,q_4) to be generic. In particular, for generic $Q \in P^3$, the projection $pr_{Q,Z=1}$

[5]You still need to check that $det(\overline{M}_{1,y_1,y_2,y_3,y_4})|_{0,0}$ does not define the zero polynomial, this is a straightforward calculation left to the reader.

is etale at $(0,0,1)$ as required,([6]).

□

6

Suppose we are given two polynomials $\{f,g\}\subset L[x]$, and $deg(f)=r$, $deg(g)=s$. Suppose f and g have a common root, then we can find a polynomial $h(x)$ with $deg(h)\geq 1$, such that $f(x)=h(x)f_1(x)$ and $g(x)=h(x)g_1(x)$, $deg(f_1)\leq r-1$, $deg(g_1)\leq s-1$. We have that;

$$f(x)g(x)=h(x)f_1(x)g(x)=g(x)f(x)=h(x)g_1(x)f(x)$$

$$h(x)(f_1(x)g(x)-g_1(x)f(x))=0$$

so $g_1(x)f(x)-f_1(x)g(x)=0$, in particular there exist polynomials $\{c,d\}\subset L[x]$, with $deg(c)\leq s-1$, $deg(d)\leq r-1$, such that $c(x)f(x)+d(x)g(x)=0$, $(*)$. Conversely, suppose $(*)$ holds, then, as $f(x)|d(x)g(x)$, and $deg(d)\leq r-1$, we have that $f(x)$ and $g(x)$ share a common root. Letting;

$$f(x)=\sum_{i=0}^{r}a_ix^i,\ g(x)=\sum_{j=0}^{s}b_jx^j$$

$$c(x)=\sum_{k=0}^{s-1}c_kx^k,\ d(x)=\sum_{l=0}^{r-1}d_lx^l$$

we have that $(*)$ holds iff there exists $\overline{w}\neq\overline{0}$, with $\overline{M}^t\centerdot\overline{w}=\overline{0}$, where;

$$\overline{M}=\begin{pmatrix} a_0 & a_1 & \dots & \dots & a_r & 0 & \dots & \dots & 0\\ 0 & a_0 & a_1 & \dots\dots & a_{r-1} & a_r & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & \dots & a_r\\ b_0 & b_1 & \dots & \dots & b_{s-1} & b_s & 0 & \dots & 0\\ 0 & b_0 & b_1 & \dots & \dots & \dots & b_s & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \dots & 0 & b_0 & b_1 & \dots & b_s \end{pmatrix}$$

$$\overline{w}=\begin{pmatrix} c_0\\ \dots\\ c_{s-1}\\ d_0\\ \dots\\ d_{r-1}\end{pmatrix}$$

iff $det(\overline{M})=0$. Generalising the above argument, if $r\geq s$, and $1\leq q\leq s$, we have that f and g share at least q common roots iff there exist polynomials $\{c_q,d_q\}\subset L[x]$, with $deg(c)\leq s-q$, $deg(d)\leq r-q$, such that $c_q(x)f(x)+d_q(x)g(x)=0$, $(*)$, iff $\overline{M}^t\centerdot\overline{w}_q=\overline{0}$, where;

$$\overline{w}_q = \begin{pmatrix} c_0 \\ \ldots \\ c_{s-q} \\ 0 \\ \ldots \\ 0 \\ d_0 \\ \ldots \\ d_{r-q} \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

iff $det(\overline{M}_q) = 0$, where $\overline{M}_q$ is the $(r+s) - 2(q-1) \times (r+s) - 2(q-1)$ matrix obtained from $\overline{M}$, by deleting rows $s-q+2 \leq i \leq s$, $s+r-q+2 \leq i \leq r+s$, and columns $s-q+2 \leq j \leq s$, $s+r-q+2 \leq j \leq r+s$. If Y' is an irreducible algebraic curve, defined in affine coordinates (x,y,z) by $f(x,y,z) = g(x,y,z) = 0$, with $f(0,0,0) = g(0,0,0) = 0$, we have;

$f(x,y,z) = \sum_{i=0}^{r} a_i(x,y)z^i$

$g(x,y,z) = \sum_{i=0}^{s} b_i(x,y)z^i$

then, by the above, the projected curve $pr_z(Y')$, is defined by $det(\overline{M}_{x,y}) = 0$, where $\overline{M}_{x,y}$ is obtained by substituting the polynomials $\{a_i(x,y), b_i(x,y)\} \subset L[x,y]$, for the coefficients $\{a_i, b_j\} \subset L$ in $\overline{M}$. Moreover, the condition that pr_z is biunivocal, is guaranteed by $det(\overline{M}_{2,x,y}) \neq 0 \cap pr_z(Y') \neq \emptyset$, where $\overline{M}_{2,x,y}$ is obtained, in the same way, substituting into $\overline{M}_2$. This follows, as, in this case, $pr_z(Y')$ being irreducible, $(det(\overline{M}_{2,x,y}) \cap pr_z(Y')) \subset pr_z(Y')$, hence, we can find a generic $\overline{x}_0 \in pr_z(Y')$ with, $det(\overline{M}_{2,x,y}) \neq 0$. In particular there exists an open subset $U \subset pr_z(Y')$, with $Card(pr^{-1}(x) \cap pr_z^{-1}(U) \cap Y') = 1$, for $x \in U$, $(*)$. If there exists $z_0 \neq 0$, with $(0,0,z_0) \in Y'$, then we can choose $(\epsilon_1, \epsilon_2, z_0') \in pr_z^{-1}(U) \cap Y' \cap \mathcal{V}_{(0,0,z_0)}$, $(\epsilon_1, \epsilon_2, \epsilon_3) \in pr_z^{-1}(U) \cap Y' \cap \mathcal{V}_{(0,0,0)}$, see [4], contradicting $(*)$. We now claim that there exists a sequence of polynomials $\{\lambda_0, \lambda_{s-2}\} \subset L[x,y]$, such that, if $\lambda_i(0,0) \neq 0$, for $1 \leq i \leq s-2$, then there exists a rational function $a(x,y) = \frac{u(x,y)}{v(x,y)}$, with $v(0,0) \neq 0$, and an open subset $V \subset pr_z(Y')$, with $(0,0) \in V$, $v|_V \neq 0$, such that $Im(b|_V) \subset Y'$ where $b(x,y) = (x,y,a(x,y))$, in particular, pr_z is birational. This uses Sylvester's method. By the Euclidean algorithm, we have that there exist $\{h_1, g_1\} \subset L[x]$; with $deg(h_1) = r-s$ and $deg(g_1) < deg(g)$ such that;

$f(x) = h_1(x)g(x) + g_1(x)$

Writing;

$h_1(x) = \sum_{k=0}^{r-s} c_k x^k \; g_1(x) = \sum_{l=0}^{s-1} d_l x^l$

we have that;

$\sum_{j=0}^{s}\sum_{k=0}^{r-s} b_j c_k b_j c_k x^{j+k} = (\sum_{i=0}^{s-1} w_i x^i) + (\sum_{i=s}^{r} a_i x^i)$

The requirement that $\sum_{j+k=i} b_j c_k = a_i$, $(s \leq i \leq r)$, is given by the matrix equation $\overline{N}_0 \cdot \overline{c}_0 = \overline{a}_0$, where;

$$\overline{N}_0 = \begin{pmatrix} b_s & b_{s-1} & \dots & b_{2s-r} \\ 0 & b_s & \dots & b_{2s-r+1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_s \end{pmatrix}$$

$$\overline{c}_0 = \begin{pmatrix} c_0 \\ \dots \\ c_{r-s} \end{pmatrix}$$

$$\overline{a}_0 = \begin{pmatrix} a_s \\ \dots \\ a_r \end{pmatrix}$$

In particular, we have that $\overline{c}_0 = \overline{N_0}^{-1} \cdot \overline{a}_0$, (i). We then have, if $s \leq r - s$ $\overline{d}_{0,1} = \overline{B}_{0,1}\overline{c}_0 - \overline{a}_{0,1}$, and, if $s > r - s$, $\overline{d}_{0,2} = \overline{B}_{0,2}\overline{c}_{0,2} - \overline{a}_{0,2}$, (ii), where;

$$\overline{d}_{0,1} = \begin{pmatrix} d_0 \\ \dots \\ d_{s-1} \\ 0 \\ \dots \end{pmatrix}$$

$$\overline{B}_{0,1} = \begin{pmatrix} b_0 & 0 & \dots & \dots & \dots & \dots & 0 \\ b_1 & b_0 & 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_s & b_{s-1} & \dots & b_0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

$$\overline{a}_{0,1} = \begin{pmatrix} a_0 \\ \dots \\ a_{s-1} \\ 0 \\ \dots \end{pmatrix}$$

$$\overline{d}_{0,2} = \begin{pmatrix} d_0 \\ \dots \\ d_{s-1} \end{pmatrix}$$

$$\overline{B}_{0,2} = \begin{pmatrix} b_0 & 0 & \dots & \dots & 0 \\ b_1 & b_0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_s & b_{s-1} & \dots & b_1 & b_0 \end{pmatrix}$$

Given a generic choice of plane W passing through S, distinct from H_λ, and, using the Lemma 5.8, restricting the parameters $\{q_1,\ldots,q_4\}$ defining the point of projection, to ensure that $P\in H_\lambda$, and avoiding the finitely many lines $\{l_{p_i,p_j}:1\leq i\leq s\}\cup\{l_{O,p_i}:1\leq i\leq s\}$, we can find $P\in H_\lambda$ such that the projection $pr_{P,W}|Y$ is birational, and,

---

$$\overline{c}_{0,2}=\begin{pmatrix}c_0\\ \ldots\\ c_{r-s}\\ 0\\ \ldots\\ 0\end{pmatrix}$$

$$\overline{a}_{0,2}=\begin{pmatrix}a_0\\ \ldots\\ a_{r-s}\\ 0\\ \ldots\\ 0\end{pmatrix}$$

Let $\lambda_{0,1}=\overline{d}_{0,1}(s)$, $\lambda_{0,2}=\overline{d}_{0,2}(s)$, then, if either $\lambda_{0,1}\neq 0$ or $\lambda_{0,2}\neq 0$, we obtain that $deg(g_1)=s-1$. Repeating the algorithm, with g replacing f and g_1 replacing g, we let $\lambda_{1,2}=\overline{d}_{1,1}(s-1)$, where the vector $\overline{d}_{1,1}$ is obtained in the same way from $\{\overline{c}_1,\overline{N}_{1,1},\overline{B}_{1,1},\overline{c}_{1,2},\overline{a}_{1,2}\}$ as $\overline{d}_{0,1}$ is obtained from $\{\overline{c}_0,\overline{N}_{0,1},\overline{B}_{0,1},\overline{c}_{0,2},\overline{a}_{0,2}\}$, and where $\{\overline{d}_{1,1},\overline{c}_1,\overline{N}_{1,1},\overline{B}_{1,1},\overline{c}_{1,2},\overline{a}_{1,2}\}$ correspond to coefficients from from $\{g_2,h_2,g_1,g_1,h_2,g\}$, in the decomposition $g=h_2g_1+g_2$. If $\lambda_{1,2}\neq 0$, we obtain that $deg(g_2)=s-2$. Continuing in this way, we define $\{\lambda_{i,2}:0\leq i\leq s-2\}$, by $\lambda_{i,2}=\overline{d}_{i,1}(s-i)$, so that, if $\{\lambda_{i,2}:0\leq i\leq s-2\}\cap\{0\}=\emptyset$, $(**)$, then we obtain a chain $\{g_i,h_i:1\leq i\leq s\}$, such that $deg(g_i)=s-i$, $deg(h_i)=1$, with $g_i=g_{i+1}h_{i+2}+g_{i+2}$, for $0\leq i\leq s-3$. The condition that $\{f,g\}$ has exactly one root is common is given by the conditions that $det(\overline{M}_1)=0$, $det(\overline{M}_2)=0$, hence, the highest common factor of f and g has degree 1. If, in addition, $(**)$ holds, $(***)$, the Euclidean algorithm terminates at the step $i=s-3$, with the hcf given by g_{s-1}, and, we obtain the common root as $\overline{d}_{s-2,1}(1)$. By repeating the steps $(i),(ii)$, we obtain polynomials $\{p_1,p_2\}\in\mathcal{Z}[y_0,\ldots,y_r,y_{r+1},\ldots,y_{r+s+2}]$, with;

$$\overline{d}_{s-2,1}(1)=\frac{p_1(a_0,\ldots,a_r,b_0,\ldots,b_s)}{p_2(a_0,\ldots,a_r,b_0,\ldots,b_s)}$$

Moreover, the polynomials are independent of the particular choice of coefficients $\{a_0,\ldots,a_r,b_0,\ldots,b_s\}$, provided that $(***)$ holds. If $(***)$ holds, replacing the coefficients with $\{a_0(0,0),\ldots,a_r(0,0),b_0(0,0),\ldots,b_s(0,0)\}$, corresponding to the curve $Y'\subset A^3$ defined by $f(x,y,z)=g(x,y,z)=0$, we let;

$$u(x,y)=p_1(a_0(x,y),\ldots,a_r(x,y),b_0(x,y),\ldots,b_s(x,y))$$

$$v(x,y)=p_2(a_0(x,y),\ldots,a_r(x,y),b_0(x,y),\ldots,b_s(x,y))$$

Then there must exist an open set $V\subset pr_z(Y')$, with $(0,0)\in V$, having the required properties.

therefore, etale, in a neighborhood of $H_\lambda \cap Y$, ([7]). Moreover, by $(***)$, choosing the line $l = pr_{P,W}(H_\lambda)$ to be $x = 0$ in a suitable coordinate system (x, y), we can assume that all the branches of $Y' = pr_{P,W}(Y)$, centred along $x = 0$, are nonsingular, intersect the line transversely, with $pr_{P,W}(S) = (0, 0)$, and $[0 : 1 : 0] \notin Y'$. Using Theorem 1.8 of [5], the fact that $pr^{-1}_{P,W}$ is etale at $(0, 0)$, Definition 5.2, and the fact that $pr^{-1}_{P,W} : (Y', (0, 0)) \rightarrow (Y, S)$ is birational, it is sufficient to show that the local algebraic and geometric multiplicities coincide for the cover $g_2 \circ pr^{-1}_{P,W} : (Y', (0, 0)) \rightarrow (Z, 0)$. We let $F(x, y)$ define the projected curve Y', in the coordinate system (x, y). By Newton's Theorem, we have that $F(x, y) = \prod_{i=1}^{n}(Y - \eta_i(X))$, with the factors $\{\eta_j : 1 \leq j \leq r\}$ corresponding to the r branches of Y' centred at $(0, 0)$. Using the method of [10], we can find an irreducible curve $Y'' \subset P^2$, in coordinates (x, z), and a locally etale morphism $pr_x : Y'' \rightarrow (A^1, 0)$ such that $\eta_i \in L[U]$, for $(0, 0) \in U \subset Y''$. Then, using the coordinates (x, y, z) introduced above, $pr_{(x,y)} : (Y'' \times A^1, (0, 0, 0)) \rightarrow (A^1 \times A^1, (0, 0))$ is etale, in particularly, by Theorem 1.4 of [10], $pr_{(x,y)}|_{A^1 \times A^1}$ is Zariski unramified at $(0, 0, 0)$. Consequently, $\bigcup_{1 \leq i \leq r} graph(\eta_i)$ is Zariski unramified over Y' at $((0, 0), 0)$, $(\dagger\dagger\dagger)$, in the sense that, for $1 \leq i \leq r$, there exists a unique branch γ'_i of $\bigcup_{1 \leq i \leq r} graph(\eta_i)$, centred at $((0, 0), 0)$, projecting onto each of the branches $\{\gamma_i : 1 \leq i \leq r\}$, centred at $(0, 0)$, and $Mult_{\gamma'_i/\gamma_i}(\bigcup_{1 \leq i \leq r} graph(\eta_i)/Y') = 1$, see Definition 5.2. We have that;

$$\frac{L[[x,y]]}{<F(x,y)>} \cong \frac{L[[x,y]]}{<\prod_{i=1}^{n}(y-\eta_i(x))>}$$

hence, $\hat{\mathcal{O}}_{(0,0),Y'} \cong \hat{\mathcal{O}}_{(0,0,0),\bigcup_{1 \leq i \leq r} graph(\eta_i)}$, ([8])

Letting $Y''' = \bigcup_{1 \leq i \leq r} graph(\eta_i)$, it follows that the morphism $pr_{(x,y)} : Y''' \rightarrow Y'$ is etale, at $(0, 0, 0)$, and, hence, we have, again using Theorem

[7] Again, one needs to make simple checks that certain definable varieties do not contain the hyperplane H_λ

[8] As;

$$\hat{\mathcal{O}}_{(0,0),Y'} \cong \left(\frac{L[x,y]}{<F(x,y)>}\right)\hat{}_{(0,0)} \cong \left(\frac{L[x,y]_{(0,0)}}{<F(x,y)>}\right)\hat{} \cong \frac{L[x,y]\hat{}_{(0,0)}}{<F(x,y)>} \cong \frac{L[[x,y]]}{<F(x,y)>}$$

and;

$$\hat{\mathcal{O}}_{(0,0,0),\bigcup_{1 \leq i \leq r} graph(\eta_i)} \cong \left(\frac{R[Y''][y]}{<\prod_{i=1}^{r}(y-\eta_i(x))>}\right)\hat{}_{(0,0,0)} \cong \left(\frac{R[Y''][y]_{(0,0,0)}}{<\prod_{i=1}^{r}(y-\eta_i(x))>}\right)\hat{}$$

$$\cong \left(\frac{R[Y''][y]\hat{}_{(0,0,0)}}{<\prod_{i=1}^{r}(y-\eta_i(x))>}\right) \cong \left(\frac{R[Y'']\hat{}_{(0,0)}[[y]]}{<\prod_{i=1}^{r}(y-\eta_i(x))>}\right) \cong \frac{L[[x,y]]}{<\prod_{i=1}^{r}(y-\eta_i(x))>} \cong \frac{L[[x,y]]}{<\prod_{i=1}^{n}(y-\eta_i(x))>}$$

because $L[x, y]_{(0,0)} \subset L[[x, y]]$, $R[Y'']\hat{}_{(0,0)} \cong L[[x]]$, by construction of Y'', and, for $r + 1 \leq i \leq n$, $y - \eta_i(x)$ are units in the ring $L[[x, y]]$.

1.8 of [5], that $mult^{alg}_{(0,0,0)}(Y'''/Z) = mult^{alg}_{(0,0)}(Y'/Z)$. Using (†††), and Definition 5.2, we have that $mult_{(0,0,0),0}(Y'''/Z) = mult_{(0,0,0),0}(Y'/Z)$. Moreover, each of the branches of Y''', centred at $(0,0,0)$, is nonsingular, $(****)$. In order to see this, observe that $(0,0)$ is a nonsingular point of Y'', hence, we can find a representation, in the sense of Theorem 6.1 of [6], $(x,t(x))$ for the unique branch γ, centred at $(0,0)$, in Y'', with $t(x) \in L[[x]]$, and, then, for $1 \leq i \leq r$, $\overline{v}(x) = (x,t(x),\eta_i(x))$ is a representation of the branch γ_i, centred at $(0,0,0)$, of Y'''. As $(\overline{v})'(0) = (1,t'(0),\eta_i'(0)) \neq \overline{0}$, a simple calculation, using Theorem 6.1 of [6], gives $(****)$. We can, therefore, assume that Y' is reducible, with r components $\{Y_i' : 1 \leq i \leq r\}$, each of which is nonsingular at S. Using the method of [5], we can find locally etale maps $\{\theta_i : 1 \leq i \leq r\}$, $\theta_i : (Y_i', S) \rightarrow (C_{m_i}, (0,0))$, for $C_{m_i} = Spec(\frac{L[x,y]}{y^{m_i}-x})$, such that $g_2|_{Y_i'} = pr_x \circ \theta_i$, where;

$$m_i = mult^{alg}_S(Y_i'/Z) = mult^{alg}_S(C_{m_i}/Z) = mult_{(0,0),0}(C_{m_i}/Z) = mult_{S,0}(Y_i'/Z)$$
$(****)$

Enumerating the components of Y', for which $m_k = mult^{alg}_S(Y_i'/Z)$, as $\{Y_s^{m_k} : s \in I_k\}$, we can take $C_{m_i} = C_s^{m_k} = Spec(\frac{L[x,y]}{y^{m_k}-sx})$, so that the curves $\{C_{m_i} : 1 \leq i \leq r\}$ are distinct, and preserve the conditions in $(****)$. We let $C = \bigcup_{1\leq i\leq r} C_{m_i}$. We let $\theta : ((Y'' \setminus \{\bigcup_{1\leq i<j\leq n}(Y_i \cap Y_j)\}) \cup \{S\}, S) \rightarrow C$ be defined by;

$$\theta(S) = (0,0)$$

$$\theta(x) = \theta_i(x), \text{ if } x \in Y_i'$$

We have, using the Chinese remainder theorem, and the fact that the components $\{y - \eta_i(x) : 1 \leq i \leq r\}$ $\{y^{m_k} - sx : 1 \leq k \leq c, 1 \leq s \leq r\}$, where $c = max_{1\leq i\leq r} m_i$, are pairwise coprime in the ring $L[[x,y]]$, that $\hat{\mathcal{O}_{S,Y'}} \cong \prod_{i=1}^r \hat{\mathcal{O}_{S,Y_i}}$, $\hat{\mathcal{O}_{(0,0),C}} \cong \prod_{i=1}^r \hat{\mathcal{O}_{(0,0),C_{m_i}}}$. It follows, as each θ_i is locally etale at S, that θ induces an isomorphism $\hat{\mathcal{O}_{(0,0),C}} \cong \hat{\mathcal{O}_{S,Y'}}$, hence θ is locally etale at S. Again, using Theorems 1.4 and 1.8 of [5], the conditions $(****)$ and Definition 5.2, we have that, it is sufficient to verify that $mult^{alg}_{(0,0)}(C/Z) = mult_{(0,0),0}(C/Z)$, that is;

$$length(k[x,y]/ < \prod(y^{m_1} - x)\ldots(y^{m_s} - x), x >) = (\sum_{i=1}^r m_i)$$

This is a straightforward calculation left to the reader.

Case 2. Y has some singular branches. Again, wlog, assume that $Y \subset P^3$. We follow the same method as in Case 1, using the method of projections and Sylvester's theory to reduce to $Y \subset P^2$. Let $\{s_j : 1 \leq j \leq r\}$ enumerate the orders of the r branches $\{\gamma_S^i : 1 \leq i \leq r\}$ of Y, centred at S. Again, we obtain that, for a generic hyperplane H_λ;

a. $I_{italian}(S, \gamma_S^j, Y, H_\lambda) = s_j$, for the branches γ_S^j, $1 \leq j \leq t$, passing through S.

b. The remaining intersections $\{p_i : 1 \leq i \leq s\} = H_\lambda \cap Y$ are non-singular, and $I_{italian}(p_i, C, H_\lambda) = 1$, $1 \leq i \leq s$, $(*****)$

Again, choosing P generically in H_λ, and W a hyperplane distinct from H_λ, passing through S, we obtain, using Lemma 5.8, and Theorem 6.4 of [6], that, for a choice of coordinate system (x, y) on W, with the line $pr_{P,W}(H_\lambda)$ corresponding to the line $x = 0$, and $(0, 0)$ corresponding to S, that the tangent lines of the branches of $Y' = pr_{P,W}(Y)$, centred at $(0, 0)$, are all transverse to the line $x = 0$, have the same orders $\{s_i : 1 \leq i \leq r\}$, the remaining intersections of Y' with $x = 0$ are all in finite position, define nonsingular points, and are transverse to $x = 0$. Again, using Puiseux Series, we can find $m \in \mathcal{Z}_{\geq 1}$, such that Y' is defined by $F(x, y) = \prod_{i=1}^{n}(y - \eta_i(x^{\frac{1}{m}}))$, $(******)$. Letting C_m be defined by $z^m - x = 0$, we have that that $\eta_i(x^{\frac{1}{m}}) \in \hat{O}_{C_m,(0,0)}$, hence, similarly to the above, we can find $Y'' \subset P^3$, in coordinates (w, x, z), such that the projection $pr_{(x,z)} : (Y'', (0, 0, 0)) \rightarrow (C_m, (0, 0))$ is etale at $(0, 0, 0)$, with $\eta_i(x^{\frac{1}{m}}) \in O_{Y'',(0,0,0)}$, for $1 \leq i \leq n$. Then $(Y'' \times A^1, (0, 0, 0, 0)$, in coordinates (w, x, y, z), is locally an etale cover of $(C_m \times A^1, (0, 0, 0))$. If $(w_0, x_0, y_0, z_0) \in graph(\eta_i(x^{\frac{1}{m}}))$, then $y_0 = \eta_i(x_0, w_0, z_0)$, with $(x_0, w_0, z_0) \in Y''$, $(x_0, z_0) \in C_m$, $z_0^m = x_0$, $(*******)$. Moreover;

$$F(x_0, y_0) = F(x_0, \eta_i(x_0, w_0, z_0)) = F(x_0, \eta_i(x_0^{\frac{1}{m}})) = 0$$

by $(******), (*******)$.

Define $pr : graph(\eta_i(x^{\frac{1}{m}})) \rightarrow C_m \times_{A_1} Y'$ by $pr(w_0, x_0, y_0, z_0) = (x_0, y_0, z_0)$, where $z_0^m = y_0$, $y_0 = \eta_i(w_0, x_0, z_0)$. Then $F(x, y) = 0$, $F(x, \eta_i(x^{\frac{1}{m}}) = 0$ implies $F(x_0, z_0), (*), = F(x_0, \eta_i(x_0, y_0, w_0)) = F(x_0, \eta_i(x_0, x_0^{\frac{1}{m}}, w_0)) =$

0, and $y_0^m = x_0, (**)$, so $(x_0, y_0, z_0) \in D \times_{A^1} Y'$.

..... Sim (ii), for each i $pr : graph(\eta_i(x^{\frac{1}{m}}))$ is Zar.unr over $(y^m - x) = 0 \times_{A_1} Y'$ at $(0,0,0,0)$. Moreover, if $W = \bigcup_{1 \leq i \leq n} graph(\eta_i(x^{\frac{1}{m}}))$, then $pr : W \rightarrow (y^m - x) = 0 \times_{A_1} Y'$ is etale at $(0,0,0,0)$, as $\frac{L[x,y]}{(y^m-x)} \otimes_{L[x]} \frac{L[x,z]}{F(x,z)} \cong \frac{L[x,y,z]}{<y^m-x,F(x,z)>}$, then completion;

$$\frac{L[[x,y,z]]}{<y^m-x,F(x,z)>}$$

$$\cong \frac{L[[x,y,z]]}{<y^m-x,\prod_{i=1}^n(z-\eta_i(y))>}$$

$$\cong \frac{\hat{O_{D,(0,0)}}[z]}{\prod_{i=1}^n(z-\eta_i(y))}$$

$$\cong \frac{\hat{O_{Y'',(0,0,0)}}[z]}{\prod_{i=1}^n(z-\eta_i(y))} \cong \hat{O_{W,(0,0,0,0)}}$$

Replace Y' by W, again algebraic multiplicity preserved, local branch multiplicity preserved, check, using birational models and Zariski unramified, multiplicative over composition (geometric multiplicity) gives same problem with product W.

(vi). Biunivocal method again, to get product of curves in P^2, geometric multiplicities m_i, birationality and local uniformisers reduce to $k[x,y]/ < \prod(y^{m_1} - x)\ldots(y^{m_s} - x) >$ again, algebraic multiplicity $m_1 + \ldots m_s$.

(vii). Repeat argument (i)-(vi), for each $b_i \in g_2^{-1}(O)$, and use total algebraic multiplicity is sum of local algebraic multiplicities.
.....

□

Lemma 5.8. *Let X, Y and Z be irreducible projective algebraic curves, with morphisms $g_1 : X \rightarrow Y$, $g_2 : Y \rightarrow Z$, such that X and Z are nonsingular, and g_1 is birational. Then if $O \in Z$, the total algebraic multiplicity of g_2 over O coincides with the total algebraic multiplicity of $(g_2 \circ g_1)$ over O.*

Lemma 5.9. *Let Y and Z be irreducible projective algebraic curves, with a morphisms $g_2 : Y \rightarrow Z$, such that Z is nonsingular. Then, if $O \in Z$, the total algebraic multiplicity of $g_2^{-1}(O)$ over O coincides with the total geometric multiplicity of $g_2^{-1}(O)$ over O.*

Choose a nonsingular model X of Y, with a birational morphism $g_1 : X \rightarrow Y$. It is straightforward to see that the total geometric multiplicity is preserved, and the total algebraic multiplicity is preserved by the previous lemma. The result follows for the morphism $g_2 \circ g_1 : X \rightarrow Z$ from the result, $char(L) = 0$, in [8], that the local geometric multiplicity=local algebraic multiplicity.

Lemma 5.10. *Let $f : C_1 \rightarrow C_2$ be a finite morphism, with $O \in C_2$, and $P \in C_1$, then, if D is the local algebraic multiplicity at P;*

$D = \sum_{\gamma\in\gamma_O} Mult_{P/\gamma}(C_1/C_2) - 1$

in particular, if A is the total algebraic multiplicity of $f^{-1}(O)$ over O, B is the total geometric multiplicity of $f^{-1}(O)$ over O, $A = rB - |f^{-1}(O)|$, where r is the number of branches, centred at O.

Choose a nonsingular model $\phi_2 : C_2^{ns} \rightarrow C_2$, with $\{Q_1, \ldots, Q_r\}$ corresponding to r branches at O, and suppose that $f(P) = O$. Let $Y = C_2^{ns} \times_{C_2} C_1 = \{(x, y) \in C_2^{ns} \times C_1, \phi_2(x) = f(y)\}$, $W = Sp(\phi_1^*(O_{O,C_2})) \times Sp(O_{O,C_2})Sp(O_{P,C_1})$, with projections $pr_1 : (C_2^{ns} \times_{C_2} C_1) \rightarrow C_2^{ns}$ and $pr_2 : (C_2^{ns} \times_{C_2} C_1) \rightarrow C_1$, $pr_1 : W \rightarrow Sp(\phi_1^*(O_{O,C_2}))$, and $pr_2 : W \rightarrow Sp(O_{P,C_1})$. For a suitable affine presentation, we have $R(Y) = R(C_2^{ns}) \otimes_{R(C_2)} R(C_1)$, $R(W) = \phi_1^*(O_{O,C_2}) \otimes_{O_{O,C_2}} O_{P,C_1}$. Then we have that $\phi_1^*(O_{O,C_2}) \otimes_{O_{O,C_2}} O_{P,C_1}/pr_2^* f^* m_O \cong R(C_2^{ns}) \otimes_{O_{O,C_2}} O_{P,C_1}/pr_1^* \phi_1^* m_O$.

We claim that $\phi_1^*(O_{O,C_2}) \otimes_{O_{O,C_2}} O_{P,C_1}/pr_2 f^* m_O \cong \phi_1^*(O_{O,C_2})/\phi_1^* m_O) \oplus O_{P,C_1}/f^* m_O$, and, as ϕ_1 is birational, that $length(\phi_1^*(O_{O,C_2})/\phi_1^* m_O) = 1$, $(*)$

so that, $length(O_{P,C_1}/f^* m_O) = length(\phi_1^*(O_{O,C_2}) \otimes_{O_{O,C_2}} O_{P,C_1}/pr_2 f^* m_O) - 1$ (using Lemma 5.8)

$= length(\phi_1^*(O_{O,C_2}) \otimes_{O_{O,C_2}} O_{P,C_1}/pr_1^* \phi_1^* m_O) - 1$ (commuting diagram for products)

$= length(\phi_1^*(O_{O,C_2}) \otimes_{O_{O,C_2}} O_{P,C_1}/pr_1^*(m_{Q_1} \cap \ldots \cap m_{Q_r})) - 1$

$= length(\oplus_{1\leq i\leq r} \phi_1^*(O_{O,C_2}) \otimes_{O_{O,C_2}} O_{P,C_1}/pr_i^*(m_{Q_i})) - 1$

$= \sum_{1\leq i\leq r} length(\phi_1^*(O_{O,C_2}) \otimes_{O_{O,C_2}} O_{P,C_1}/pr_i^*(m_{Q_i})) - 1$

by the CRT, as the ideals $pr_i^*(m_{Q_i})$ are coprime in $R(C_2^{ns}) \otimes_{O_{O,C_2}} O_{P,C_1}$. The local result then follows, by 5.8 and 5.9. In particular;

Lemma 5.11. *Let X, Y and Z be irreducible projective algebraic curves, with morphisms $g_1 : X \rightarrow Y$, $g_2 : Y \rightarrow Z$, such that X is nonsingular, and g_1 is birational. Then, if $O \in Z$, the total algebraic multiplicity of $g_2^{-1}(O)$ over O coincides with the total algebraic multiplicity of $(g_2 \circ g_1)^{-1}(O)$ over O.*

By the previous lemma, and the fact that geometric multiplicity is preserved.

Definition 5.12. *If $f : C_1 \rightarrow C_2$ is a finite morphism between projective algebraic curves, we say that f is geometrically flat, if the total geometric multiplicity is independent of the choice of branch $\gamma \in \gamma_O$, for any $O \in C_2$, and, flat, if the total algebraic multiplicity is independent of the choice of $O \in C_2$.*

We have ,by Definition 5.12 and Lemma 5.4, that;

Theorem 5.13. *If f is a finite morphism between irreducible projective algebraic curves, then f is geometrically flat.*

Remarks 5.14. *It is known that when C_2 is nonsingular, f is flat. Counterexample to stronger result?*

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