SOME NOTES ON NON-STANDARD DIFFERENTIATION AND INTEGRATION

TRISTRAM DE PIRO

Abstract. This short paper provides some basic results required to define non-standard differentiation and integration.

We assemble here some basic results concerning non-standard differentiation and integration. We will assume some familiarity with ideas in mathematical logic and properties of the real numbers $\mathcal{R}$. A far more detailed account can be found in [2].

Lemma 0.1. *There exists a structure* $(\mathcal{R}^*, +, ., <)$*, that is an elementary extension of the real numbers* $\mathcal{R}$*, with the property that it contains an element* ϵ*, such that;*

$0 < \epsilon < r$*, for every* $r \in \mathcal{R}$ *with* $0 < r$ $(\dagger)$

Proof. The proof is a straightforward application of the compactness theorem from first order logic. One considers the following theory in the language $\mathcal{L}_{\mathcal{R}} \cup \{c\}$;

$Th(\mathcal{R}, +, ., <) \cup \{0 < c < r : r \in \mathcal{R}\}$

Any finite set S of sentences from this theory is satisfied by the structure $(\mathcal{R}, +, ., <)$ itself, as one can interpret the constant c to be any real number r', such that;

$0 < r' < min_{1 \leq i \leq n} r_i$

where $r_i \in \mathcal{R}$ are the finitely many constant terms appearing in the set S. Hence, the theory has a model $(\mathcal{R}^*, +, ., <)$ with the desired properties. □

Thanks to my colleague Professor Carlo Toffalori, at The University of Camerino, for a helpful discussion and the use of his office from 14th-16th May, 2008.

Definition 0.2. *We will refer to any element $\epsilon \in \mathcal{R}^*$, with the property (†), as a positive infinitesimal, and, any element $r' \in \mathcal{R}^*$, with the property that there exist $\{r_1, r_2\} \subset \mathcal{R}$, such that $r_1 < r' < r_2$, as bounded. We refer to elements in $\mathcal{R}^*$, but not in $\mathcal{R}$, as non-standard. We refer to any structure $\mathcal{R}^*$ satisfying Lemma 0.1, as a non-standard extension of $\mathcal{R}$.*

We then have the following;

Lemma 0.3. *For* every *bounded $r' \in \mathcal{R}^*$, there exists a* unique *$r \in \mathcal{R}$, such that;*

$r' = r + \epsilon$ or $r' = r - \epsilon$ or $r' = r$

where ϵ is a positive infinitesimal.

Proof. As r' is bounded, the sets;

$S_1 = \{r \in \mathcal{R} : r \leq r'\}$, $S_2 = \{r \in \mathcal{R} : r' \leq r\}$

are both non-empty. By straightforward properties of $\mathcal{R}$, there exist $\{s_1, s_2\} \subset \mathcal{R}$, such that;

$s_1 = sup(S_1)$ and $s_2 = inf(S_2)$

It is easy to see that $s_1 = s_2$, using the fact that $\mathcal{R}^*$ satisfies the first-order property of $\mathcal{R}$;

$\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$

Let $r = s_1 = s_2$, then clearly r must belong to either S_1 or S_2, the existence result then follows from considering the alternatives $r \in S_1 \cap S_2^c$, $r \in S_1^c \cap S_2$ or $r \in S_1 \cap S_2$. For uniqueness, suppose that there existed distinct elements $\{r_1, r_2\} \subset \mathcal{R}$, and, without loss of generality, positive infinitesimals $\{\epsilon_1, \epsilon_2\}$, such that $r = r_1 + \epsilon_1 = r_2 + \epsilon_2$. Then;

$(r_1 - r_2) = (\epsilon_2 - \epsilon_1)$

However, it is straightforward to show that if $\epsilon_2 - \epsilon_1 \neq 0$, there exists a positive infinitesimal c, such that $\epsilon_2 - \epsilon_1 = c$ or $\epsilon_2 - \epsilon_1 = -c$. In any case, this contradicts the fact that $r_1 - r_2 \neq 0$.

□

Definition 0.4. *Given a bounded $r' \in \mathcal{R}^*$, we refer to the r provided by the previous lemma as the standard part $st(r')$ of r'. Given $r \in \mathcal{R}$, we refer to the set;*

$$\mathcal{V}_r = \{r' \in \mathcal{R}^* : st(r') = r\}$$

as an infinitesimal neighborhood of r.

We then have;

Lemma 0.5. *The infinitesimal neighborhoods $\mathcal{V}_r$, for $r \in \mathcal{R}$, define a partition of the bounded elements of $\mathcal{R}^*$. Moreover, each infinitesimal neighborhood $\mathcal{V}_r$ contains non-standard elements, that is elements distinct from r.*

Proof. If $r_1 \neq r_2$ belong to $\mathcal{R}$, then the infinitesimal neighborhoods $\mathcal{V}_{r_1}$ and $\mathcal{V}_{r_2}$ are disjoint, by the uniqueness part of Lemma 0.3. It follows easily that the infinitesimal neighborhoods define a partition of the bounded elements of $\mathcal{R}$. If ϵ is a positive infinitesimal, then $r + \epsilon$ is a non-standard element, belonging to the infinitesimal neighborhood $\mathcal{V}_r$ of r. $\square$

We then consider how to define real functions on a non-standard extension $\mathcal{R}^*$;

Lemma 0.6. *Let $f(x)$ be a real function on the bounded interval $(a, b) \subset \mathcal{R}$, then f extends to a function on the bounded interval $(a, b) \subset \mathcal{R}^*$, for a non-standard extension of $\mathcal{R}$*

Proof. One should go through the proof of Lemma 0.1, replacing the structure $(\mathcal{R}, +, ., <)$ with $(\mathcal{R}, +, ., <, f)$. The result then follows from the fact that the statement;

$$\forall x[(a < x < b) \rightarrow \exists! y(f(x) = y)]$$

is elementary for the theory $Th(\mathcal{R}, +, ., <, f)$.

$\square$

With this basic set up, we can then define non-standard versions of continuity and differentiation;

Definition 0.7. *Non-Standard Continuity*

Let $f(x)$ be a real function on the bounded interval $(a,b) \subset \mathcal{R}$, and consider its extension to $\mathcal{R}^*$. We say that f is non-standard continuous at $x \in (a,b)$, for $x \in \mathcal{R}$, if;

$f(x+\epsilon) \in \mathcal{V}_{f(x)}$, *for any* $\epsilon \in \mathcal{V}_0$

We then have;

Lemma 0.8. *f is non-standard continuous at $x \in (a,b)$ iff it is continuous at $x \in (a,b)$.*

Proof. The proof is an elementary exercise, using the method of Lemma 0.3 and the usual definition of continuity in terms of limits. □

Definition 0.9. *Non-Standard Differentiation*

Let hypothesis be as in the previous definition. Then we say that f is non-standard differentiable, at $x \in (a,b)$, if there exists $c \in \mathcal{R}$, such that;

$\frac{f(x+\epsilon)-f(x)}{\epsilon} \in \mathcal{V}_c$, *for every infinitesimal* $\epsilon \in \mathcal{V}_0$.

We then define the non-standard derivative of f at x to be c.

Lemma 0.10. *f is non-standard differentiable at $x \in (a,b)$ iff it is differentiable at $x \in (a,b)$. Moreover, in this case, the non-standard derivative equals the usual derivative.*

Proof. Again, the proof is an easy consequence of the usual definition of differentiability at $x \in (a,b)$ in terms of limits, that;

$lim_{h\rightarrow 0}\frac{f(x+h)-f(x)}{h}$

exists, and arguments, using the method of Lemma 0.3.

□

We now consider the non-standard approach to integration;

Definition 0.11. *Let $f(x)$ be a continuous function on the bounded interval $[a,b]$. For a real number c, with $0 < c < (b-a)$, we define the Riemann sum $R_f(c)$ to be;*

$R_f(c) = \sum_{j=0}^{N(c)} f(a + jc)c$

where $N(c)$ is the greatest positive integer n such that $(a + nc) < b$.

We then have;

Lemma 0.12. *Let ϵ be any positive infinitesimal, and let $R_f(\epsilon)$ be its value in $\mathcal{R}^*$, as guaranteed by Lemma 0.6. Then $R_f(\epsilon)$ is bounded, and, moreover, the standard part $st(R_f(\epsilon))$ coincides with the usual definite integral;*

$\int_a^b f(x)dx$

Proof. Let $\{S, T\}$ be any real numbers such that;

$S > max_{x\in[a,b]}f(x)$

$T < min_{x\in[a,b]}f(x)$

Then a straightforward calculation shows that $T(b - a) < R_f(c) < S(b - a)$, for $0 < c < (b - a)$. It follows easily, by formulating this statement in $\mathcal{L}_R$, that $T(b - a) < R_f(\epsilon) < S(b - a)$, for any positive infinitesimal ϵ. Hence, the boundedness result follows. One can show easily that;

$\int_a^b f(x)dx = lim_{n\to\infty}R_f(c_n)$

where $c_n = \frac{(b-a)}{n}$. It is a simple calculation, using the method of Lemma 0.3, to show that this must coincide with the standard part $st(R_f(\epsilon))$, for any positive infinitesimal ϵ.

□

The following definition and lemma are then obvious;

Definition 0.13. *Non-Standard Integration*

Let f be a continuous function on the bounded interval $[a, b]$, then we define its non-standard integral to be $st(R_f(\epsilon))$, where ϵ is any positive infinitesimal.

Lemma 0.14. *For a continuous function f on the bounded interval $[a,b]$, the non-standard integral coincides with the usual (Riemann or Lebesgue)integral.*

Proof. The proof follows immediately from the previous definition and Lemma 0.12. ☐

For future applications, it will be convenient to work with a particular class of non-standard extensions of $\mathcal{R}$. Namely, we consider the structure $\mathcal{R}_{an} = (\mathcal{R}, +, ., -, 0, 1, <; \mathcal{F})$ where $\mathcal{F}$ denotes the class of functions $\bar{f} : \mathcal{R}^m \rightarrow \mathcal{R}$ of the following form;

For any *analytic* function $f : U \rightarrow \mathcal{R}$, where U is an open neighborhood of a product $I_1 \times \ldots \times I_m$ of closed intervals, let;

$$\begin{aligned} \bar{f}(\bar{x}) &= f(\bar{x}) \text{ if } \bar{x} \in I_1 \times \ldots \times I_m \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The theory $Th(\mathcal{R}_{an})$ is model complete, due to a theorem of Gabrielov, see [1]. We then make the following;

Definition 0.15. *We define any structure $(\mathcal{R}^*, +, ., -, 0, 1, <; \mathcal{F})$, that is an elementary extension of $\mathcal{R}_{an}$, and contains an element ϵ such that;*

$0 < \epsilon < r$, for every $r \in \mathcal{R}$

to be an analytic non-standard extension of $\mathcal{R}$.

Remarks 0.16. *Following the proof of Lemma 0.1, it is easily shown that analytic non-standard extensions exist. It is an easy exercise to show that we can obtain the remaining results of the paper, provided we restrict ourselves to analytic functions on a bounded open (or closed) interval in $\mathcal{R}$.*

References

[1] A. Gabrielov, Projections of Semi-Analytic Sets, Functional Anal. Appl. 2, (1968), 282-291.

[2] Abraham Robinson, Non-Standard Analysis, Princeton University Press, (1996)

55B Via Ludovica Albertoni, Rome, 00152
E-mail address: tristam.depiro@unicam.it