DE BROGLIE'S HYPOTHESIS AND THE WAVE EQUATION

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Abstract.

Lemma 0.1. *We have that;*

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy = \frac{n!\pi}{2^n[(\frac{n}{2})!]^2}$, *if* n *is even*

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy = \frac{[\frac{n-1}{2}!]^2 2^n}{n!}$, *if* n *is odd*

Proof. Let $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy$, then for $n \geq 2$, we have that, using integration by parts;

$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy$

$= [cos^{n-1}(y)sin(y)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int -\frac{\pi}{2}^{\frac{\pi}{2}}(n-1)cos^{n-2}(y)sin^2(y)dy$

$= \int -\frac{\pi}{2}^{\frac{\pi}{2}}(n-1)cos^{n-2}(y)(1 - cos^2(y))dy$

$= (n-1)I_{n-2} - (n-1)I_n$

so that, rearranging;

$I_n = \frac{n-1}{n} I_{n-2}$

and, using the fact $I_0 = \pi$, $I_1 = 2$, we have that, for n even;

$I_n = \frac{n!}{2^n[(\frac{n}{2})!]^2}\pi$

and, for n odd;

$I_n = \frac{[\frac{n-1}{2}!]^2 2^n}{n!}$

□

Lemma 0.2. *Let $n \in \mathcal{N}$, $\epsilon > 0$, and let $\gamma_{n,\epsilon}$ be defined by;*

$\gamma_{n,\epsilon}(x) = \frac{1}{\epsilon}cos^n(\frac{\pi x}{2\epsilon})$, *for* $x \in [-\epsilon, \epsilon]$

$\gamma_{n,\epsilon}(x) = 0$, *otherwise*

Then $\gamma_{n,\epsilon}$ has the following properties;

(i). $\gamma_{n,\epsilon} \in C^{n-1}(\mathcal{R})$.

(ii). $\gamma_{n,\epsilon} \geq 0$.

(iii). $\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{n!}{2^{n-1}[(\frac{n}{2})!]^2}$, *n even*

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{[\frac{n-1}{2}!]^2 2^{n+1}}{\pi n!}$, *n odd*

(iv) $\gamma_{n,\epsilon}$ *is supported on* $[-\epsilon, \epsilon]$.

Proof. (ii) is clear as $cos(y) \geq 0$ for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, (iv) is clear by the definition of $\gamma_{n,\epsilon}$. To prove (i), it is sufficient to show that;

$$cos^n(\frac{\pi x}{2\epsilon})^{(m)}(\epsilon) = cos^n(\frac{\pi x}{2\epsilon})^{(m)}(-\epsilon) = 0$$

for $0 \leq m \leq n-1$. We can prove this by induction on n, as for $n = 1$, we have that;

$$cos(\frac{\pi x}{2\epsilon})(\epsilon) = cos(\frac{\pi}{2}) = cos(\frac{\pi x}{2\epsilon})(-\epsilon) = cos(-\frac{\pi}{2}) = 0$$

and, if the inductive hypothesis holds for $n \in \mathcal{N}$, then, for $1 \leq m \leq n$;

$$cos^{n+1}(\frac{\pi x}{2\epsilon})^{(m)}(\epsilon)$$

$$= -[\frac{\pi(n+1)}{2\epsilon}cos^n(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon})]^{(m-1)}(\epsilon)$$

$$= -\frac{\pi(n+1)}{2\epsilon}[\sum_{k=0}^{m-1} C_k^{m-1} cos^n(\frac{\pi x}{2\epsilon})^{(m-1-k)} sin(\frac{\pi x}{2\epsilon})^{(k)}](\epsilon)$$

$$= 0$$

and similarly;

$cos^{n+1}(\frac{\pi x}{2\epsilon})^{(m)}(-\epsilon) = 0$

while, clearly;

$cos^{n+1}(\frac{\pi x}{2\epsilon})(\epsilon) = cos^{n+1}(\frac{\pi x}{2\epsilon})(-\epsilon) = 0$

To prove (iii), we have that, for $n \in \mathcal{N}$;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx$

$= \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} cos^n(\frac{\pi x}{2\epsilon})$

$= \frac{1}{\epsilon}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)\frac{2\epsilon}{\pi}dy$, $(y = \frac{\pi x}{2\epsilon})$

$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(y)dy$

so that, using Lemma 0.1, for n even;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{2}{\pi}\frac{n!}{2^n[(\frac{n}{2})!]^2}\pi$

$= \frac{n!}{2^{n-1}[(\frac{n}{2})!]^2}$

and, for n odd;

$\int_{\mathcal{R}} \gamma_{n,\epsilon}(x)dx = \frac{2}{\pi}\frac{[\frac{n-1}{2}!]^2 2^n}{n!}$

$= \frac{[\frac{n-1}{2}!]^2 2^{n+1}}{\pi n!}$

□

Lemma 0.3. *Let $\delta_{n,\epsilon}(x)$ be defined by;*

$\delta_{n,\epsilon}(x) = \frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\gamma_{n,\epsilon}$, *for n even*

and by;

$\delta_{n,\epsilon}(x) = \frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n,\epsilon}$, *for n odd*

Then the properties $(i), (ii), (iv)$ of Lemma 0.2 hold, with (iii) changed to;

$(iii)'$. $\int_{\mathcal{R}}\gamma_{n,\epsilon}(x)dx=1$, $n\in\mathcal{N}$

and, for $n\in\mathcal{N}$;

$lim_{\epsilon\rightarrow 0}\delta_{n,\epsilon}=\delta$

in the sense of distributions, where δ is the Dirac delta function on $\mathcal{R}$.

Proof. The first claim is clear as we have just normalised $\gamma_{n,\epsilon}$. For the remaining claim, let $f\in C_c^\infty(\mathcal{R})$, and write;

$f=f^+ + f^-$

where;

$f^+(x)=f(x)$, if $f(x)\geq 0$

$f^+(x)=0$ otherwise

$f^-(x)=f(x)$, if $f(x)\leq 0$

$f^-(x)=0$ otherwise

Then, using properties $(ii),(iii)',(iv)$ of $\delta_{n,\epsilon}$ and continuity of f;

$min_{[-\epsilon,\epsilon]}f^+ + min_{[-\epsilon,\epsilon]}f^- \leq \delta_{n,\epsilon}(f)=\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f^+(x)dx+\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f^-(x)dx$

$\leq max_{[-\epsilon,\epsilon]}f^+ + max_{[-\epsilon,\epsilon]}f^-$

with;

$lim_{\epsilon\rightarrow 0}min_{[-\epsilon,\epsilon]}f^+ + min_{[-\epsilon,\epsilon]}f^- = lim_{\epsilon\rightarrow 0}max_{[-\epsilon,\epsilon]}f^+ + max_{[-\epsilon,\epsilon]}f^-$

$=f(0)$

so that $lim_{\epsilon\rightarrow 0}\delta_{n,\epsilon}(f)=f(0)=\delta(f)$, as required.

□

Lemma 0.4. *We define the time derivative* δ_t' *of the delta function* δ *to be;*

$\frac{d}{dt}\delta(x-vt)$

where v *is the velocity, so that;*

$\delta_t' = -v\delta'$

in the sense of distributions. Similarly, we define the time derivative $\delta_{n,\epsilon,t}'$ *of the approximations by;*

$\frac{d}{dt}\delta_{n,\epsilon}(x-vt)$

so that, by the chain rule;

$\delta_{n,\epsilon,t}'(x) = -v\delta_{n,\epsilon}'(x)$

Then;

$lim_{\epsilon\rightarrow 0}\delta_{n,\epsilon,t}' = \delta_t'$

in the sense of distributions.

Moreover, for n *even,* $n \geq 2$*;*

$\delta_{n,\epsilon,t}'(x) = \frac{v2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}sin(\frac{\pi x}{2\epsilon})$

and, for n *odd,* $n \geq 3$*;*

$\delta_{n,\epsilon,t}'(x) = \frac{v\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}sin(\frac{\pi x}{2\epsilon})$

In particularly, for $n \geq 2$*;*

$\delta_{n,\epsilon,t}' \in C_c^{n-2}(\mathcal{R})$

Proof. For the first claim, let $f \in C_c^\infty(\mathcal{R})$, then, using integration by parts, (iv) of Lemma 0.3;

$\delta_{n,\epsilon,t}'(f) = -v\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}'(x)f(x)dx$

$= -v([\delta'_{n,\epsilon}(x)f(x)]_{-\epsilon}^{\epsilon} - \int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f'(x)dx)$

$= v\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f'(x)dx$

so that, using the main result of Lemma 0.3;

$lim_{\epsilon\rightarrow 0}\delta'_{n,\epsilon,t}(f)$

$= vlim_{\epsilon\rightarrow 0}\int_{-\epsilon}^{\epsilon}\delta_{n,\epsilon}(x)f'(x)dx$

$= vf'(0)$

$= -v\delta'(f)$

$= \delta'_t(f)$

as required. For the second claim, we have that, for n even, $n \geq 2$;

$\delta'_{n,\epsilon,t}(x) = -v\delta'_{n,\epsilon}(x)$

$= -v\frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\gamma'_{n,\epsilon}(x)$

$= -v\frac{2^{n-1}[\frac{n}{2}!]^2}{n!}(-ncos^{n-1}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon})\frac{\pi}{2\epsilon^2})$

$= v\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{1}{\epsilon}cos^{n-1}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon}))$

$= \frac{v2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}(x)sin(\frac{\pi x}{2\epsilon})$

and, for n odd, $n \geq 3$;

$\delta'_{n,\epsilon,t}(x) = -v\delta'_{n,\epsilon}(x)$

$= -v\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma'_{n,\epsilon}(x)$

$= -v\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}}(-ncos^{n-1}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon})\frac{\pi}{2\epsilon^2})$

$= v\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{1}{\epsilon}cos^{n-1}(\frac{\pi x}{2\epsilon})sin(\frac{\pi x}{2\epsilon}))$

$= \frac{v\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}(x)sin(\frac{\pi x}{2\epsilon})$

For the final claim, we have that, by the product rule, for $0 \leq m \leq n-2$;

$(\gamma_{n-1,\epsilon}(x)sin(\frac{\pi x}{2\epsilon}))^{(m)}$

$= \sum_{k=0}^{m} \gamma_{n-1,\epsilon}^{(m-k)}(sin(\frac{\pi x}{2\epsilon}))^{k}$

and use the fact that $\gamma_{n-1,\epsilon} \in C_c^{n-2}(\mathcal{R})$ □

Lemma 0.5. *Let D denote the 3-dimensional Dirac delta function. For $n \in \mathcal{N}, \epsilon > 0$, let $D_{n,\epsilon}$ be defined by;*

$D_{n,\epsilon}(x,y,z) = \delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)$

Then $D_{n,\epsilon}$ has the following properties;

(i). $D_{n,\epsilon} \in C^{n-1}(\mathcal{R}^3)$.

(ii). $D_{n,\epsilon} \geq 0$.

(iii). $\int_{\mathcal{R}^3} D_{n,\epsilon}(x,y,z)dxdydz = 1$

(iv) $D_{n,\epsilon}$ is supported on $[-\epsilon,\epsilon]^3$.

Moreover, for $n \in \mathcal{N}$, $lim_{\epsilon \to 0} D_{n,\epsilon} = D$ in the sense of distributions, $(*)$.

We define the time derivative D'_t of the delta function D to be;

$\frac{d}{dt}D(\overline{x} - \overline{v}t)$

where $\overline{v}$ is the velocity vector, so that;

$D'_t = -v_1 D_x - v_2 D_y - v_3 D_z$

in the sense of distributions. Similarly, we define the time derivative $D'_{n,\epsilon,t}$ of the approximations by;

$\frac{d}{dt}D_{n,\epsilon}(\overline{x} - \overline{v}t)$

so that, by the chain rule;

$D'_{n,\epsilon,t}(\overline{x})=-v_1D_{n,\epsilon,x}(\overline{x})-v_2D_{n,\epsilon,y}(\overline{x})-v_3D_{n,\epsilon,z}(\overline{x})$

$=-v_1\delta'_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)-v_2\delta_{n,\epsilon}(x)\delta'_{n,\epsilon}(y)\delta_{n,\epsilon}(z)-v_3\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta'_{n,\epsilon}(z)$

Then;

$lim_{\epsilon\rightarrow 0}D'_{n,\epsilon,t}=D'_t$, $(**)$

in the sense of distributions.

Finally, we have that;

$D_{n,\epsilon}(x,y,z)=(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^3\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)$, *for n even*

and;

$D_{n,\epsilon}(x,y,z)=(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^3\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)$, *for n odd*

We have that;

$D'_{n,\epsilon,t}=\frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^2[v_1\gamma_{n-1,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi x}{2\epsilon})$

$+v_2\gamma_{n,\epsilon}(x)\gamma_{n-1,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi y}{2\epsilon})+v_3\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})]$

for n even.

$D'_{n,\epsilon,t}=\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^2[v_1\gamma_{n-1,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi x}{2\epsilon})$

$+v_2\gamma_{n,\epsilon}(x)\gamma_{n-1,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi y}{2\epsilon})+v_3\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})]$

for n odd.

In particularly, $D'_{n,\epsilon,t}\in C_c^{n-2}(\mathcal{R}^3)$.

Proof. For the first claim, (i) follows from the fact that $\delta_{n,\epsilon}\in C^{n-1}(\mathcal{R})$ and the fact that if $i+j+k=n-1$;

$\frac{\partial^{i+j+k}D_{n,\epsilon}}{\partial x^i\partial y^j\partial z^k}(x,y,z)=\delta_{n,\epsilon}^{(i)}(x)\delta_{n,\epsilon}^{(j)}(y)\delta_{n,\epsilon}^{(k)}(z)$

(ii) is trivial from the corresponding property of the $\delta_{n,\epsilon}$. (iii) follows form Fubini's theorem;

$$\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]} D_{n,\epsilon}(x,y,z)dxdydz = \int_{[-\epsilon,\epsilon]}\delta_{n,\epsilon}(x)dx\int_{[-\epsilon,\epsilon]}\delta_{n,\epsilon}(y)dy\int_{[-\epsilon,\epsilon]}\delta_{n,\epsilon}(z)dz$$

and the corresponding property (iii) of $\delta_{n,\epsilon}$. (iv) is again trivial from the the corresponding property of $\delta_{n,\epsilon}$. The claim $(*)$ is a consequence of properties $(ii),(iii),(iv)$, see the proof above for the 1-dimensional case. For the claim $(**)$, we have that, for $f\in C_c^\infty(\mathcal{R}^3)$, using integration by parts and Tonelli's theorem;

$$D'_{n,\epsilon,t}(f) = \int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}[-v_1\delta'_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)-v_2\delta_{n,\epsilon}(x)\delta'_{n,\epsilon}(y)\delta_{n,\epsilon}(z)$$

$$-v_3\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta'_{n,\epsilon}(z)]fdxdydz$$

$$= \int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}\int_{[-\epsilon,\epsilon]}[v_1\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)f_x + v_2\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)f_y$$

$$+v_3\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)]f_zdxdydz$$

so that, by the claim $(*)$;

$$lim_{\epsilon\rightarrow 0}D'_{n,\epsilon,t}(f) = v_1f_x(0)+v_2f_y(0)+v_3f_z(0)$$

$$= (-v_1D_x - v_2D_y - v_3D_z)(f)$$

$$= D'_t(f)$$

The first computational claim follows from Lemma 0.3 and the definition of $D_{n,\epsilon}$. For the second computational claim;

$$D'_{n,\epsilon,t}(\overline{x}) = -v_1\delta'_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta_{n,\epsilon}(z)-v_2\delta_{n,\epsilon}(x)\delta'_{n,\epsilon}(y)\delta_{n,\epsilon}(z)-v_3\delta_{n,\epsilon}(x)\delta_{n,\epsilon}(y)\delta'_{n,\epsilon}(z)$$

so that, for n even;

$$D'_{n,\epsilon,t}(\overline{x}) = -v_1[-\tfrac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}(x)sin(\tfrac{\pi x}{2\epsilon})][(\tfrac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(y)][(\tfrac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(z)]$$

$$-v_2[(\tfrac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(x)][-\tfrac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}(y)sin(\tfrac{\pi y}{2\epsilon})][(\tfrac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(z)]$$

$$-v_3[(\tfrac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(x)][(\tfrac{2^{n-1}[\frac{n}{2}!]^2}{n!})\gamma_{n,\epsilon}(y)][-\tfrac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}\gamma_{n-1,\epsilon}(z)sin(\tfrac{\pi z}{2\epsilon})]$$

$= v_1 \frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^2\gamma_{n-1,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi x}{2\epsilon})$

$+v_2 \frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^2\gamma_{n,\epsilon}(x)\gamma_{n-1,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi y}{2\epsilon})$

$+v_3 \frac{2^{n-1}\pi[\frac{n}{2}!]^2}{2\epsilon(n-1)!}(\frac{2^{n-1}[\frac{n}{2}!]^2}{n!})^2\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})$

and, for n odd;

$D'_{n,\epsilon,t}(\overline{x}) = -v_1[-\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}(x)sin(\frac{\pi x}{2\epsilon})][(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(y)][(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(z)]$

$-v_2[(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(x)][-\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}(y)sin(\frac{\pi y}{2\epsilon})][(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(z)]$

$-v_3[(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(x)][(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})\gamma_{n,\epsilon}(y)][-\frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})]$

$= v_1 \frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^2\gamma_{n-1,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi x}{2\epsilon})$

$+v_2 \frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^2\gamma_{n,\epsilon}(x)\gamma_{n-1,\epsilon}(y)\gamma_{n,\epsilon}(z)sin(\frac{\pi y}{2\epsilon})$

$+v_3 \frac{\pi^2 n!n}{2\epsilon[(\frac{n-1}{2})!]^2 2^{n+1}}(\frac{\pi n!}{[(\frac{n-1}{2})!]^2 2^{n+1}})^2\gamma_{n,\epsilon}(x)\gamma_{n,\epsilon}(y)\gamma_{n-1,\epsilon}(z)sin(\frac{\pi z}{2\epsilon})$

The final claim follows as above, by repeated application of the product rule, using the fact that $\gamma_{n-1,\epsilon} \in C_c^{n-2}(\mathcal{R})$ and $\gamma_{n,\epsilon} \in C_c^{n-1}(\mathcal{R})$ $\square$

Definition 0.6. *For $n \geq 4$, we let $\rho_{n,\epsilon}$ be the unique charge distribution on $\mathcal{R}^3 \times \mathcal{R}_{\geq 0}$ defined by the initial conditions $\{qD_{n,\epsilon}, qD'_{n,\epsilon,t}\}$, satisfying the wave equation with velocity c, $\square^2(\rho_{n,\epsilon}) = 0$, on $\mathcal{R}^3 \times \mathcal{R}_{\geq 0}$, where q is the total charge.*

Lemma 0.7. *For $n \geq 6$, we have that;*

$\rho_{n,\epsilon}(\overline{x}, t) = \frac{1}{(2\pi)^3}\int_{\mathcal{R}^3}(A(\overline{k})e^{ikct} + B(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

where;

$A(\overline{k}) = \frac{\mathcal{F}(qD_{n,\epsilon})}{2} + \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc}$

$B(\overline{k}) = \frac{\mathcal{F}(qD_{n,\epsilon})}{2} - \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc}$

and $\mathcal{F}$ denotes the usual 3 dimensional Fourier transform;

$\mathcal{F}(g) = \int_{\mathcal{R}^3} g(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}$

for $g \in L^1(\mathcal{R}^3)$.

Proof. We have that the existence of $\rho_{n,\epsilon}$ is guaranteed by Kirchoff's formula, for $n \geq 4$, as the initial conditions $D_{n,\epsilon} \in C_c^3(\mathcal{R}^3$ and $D'_{n,\epsilon,t} \in C_c^2(\mathcal{R}^4)$ for $n \geq 4$, see [1]. Using the fact that the initial conditions $\{qD_{n,\epsilon}, qD'_{n,\epsilon,t}\}$ have compact support, $\rho_{n,\epsilon,t}$ has compact support as a process, in particularly $\rho_{n,\epsilon,t} \in L^1(\mathcal{R}^3)$, for $t \geq 0$. Using Kirchoff's formula, see [1], we have that, for $t > 0$;

$$\rho_{n,\epsilon}(\overline{x},t) = \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t}(\overline{y}) + qD_{n,\epsilon}(\overline{y}) + DqD_{n,\epsilon}(\overline{y}) \cdot (\overline{y}$$

$$-\overline{x})]dS(\overline{y})$$

Then, we have that, using the substitution $\overline{z} = \overline{y} - (h,0,0)$, $d\overline{z} = d\overline{y}$ and interchanging limits;

$$\frac{\partial \rho_{n,\epsilon}}{\partial x} = lim_{h\to 0}\frac{1}{h}(\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x}+(h,0,0),ct)}[ctqD'_{n,\epsilon,t}(\overline{y})+qD_{n,\epsilon}(\overline{y})+DqD_{n,\epsilon}(\overline{y})\cdot$$

$$(\overline{y}-(\overline{x}+(h,0,0)))]dS(\overline{y}) - \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t}(\overline{y}) + qD_{n,\epsilon}(\overline{y})+$$

$$DqD_{n,\epsilon}(\overline{y}) \cdot (\overline{y}-\overline{x})]dS(\overline{y}))$$

$$= lim_{h\to 0}\frac{1}{h}(\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t}(\overline{y}+(h,0,0)) + qD_{n,\epsilon}(\overline{y}+(h,0,0))$$

$$+DqD_{n,\epsilon}(\overline{y}+(h,0,0))\cdot(\overline{y}+(h,0,0)-(\overline{x}+(h,0,0)))]dS(\overline{y})-\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t}(\overline{y})$$

$$+qD_{n,\epsilon}(\overline{y}) + DqD_{n,\epsilon}(\overline{y}) \cdot (\overline{y}-\overline{x})]dS(\overline{y}))$$

$$= \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[lim_{h\to 0}\frac{1}{h}(ctqD'_{n,\epsilon,t}(\overline{y}+(h,0,0))-ctqD'_{n,\epsilon,t}(\overline{y}))+lim_{h\to 0}\frac{1}{h}(qD_{n,\epsilon}(\overline{y}+$$

$$(h,0,0))-qD_{n,\epsilon}(\overline{y}))+lim_{h\to 0}\frac{1}{h}(DqD_{n,\epsilon}(\overline{y}+(h,0,0))-DqD_{n,\epsilon}(\overline{y}))\cdot(\overline{y}-$$

$$\overline{x})]dS(\overline{y})$$

$$= \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}[ctqD'_{n,\epsilon,t,x}(\overline{y})+qD_{n,\epsilon,x}(\overline{y})+DqD_{n,\epsilon,x}(\overline{y})\cdot(\overline{y}-\overline{x})]dS(\overline{y})$$

so that, as the initial conditions $\{qD_{n,\epsilon}, qD'_{n,\epsilon,t}\}$ are in $C_c^{n-1}(\mathcal{R}^3)$ and $C_c^{n-2}(\mathcal{R}^3)$ respectively, it follows that for $t > 0$, $\rho_{n,\epsilon,t} \in C_c^{n-3}(\mathcal{R}^3)$. In particularly, for $n \geq 6$, $\rho_{n,\epsilon,t} \in C_c^3(\mathcal{R}^3)$, $(*)$. We have that;

$\Box^2(\rho_{n,\epsilon,t}) = \bigtriangledown^2(\rho_{n,\epsilon,t}) - \frac{1}{c^2}\frac{\partial^2 \rho_{n,\epsilon,t}}{\partial t^2} = 0$

so we can apply $\mathcal{F}$ to both sides and obtain, and differentiating under the integral sign;

$\mathcal{F}(\bigtriangledown^2(\rho_{n,\epsilon,t}))(\overline{k},t) - \frac{1}{c^2}\frac{\partial^2 \mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t)}{\partial t^2} = 0$

As $\rho_{n,\epsilon,t} \in C^2_c(\mathcal{R}^3)$, by $(*)$, we have, using integration by parts, that;

$\mathcal{F}(\bigtriangledown^2(\rho_{n,\epsilon,t}))(\overline{k},t) = -k^2\mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t)$

so that;

$-k^2\mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t) - \frac{1}{c^2}\frac{\partial^2 \mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t)}{\partial t^2} = 0$

and we can use Peano's theorem to solve the ODE in time, to obtain;

$\mathcal{F}(\rho_{n,\epsilon,t})(\overline{k},t) = A(\overline{k})e^{ikct} + B(\overline{k})e^{-ikct}$

where, at $t = 0$;

$A(\overline{k}) + B(\overline{k}) = \mathcal{F}(qD_{n,\epsilon})$

and, taking the time derivative at $t = 0$;

$ikcA(\overline{k}) - ikcB(\overline{k}) = \mathcal{F}(qD'_{n,\epsilon,t})$

Now we can solve the simultaneous equations to get the expressions for $\{A(\overline{k}), B(\overline{k})\}$, We have that $\mathcal{F}(qD_{n,\epsilon}) \in C^\infty(\mathcal{R}^3)$ as $qD_{n,\epsilon}$ has compact support and similarly for $\mathcal{F}(qD'_{n,\epsilon,t})$. Moreover, we have that, using integration by parts, for $k_1 \neq 0$;

$\mathcal{F}(qD_{n,\epsilon,x}) = ik_1\mathcal{F}(qD_{n,\epsilon})$

so that, for $k \neq 0$, using the fact that for $n \geq 6$, $qD_{n,\epsilon} \in C^5_c(\mathcal{R}^3)$, $qD'_{n,\epsilon,t} \in C^4(\mathcal{R}^3)$;

$\mathcal{F}((\bigtriangledown^2)^2 qD_{n,\epsilon}) = k^4\mathcal{F}(qD_{n,\epsilon})$

$|\mathcal{F}(qD_{n,\epsilon})| \leq \frac{|\mathcal{F}((\bigtriangledown^2)^2 qD_{n,\epsilon})|}{k^4}$

$\leq \frac{C}{k^4}$

$\mathcal{F}((\bigtriangledown^2)^2 qD'_{n,\epsilon,t}) = k^4\mathcal{F}(qD'_{n,\epsilon,t})$

$|\mathcal{F}(qD'_{n,\epsilon,t})| \leq \frac{|\mathcal{F}((\bigtriangledown^2)^2 qD'_{n,\epsilon,t})|}{k^4}$

$\leq \frac{D}{k^4}$ (†)

Converting to polar coordinates, this proves that $\{\mathcal{F}(qD_{n,\epsilon}), \mathcal{F}(qD'_{n,\epsilon,t})\} \subset L^1(\mathcal{R}^3)$. Now we can use the fact that, for $k \neq 0$;

$|A(\overline{k})| \leq \frac{|\mathcal{F}(qD_{n,\epsilon})|}{2} + \frac{|\mathcal{F}(qD'_{n,\epsilon,t})|}{2kc}$

$|B(\overline{k})| \leq \frac{|\mathcal{F}(qD_{n,\epsilon})|}{2} + \frac{|\mathcal{F}(qD'_{n,\epsilon,t})|}{2kc}$

the fact that $\{\mathcal{F}(qD_{n,\epsilon}), \mathcal{F}(qD'_{n,\epsilon,t}) \subset C^\infty(\mathcal{R}^3)$, and (†), using polar coordinates again, to prove that $\{A(\overline{k}), B(\overline{k})\} \subset L^1(\mathcal{R}^3)$. Finally, for $t > 0$;

$|\mathcal{F}(\rho_{n,\epsilon})| = |A(\overline{k})e^{ikct} + B(\overline{k}e^{-ikct}|$

$\leq |A(\overline{k})| + |B(\overline{k})|$

so that $\mathcal{F}(\rho_{n,\epsilon}) \in L^1(\mathcal{R}^3)$ for $t > 0$. It follows that we can apply the inversion theorem in the last step.

□

Lemma 0.8. *Let $\mathcal{F}$ be the 1-dimensional Fourier transform, then, for n even, $k \in \mathcal{R}$, $k \neq \frac{\pi}{\epsilon}(\frac{n}{2} - j)$, $0 \leq j \leq n$;*

$\mathcal{F}(\gamma_{n,\epsilon})(k) = -\frac{1}{2^{n-1}}\sum_{j=0}^{n} C_j^n \frac{1}{\pi(\frac{n}{2}-j)-\epsilon k}[(-1)^{\frac{n}{2}-j} sin(\epsilon k)]$

and, for $0 \leq j_0 \leq n$;

$\mathcal{F}(\gamma_{n,\epsilon})(\frac{\pi}{\epsilon}(\frac{n}{2} - j_0)) = \frac{1}{2^{n-1}}C_{j_0}^n$

In particularly;

$lim_{\epsilon \to 0}\mathcal{F}(\gamma_{n,\epsilon}) = \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$

uniformly on compact subsets of $\mathcal{R}$. *For* n *even;*

$\mathcal{F}(\delta_{n,\epsilon})=\frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\mathcal{F}(\gamma_{n,\epsilon})$

in particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\delta_{n,\epsilon})=1$

uniformly on compact subsets of $\mathcal{R}$. *For* n *even;*

$\mathcal{F}(\delta'_{n,\epsilon,t})(k)=ivk\mathcal{F}(\delta_{n,\epsilon})$

in particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\delta'_{n,\epsilon,t})(k)=ivk$

uniformly on compact subsets of $\mathcal{R}$.

Proof. We have that, for $k\in\mathcal{R}$, n even, $k\neq\frac{\pi}{\epsilon}(\frac{n}{2}-j)$, $0\leq j\leq n$, by the definition of $\gamma_{n,\epsilon}$ and $\mathcal{F}$;

$\mathcal{F}(\gamma_{n,\epsilon})=\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}cos^n(\frac{\pi x}{2\epsilon})e^{-ixk}dx$

$=\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}(\frac{e^{\frac{i\pi x}{2\epsilon}}+e^{\frac{-i\pi x}{2\epsilon}}}{2})^n e^{-ixk}dx$

$=\frac{1}{2^n\epsilon}\int_{-\epsilon}^{\epsilon}\sum_{j=0}^{n}C_j^n e^{\frac{i\pi x(n-j)}{2\epsilon}}e^{\frac{-i\pi xj}{2\epsilon}}e^{-ixk}dx$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n\int_{-\epsilon}^{\epsilon}e^{\frac{i\pi x(n-2j-\frac{2\epsilon k}{\pi})}{2\epsilon}}dx$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n[\frac{e^{\frac{i\pi x(n-2j-\frac{2\epsilon k}{\pi})}{2\epsilon}}}{\frac{i\pi}{2\epsilon}(n-2j-\frac{2\epsilon k}{\pi})}]_{-\epsilon}^{\epsilon}$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n\frac{1}{\frac{i\pi}{2\epsilon}(n-2j-\frac{2\epsilon k}{\pi})}[e^{\frac{i\pi x(n-2j-\frac{2\epsilon k}{\pi})}{2\epsilon}}]_{-\epsilon}^{\epsilon}$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n\frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik}[i^{n-2j-\frac{2\epsilon k}{\pi}}-(-i)^{n-2j-\frac{2\epsilon k}{\pi}}]$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n\frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik}[i^{n-2j}e^{-i\epsilon k}-(-i)^{n-2j}e^{i\epsilon k}]$

$=\frac{1}{2^n\epsilon}\sum_{j=0}^{n}C_j^n\frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik}[i^{n-2j}e^{-i\epsilon k}-i^{n-2j}e^{i\epsilon k}]$, $(n-2j)$ even

$= \frac{1}{2^n\epsilon}\sum_{j=0}^{n} C_j^n \frac{1}{\frac{i\pi}{2\epsilon}(n-2j)-ik}[-2i^{n-2j+1}sin(\epsilon k)]$

$= \frac{1}{2^n\epsilon}\sum_{j=0}^{n} C_j^n \frac{1}{\frac{\pi}{2\epsilon}(n-2j)-k}[-2i^{n-2j}sin(\epsilon k)]$

$= -\frac{1}{2^{n-1}\epsilon}\sum_{j=0}^{n} C_j^n \frac{1}{\frac{\pi}{2\epsilon}(n-2j)-k}[(-1)^{\frac{n}{2}-j}sin(\epsilon k)]$

$= -\frac{1}{2^{n-1}}\sum_{j=0}^{n} C_j^n \frac{1}{\pi(\frac{n}{2}-j)-\epsilon k}[(-1)^{\frac{n}{2}-j}sin(\epsilon k)]$

We know that $\mathcal{F}(\gamma_{n,\epsilon})$ is continuous, so that, for fixed ϵ, $0 \leq j_0 \leq n$, using L'Hopital's rule;

$\mathcal{F}(\gamma_{n,\epsilon})(\frac{\pi}{\epsilon}(\frac{n}{2}-j_0)) = lim_{k\rightarrow\frac{\pi}{\epsilon}(\frac{n}{2}-j_0)}\mathcal{F}(\gamma_{n,\epsilon})(k)$

$= -\frac{1}{2^{n-1}}(\sum_{j=0,j\neq j_0}^{n} C_j^n \frac{1}{\pi(j_0-j)}[(-1)^{\frac{n}{2}-j}sin(\pi(\frac{n}{2}-j_0))]$

$-\frac{1}{2^{n-1}}C_{j_0}^n\frac{1}{-\epsilon}[(-1)^{\frac{n}{2}-j_0}\epsilon cos(\pi(\frac{n}{2}-j_0))]$

$= \frac{1}{2^{n-1}}C_{j_0}^n[(-1)^{\frac{n}{2}-j_0}(-1)^{\frac{n}{2}-j_0}]$

$= \frac{1}{2^{n-1}}C_{j_0}^n$

In particular;

$\mathcal{F}(\gamma_{n,\epsilon})(0) = \frac{1}{2^{n-1}}C_{\frac{n}{2}}^n$

$= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$, $(*)$

independently of ϵ.

It follows that, for $k \neq 0$, using L'Hopital's rule;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\gamma_{n,\epsilon})(k) = lim_{\epsilon\rightarrow 0}\frac{1}{2^{n-1}}C_{\frac{n}{2}}^n\frac{1}{\epsilon k}sin(\epsilon k)$

$= \frac{1}{2^{n-1}}\frac{n!}{(\frac{n}{2}!)^2}lim_{\epsilon\rightarrow 0}\frac{1}{k}cos(\epsilon k)k$

$= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}lim_{\epsilon\rightarrow 0}cos(\epsilon k)$

$= \frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$

and clearly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\gamma_{n,\epsilon})(0)=\frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$

by $(*)$.

It follows that;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\gamma_{n,\epsilon})=\frac{n!}{2^{n-1}(\frac{n}{2}!)^2}$ (pointwise convergence)

To obtain uniform convergence on compact subsets, note that;

$\frac{1}{\epsilon k}sin(\epsilon k)$

converges uniformly to 1 on compact subsets as, for $|k|\leq K$;

$|\frac{1}{\epsilon k}sin(\epsilon k)-1|=\frac{|sin(\epsilon k)-\epsilon k|}{|\epsilon k|}$

$=\frac{|\epsilon k+O((\epsilon k)^3)-\epsilon k|}{|\epsilon k|}$

$\leq C|\epsilon k|^2$

$\leq C\epsilon^2K^2$

and $sin(\epsilon k)$

converges uniformly to 0 on compact subsets as , for $|k|\leq K$;

$|sin(\epsilon k)|\leq|\epsilon k|$

$\leq\epsilon K$

By Lemma 0.3, we have that, for n even;

$\mathcal{F}(\delta_{n,\epsilon})=\frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\mathcal{F}(\gamma_{n,\epsilon})$

so that, by the previous claim in this lemma;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\delta_{n,\epsilon})=\frac{2^{n-1}[\frac{n}{2}!]^2}{n!}\frac{n!}{2^{n-1}(\frac{n}{2}!)^2}=1$

on compact subsets of $\mathcal{R}$. We have that, using integration by parts and the definition of $\delta'_{n,\epsilon,t}$;

$\mathcal{F}(\delta'_{n,\epsilon,t})(k) = \mathcal{F}(-v\delta'_{n,\epsilon})$

$= -v(-ik)\mathcal{F}(\delta_{n,\epsilon})$

$= ivk\mathcal{F}(\delta_{n,\epsilon})$

so that, by the previous claim;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(\delta_{n,\epsilon}) = ivklim_{\epsilon\rightarrow 0}\mathcal{F}(\delta_{n,\epsilon})$

$= ivk$

uniformly on compact subsets of $\mathcal{R}$.

□

Lemma 0.9. *Let $\mathcal{F}$ be the three dimensional Fourier transform, then;*

$\mathcal{F}(D_{n,\epsilon}) = \mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$

where $\mathcal{F}_1$ is the 1-dimensional Fourier transform.

In particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(D_{n,\epsilon}) = 1$

uniformly on compact subsets of $\mathcal{R}^3$.

$\mathcal{F}(D'_{n,\epsilon,t})(k_1, k_2, k_3) = -v_1\mathcal{F}_1(\delta'_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$

$-v_2\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta'_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3) - v_3\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta'_{n,\epsilon})(k_3)$

In particularly;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(D'_{n,\epsilon,t})(\overline{k}) = i\overline{v}\centerdot\overline{k}$

uniformly on compact subsets of $\mathcal{R}^3$.

Proof. By Definition 0.5 and the theorem of Fubini;

$\mathcal{F}(D_{n\epsilon})(k_1, k_2, k_3) = \mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$

where $\mathcal{F}_1$ is the 1-dimensional Fourier transform. In particular, as the projections of a compact subset are compact, and using the result of Lemma 0.8;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(D_{n\epsilon})(\overline{k}) = 1.1.1$

$= 1$

uniformly on compact subsets of $\mathcal{R}^3$.

By definition 0.5 again and Fubini's theorem;

$\mathcal{F}(D'_{n,\epsilon,t})(k_1, k_2, k_3) = -v_1\mathcal{F}_1(\delta'_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3)$

$-v_2\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta'_{n,\epsilon})(k_2)\mathcal{F}_1(\delta_{n,\epsilon})(k_3) - v_3\mathcal{F}_1(\delta_{n,\epsilon})(k_1)\mathcal{F}_1(\delta_{n,\epsilon})(k_2)\mathcal{F}_1(\delta'_{n,\epsilon})(k_3)$

so that, using the result of Lemma 0.8, integration by parts and the previous observation;

$lim_{\epsilon\rightarrow 0}\mathcal{F}(D'_{n,\epsilon,t})(k_1, k_2, k_3) = -v_1(-ik_1).1.1 - v_2(-ik_2).1.1 - v_3(-ik_3).1.1$

$= i\overline{v}\centerdot\overline{k}$

uniformly on compact subsets of $\mathcal{R}^3$.

□

Lemma 0.10. *For the charge distribution $\rho_{n,\epsilon}$ of Lemma 0.7, we have that;*

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k}) = \frac{q}{2} + \frac{q\overline{v}\centerdot\overline{k}}{2kc}$

$lim_{\epsilon\rightarrow 0}B_{n,\epsilon}(\overline{k}) = \frac{q}{2} - \frac{q\overline{v}\centerdot\overline{k}}{2kc}$

uniformly on compact subsets of $\mathcal{R}^3 \setminus \{0\}$.

Proof. We have that, by Lemma 0.7 and the results of Lemma 0.9;

$lim_{\epsilon\rightarrow 0}A_{n,\epsilon}(\overline{k}) = lim_{\epsilon\rightarrow 0}(\frac{\mathcal{F}(qD_{n,\epsilon})}{2} + \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc})$

$= \frac{q}{2} + \frac{qi\overline{v}\centerdot\overline{k}}{2ikc}$

$= \frac{q}{2} + \frac{q\overline{v}.\overline{k}}{2kc}$

$lim_{\epsilon \rightarrow 0} B_{n,\epsilon}(\overline{k}) = lim_{\epsilon \rightarrow 0}(\frac{\mathcal{F}(qD_{n,\epsilon})}{2} - \frac{\mathcal{F}(qD'_{n,\epsilon,t})}{2ikc})$

$= \frac{q}{2} - \frac{qi\overline{v}.\overline{k}}{2ikc}$

$= \frac{q}{2} - \frac{q\overline{v}.\overline{k}}{2kc}$

uniformly on compact subsets of $\mathcal{R}^3 \setminus \{0\}$.

$\square$

Lemma 0.11. *For $n \geq 6$, for $k \in \mathcal{R}_{>0}$, we define the intensity $I_{n,\epsilon}(\overline{x}, t, k)$ at k to be;*

$I_{n,\epsilon}(\overline{x}, t, k) = \frac{1}{(2\pi)^3} \int_{S_k} (A(\overline{k})e^{ikct} + B(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}} dS(\overline{k})$

so that, by Lemma 0.7;

$\rho_{n,\epsilon}(\overline{x}, t) = \int_{\mathcal{R}} I(\overline{x}, t, k)dk$

Then;

$lim_{\epsilon \rightarrow 0} I_{n,\epsilon}(s\overline{v}, t, k) = \frac{q}{(2\pi)^3}(\frac{4\pi k}{sv} sin(svk)cos(ctk) - \frac{4\pi}{s^2vc} sin(svk)sin(ctk)$

$+ \frac{4\pi k}{cs} cos(svk)sin(ctk))$

For the electron at time t moving a distance d, we obtain local maxima in the wave number when;

$2kd = -Dtan(2kd)$

where $D = \frac{c-v}{c+v}$

is the frequency shift due to the Doppler effect, for an observer moving at speed v.

....

Proof. By Lemma 0.10 and the definition of $I_{n,\epsilon}$, we have that;

$lim_{\epsilon \rightarrow 0} I_{n,\epsilon}(s\overline{v}, t, k) = \frac{1}{(2\pi)^3} \int_{S_k} (A_{n,\epsilon}(\overline{k})e^{ikct} + B_{n,\epsilon}(\overline{k})e^{-ikct})e^{i\overline{k}.s\overline{v}} dS(\overline{k})$

$= \frac{q}{(2\pi)^3}e^{ikct}\int_{S_k}(\frac{1}{2}+\frac{\overline{v}.\overline{k}}{2kc})e^{is\overline{v}.\overline{k}}dS(\overline{k})+\frac{q}{(2\pi)^3}e^{-ikct}\int_{S_k}(\frac{1}{2}-\frac{\overline{v}.\overline{k}}{2kc})e^{is\overline{v}.\overline{k}}dS(\overline{k})$, $(*)$

Switching to polar coordinates, letting θ be the angle between $\overline{v}$ and $\overline{k} \in S_k$, we have that;

$\int_{S_k}\frac{1}{2}e^{is\overline{v}.\overline{k}}dS(\overline{k}) = \frac{1}{2}\int_0^{\pi}\int_{-\pi}^{\pi}e^{isvkcos(\theta)}k^2 sin(\theta)d\theta d\phi$

$= \pi k^2\int_0^{\pi}e^{isvkcos(\theta)}sin(\theta)d\theta$

$= \pi k^2\int_{-1}^{1}e^{isvku}du$, $(u = cos(\theta), du = -sin(\theta)d\theta)$

$= \pi k^2[\frac{e^{isvku}}{isvk}]_{-1}^{1}$

$= \frac{\pi k^2}{isvk}(e^{isvk}-e^{-isvk})$

$= \frac{\pi k^2}{isvk}2isin(svk)$

$= \frac{2\pi ksin(svk)}{sv}$

and;

$\int_{S_k}\frac{\overline{v}.\overline{k}}{2kc}e^{is\overline{v}.\overline{k}}dS(\overline{k}) = \frac{1}{2}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{vkcos(\theta)}{2kc}e^{isvkcos(\theta)}k^2 sin(\theta)d\theta d\phi$

$= \frac{\pi vk^2}{c}\int_0^{\pi}e^{isvkcos(\theta)}cos(\theta)sin(\theta)d\theta$

$= \frac{\pi vk^2}{c}\int_{-1}^{1}ue^{isvku}du$, $(u = cos(\theta), du = -sin(\theta)d\theta)$

$= \frac{\pi vk^2}{c}([\frac{ue^{isvku}}{isvk}]_{-1}^{1}-\int_{-1}^{1}\frac{e^{isvku}}{isvk}du)$

$= \frac{\pi vk^2}{c}(\frac{e^{isvk}+e^{-isvk}}{isvk}-[\frac{e^{isvku}}{(isvk)^2}]_{-1}^{1})$

$= \frac{\pi vk^2}{cisvk}(2cos(svk))-\frac{\pi vk^2}{c(isvk)^2}2isin(svk)$

$= \frac{2\pi isin(svk)}{s^2vc}-\frac{2\pi ikcos(svk)}{cs}$

It follows from $(*)$, that;

$lim_{\epsilon\rightarrow 0}I_{n,\epsilon}(s\overline{v},t,k) = \frac{q}{(2\pi)^3}e^{ikct}(\frac{2\pi ksin(svk)}{sv}+\frac{2\pi isin(svk)}{s^2vc}-\frac{2\pi ikcos(svk)}{cs})$

$+\frac{q}{(2\pi)^3}e^{-ikct}(\frac{2\pi k sin(svk)}{sv} - \frac{2\pi i sin(svk)}{s^2vc} + \frac{2\pi i k cos(svk)}{cs})$

$= \frac{q}{(2\pi)^3}(\frac{2\pi k}{sv}sin(svk)2cos(kct)+\frac{2\pi i^2}{s^2vc}sin(svk)2sin(kct)-\frac{2\pi ik}{cs}cos(svk)2isin(kct))$

$= \frac{q}{(2\pi)^3}(\frac{4\pi k}{sv}sin(svk)cos(kct)-\frac{4\pi}{s^2vc}sin(svk)sin(kct)+\frac{4\pi k}{cs}cos(svk)sin(kct))$

(†)

The electron wave propagates at speed c, so we are interested in the case when $|s\overline{v}| = ct$, so that $s = \frac{ct}{v}$. Making this substitution in (†), we obtain that;

$lim_{\epsilon\rightarrow 0}I_{n,\epsilon}(\frac{ct}{v}\overline{v}, t, k) = \frac{q}{(2\pi)^3}(\frac{4\pi k}{ct}sin(kct)cos(kct)-\frac{4\pi v}{c^3t^2}sin^2(kct)+\frac{4\pi kv}{c^2t}cos(kct)sin(kct))$

$= \frac{q}{(2\pi)^3}([\frac{4\pi k}{ct} + \frac{4\pi kv}{c^2t}]sin(kct)cos(kct) - \frac{4\pi v}{c^3t^2}sin^2(kct))$

$= \frac{q}{(2\pi)^3}([\frac{2\pi k}{ct} + \frac{2\pi kv}{c^2t}]sin(2kct) - \frac{4\pi v}{c^3t^2}(\frac{1-cos(2kct)}{2}))$

$= \frac{q}{(2\pi)^3}([\frac{2\pi k}{ct} + \frac{2\pi kv}{c^2t}]sin(2kct) + \frac{2\pi v}{c^3t^2}cos(2kct) - \frac{2\pi v}{c^3t^2})$

We look for a local maximum in k, so that;

$\frac{d}{dk}lim_{\epsilon\rightarrow 0}I_{n,\epsilon}(\frac{ct}{v}\overline{v}, t, k) = \frac{d}{dk}(\frac{q}{(2\pi)^3}([\frac{2\pi k}{ct} + \frac{2\pi kv}{c^2t}]sin(2kct) + \frac{2\pi v}{c^3t^2}cos(2kct)$

$-\frac{2\pi v}{c^3t^2}))$

$= \frac{q}{(2\pi)^3}([\frac{2\pi}{ct}+\frac{2\pi v}{c^2t}]sin(2kct)+2ct[\frac{2\pi k}{ct}+\frac{2\pi kv}{c^2t}]cos(2kct)-2ct\frac{2\pi v}{c^3t^2}sin(2kct))$

$= \frac{q}{(2\pi)^3}([\frac{2\pi}{ct} + \frac{2\pi v}{c^2t}]sin(2kct) + [4\pi k + \frac{4\pi kv}{c}]cos(2kct) - \frac{4\pi v}{c^2t}sin(2kct))$

$= \frac{q}{(2\pi)^3}([\frac{2\pi}{ct} - \frac{2\pi v}{c^2t}]sin(2kct) + [4\pi k + \frac{4\pi kv}{c}]cos(2kct))$

$= 0$

so that, rearranging;

$k = -Dtan(2kct)$

where $D = \frac{\frac{2\pi}{ct}-\frac{2\pi v}{c^2t}}{4\pi(1+\frac{v}{c})} = \frac{c-v}{2ct(c+v)}$

Now use the fact that the distance at time t is $d = ct$.

....

$\square$

References

[1] Partial Differential Equations, 2nd Edition, L. Evans, AMS.

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