

CONSERVATION OF ENERGY IN ATOMIC SYSTEMS

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ABSTRACT.

Lemma 0.1. *Let $(\rho, \bar{J}, \bar{E}, \bar{B})$ be an electromagnetic system, satisfying Maxwell's equations on $\mathcal{R}^4$, then if the system is in thermal equilibrium and constant temperature, with U_{mech} the mechanical energy density, then;*

$$\nabla(U_{mech}) = \frac{\bar{f}}{3}$$

where $\bar{f} = \rho\bar{E} + \bar{J} \times \bar{B}$ is force density.

Proof. By the first law of thermodynamics;

$$dU_{mech} = dU = dQ - pdV = -pdxdydz$$

where p is electromagnetic pressure, dV is an element of volume and we have used the fact that temperature is constant and no heat is released to show that $dQ = 0$. It follows that;

$$\frac{\partial U_{mech}}{\partial x} = \frac{dU}{dx} = -pdydz$$

where p is the pressure on the wall $dydz$. By the definition of pressure as $p = \frac{F}{A}$, where F is the force exerted by the system and A is area, we have that $p = -\frac{\bar{f}_1}{3dydz}$, where 3 is included to model the fact that if the particles are in thermal equilibrium, they are in random motion like an ideal gas, and a third of the particles will exert a force on the wall $dydz$. It follows that;

$$\frac{\partial U_{mech}}{\partial x} = \frac{\bar{f}_1}{3dydz}dydz = \frac{\bar{f}_1}{3}$$

The argument for the partial derivatives $\{\frac{\partial U_{mech}}{\partial y}, \frac{\partial U_{mech}}{\partial z}\}$ is the same, which gives the result by the definition of $\nabla(U_{mech})$. $\square$

Definition 0.2. *By an atomic system, we mean charge and current $(\rho, \bar{J})$ on $\mathcal{R}^4$, satisfying the continuity equation;*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{J} = 0$$

confined to a bounded volume V , with $\bar{J}|_{\partial V} = \bar{0}$ in the base frame S . For the purposes of this paper, we restrict ourselves to the case when $V = B(\bar{0}, r)$, $r > 0$, with $\delta V = S(\bar{0}, r)$. We remark that we constructed such $(\rho, \bar{J})$ in [1] and [5], with $(\rho, \bar{J})$, satisfying additional wave like relations.

We say that the system is non radiating if for every volume $U_{t, \bar{v}}$ inside $V_{t, \bar{v}}$, travelling with velocity $-\bar{v}$ in an inertial frame $S_{\bar{v}}$, with velocity $\bar{v}$ relative to S , there exist electromagnetic fields $(\bar{E}_{\bar{v}}, \bar{B}_{\bar{v}})$ on $V_{t, \bar{v}}$, satisfying Maxwell's equations, with the property that;

$$\frac{d}{dt} \int_{U_{t, \bar{v}}} (U_{el} + U_{mech}) dV_{t, \bar{v}} = 0$$

where $U_{el} = \frac{1}{2}(\epsilon_0 E_{\bar{v}}^2 + \frac{1}{\mu_0} B_{\bar{v}}^2)$ is the energy stored in the fields and $E_{\bar{v}} = |\bar{E}_{\bar{v}}|$, $B_{\bar{v}} = |\bar{B}_{\bar{v}}|$.

Intuitively, the condition means, that energy is conserved and exchanged between the mechanical energy of electrons and the energy stored in the fields, with no flux of energy within the moving volume itself or to the outside.

Lemma 0.3. *A non radiating atomic system as in Definition 0.2 satisfies the wave like relations of [1], further noted at the beginning of [3].*

Proof. By the Reynolds transport theorem, Poynting's theorem, the divergence theorem and elementary vector identities, we have that;

$$\begin{aligned} \frac{d}{dt} \int_{U_{t, \bar{v}}} (U_{el} + U_{mech}) dV_{t, \bar{v}} &= \int_{U_{t, \bar{v}}} \left(\frac{\partial U_{el}}{\partial t} + \frac{\partial U_{mech}}{\partial t} \right) dV_{t, \bar{v}} \\ &+ \int_{\delta U_{t, \bar{v}}} \hat{n} \cdot (-\bar{v})(U_{el} + U_{mech}) d\delta U_{t, \bar{v}} \\ &= \int_{U_{t, \bar{v}}} -\frac{1}{\mu_0} (\nabla \cdot \bar{E}_{\bar{v}} \times \bar{B}_{\bar{v}}) dV_{t, \bar{v}} + \int_{U_{t, \bar{v}}} \nabla \cdot (-\bar{v}(U_{el} + U_{mech})) dV_{t, \bar{v}} \\ &= \int_{U_{t, \bar{v}}} -\frac{1}{\mu_0} (\nabla \cdot \bar{E}_{\bar{v}} \times \bar{B}_{\bar{v}}) - \bar{v} \cdot \nabla (U_{el} + U_{mech}) dV_{t, \bar{v}} \end{aligned}$$

$$= 0$$

so that, as $U_{t,\bar{v}}$ was arbitrary inside $V_{t,\bar{v}}$, we must have that, for every inertial frame $S_{\bar{v}}$, there exist $(\bar{E}_{\bar{v}}, \bar{B}_{\bar{v}})$ on $V_{t,\bar{v}}$ such that;

$$-\frac{1}{\mu_0}(\nabla \cdot \bar{E}_{\bar{v}} \times \bar{B}_{\bar{v}}) - \bar{v} \cdot \nabla(U_{el} + U_{mech}) = 0$$

so that, using results of [4] and [5], we have that in the frames $\{S_{1,\infty}, S_{2,\infty}, S_{3,\infty}\}$, there exist;

$$\{(\bar{E}_{1,\infty}, \bar{B}_{1,\infty}), (\bar{E}_{2,\infty}, \bar{B}_{2,\infty}), (\bar{E}_{3,\infty}, \bar{B}_{3,\infty})\}$$

with corresponding;

$$\{(\rho_{1,\infty}, \bar{J}_{1,\infty}), (\rho_{2,\infty}, \bar{J}_{2,\infty}), (\rho_{3,\infty}, \bar{J}_{3,\infty})\}$$

$$\{U_{el,1,\infty}, U_{mech,1,\infty}, U_{el,2,\infty}, U_{mech,2,\infty}, U_{el,3,\infty}, U_{mech,3,\infty}\}$$

on;

$$\{V_{t_{\infty},1,\infty}, V_{t_{\infty},2,\infty}, V_{t_{\infty},3,\infty}\}$$

such that;

$$-\frac{1}{\mu_0}(\nabla_{1,\infty} \cdot \bar{E}_{1,\infty} \times \bar{B}_{1,\infty}) - \bar{e}_{1,\infty} \cdot \nabla_{1,\infty}(U_{el,1,\infty} + U_{mech,1,\infty}) = 0$$

$$-\frac{1}{\mu_0}(\nabla_{2,\infty} \cdot \bar{E}_{2,\infty} \times \bar{B}_{2,\infty}) - \bar{e}_{2,\infty} \cdot \nabla_{2,\infty}(U_{el,2,\infty} + U_{mech,2,\infty}) = 0$$

$$-\frac{1}{\mu_0}(\nabla_{3,\infty} \cdot \bar{E}_{3,\infty} \times \bar{B}_{3,\infty}) - \bar{e}_{3,\infty} \cdot \nabla_{3,\infty}(U_{el,3,\infty} + U_{mech,3,\infty}) = 0$$

It follows that, noting that, in the above equations, the left hand term is finite, while the right hand term is infinite unless, in the frames $\{S_{1,\infty}, S_{2,\infty}, S_{3,\infty}\}$ respectively, we have that;

$$(\nabla_{1,\infty}(U_{el,1,\infty} + U_{mech,1,\infty}))_1 = 0$$

$$(\nabla_{2,\infty}(U_{el,2,\infty} + U_{mech,2,\infty}))_2 = 0$$

$$(\nabla_{3,\infty}(U_{el,3,\infty} + U_{mech,3,\infty}))_3 = 0$$

Using Lemma 0.1, we obtain, in the frames $\{S_{1,\infty}, S_{2,\infty}, S_{3,\infty}\}$ respectively;

$$\nabla_{1,\infty}(U_{el,1,\infty}) + \frac{1}{3}(\bar{f}_{1,\infty})_1 = 0$$

$$\nabla_{2,\infty}(U_{el,2,\infty}) + \frac{1}{3}(\bar{f}_{2,\infty})_2 = 0$$

$$\nabla_{3,\infty}(U_{el,3,\infty}) + \frac{1}{3}(\bar{f}_{3,\infty})_3 = 0$$

and, by the transformation rules for infinite boosts, given in Lemma 0.2 of [5], the transformation of $\bar{f}$ to $(\bar{E}, \bar{J})$ in the base frame S , given in Lemma 0.2 of [5], and noting the improvement in Lemma 0.5 of [5], we obtain, in the base frame S , $(\bar{E}, \bar{B})$ such that;

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0|(e_1, icb_3, -icb_2)|^2 + \frac{1}{\mu_0}|(b_1, -\frac{i}{c}e_3, \frac{i}{c}e_2)|^2)) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0|(-icb_3, e_2, icb_1)|^2 + \frac{1}{\mu_0}|(\frac{i}{c}e_3, b_2, -\frac{i}{c}e_1)|^2)) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0|(icb_2, -icb_1, e_3)|^2 + \frac{1}{\mu_0}|(-\frac{i}{c}e_2, \frac{i}{c}e_1, b_3)|^2)) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

and, expanding, we obtain;

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0(e_1^2 - c^2(b_2^2 + b_3^2)) + \frac{1}{\mu_0}(b_1^2 - \frac{1}{c^2}(e_2^2 + e_3^2)))) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0(e_2^2 - c^2(b_1^2 + b_3^2)) + \frac{1}{\mu_0}(b_2^2 - \frac{1}{c^2}(e_1^2 + e_3^2)))) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

$$-\frac{i}{c}\frac{\partial}{\partial t}(\frac{1}{2}(\epsilon_0(e_3^2 - c^2(b_1^2 + b_2^2)) + \frac{1}{\mu_0}(b_3^2 - \frac{1}{c^2}(e_1^2 + e_2^2)))) + \frac{i}{3c}(\bar{E}, \bar{J}) = 0$$

Adding the relations, we obtain that;

$$-\frac{i}{c}\frac{d}{dt}(U_{el} - 2U_{el} - U_{mech}) = 0$$

in the base frame S , so that;

$$\frac{d}{dt}(U_{el} + U_{mech}) = 0$$

and, using Poynting's theorem;

$$\nabla \cdot \bar{E} \times \bar{B} = 0$$

If the atomic system is non radiating, it is an exercise to show that it is non radiating in $S_{\bar{v}}$, travelling with velocity $\bar{v}$ relative to S . We, therefore, conclude that for any inertial frame $S_{\bar{v}}$, there exist electromagnetic fields $(\bar{E}_{\bar{v}}, \bar{B}_{\bar{v}})$, on the moving volume $V_{t,\bar{v}}$, such that;

$$\nabla_{\bar{v}} \cdot \bar{E}_{\bar{v}} \times \bar{B}_{\bar{v}} = 0$$

By the main result in [4], this implies there exists $(\bar{E}_{\bar{v}}, \bar{B}_{\bar{v}})$, on the moving volume $V_{t,\bar{v}}$, with $\bar{B}_{\bar{v}} = \bar{0}$. By the results of [1], this implies, that the original $(\rho, \bar{J})$ in the base frame S satisfy the wave like relations explained there and at the beginning of [3].

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REFERENCES

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